Preface

Associativity is an ancient concept. However, the modern theory of the functional equation of associativity on the real line begins with a celebrated paper of J. Aczél, written in 1949, in which he gives a general representation theorem for associative functions on intervals.

The impetus for this book stems from the theory of probabilistic metric spaces. There, in connection with the triangle inequality, it became necessary to have large classes of associative functions at one's disposal, to know the properties of individual associative functions, the relationship between pairs of associative functions, etc. Subsequently, the same need arose in the Kampé de Fériet-Forte theory of information without probability, in theories of multivalued logic and fuzzy sets and in various problems in statistics.

By the early 1980's, problems centered around the associativity equation were receiving considerable attention, the literature on the subject was growing but widely scattered, and many facts that were well known to experts in the field were constantly being rediscovered by newcomers. Accordingly, at an International Symposium on Functional Equations which was held in Oberwolfach in December 1984, we decided to gather the basics of the theory in one place and, in the process, simplify many proofs and add a number of new results. But then, for personal and professional reasons (e.g., the first two authors became involved with heavy academic responsibilities and, more recently, the third author was occupied with the editing of the "Karl Menger Selecta Mathematica"), the work bogged down, at times to a standstill. In the meantime, an explosion of interest in and work on associative functions (in particular, t-norms and associative copulas) was taking place. Numerous papers appeared; a special issue of Fuzzy Sets and Systems (Vol. 104, No. 1, May 1999) was devoted to the topic;

the essentials were presented in chapters of several books, e.g., [Gottwald (1993); Fodor and Roubens (1994b); Nguyen and Walker (2000); Nelsen (1999); Hadzic and Pap (2001)]. These, as their titles indicate, are more concerned with applications than with fundamentals. Then, in 2000, the book "Triangular Norms" by E.P. Klement, R. Mesiar and E. Pap was published. Surprisingly – and contrary to what one might expect – aside from the very basic facts, many of which are contained in the book "Probabilistic Metric Spaces" by the third author and A. Sklar, there is little overlap between our book and the book by Klement, Mesiar and Pap. Indeed, the two books complement each other very well. Thus, in spite of the passage of time, we can still say that the contents of this book – which have of course been brought up to date and which include many results not heretofore published – are the foundation on which all the other developments rest.

This book is divided into four chapters. Chapter 1 is introductory. In it, we first present a brief overview of the basic facts concerning some of the classical functional equations and the associated inequalities: they all play an important role in our studies. The bulk of the chapter is devoted to introducing the associative functions which will be our primary concern in the remainder of the book, the so-called t-norms, and two other classes of functions, s-norms and copulas. We define these functions, derive some of their basic properties, and establish some of the relations within and among these classes. In this chapter, we also establish the terminology and the notational conventions which we use in the sequel.

Chapter 2 is devoted to the basic representation theorem for associative functions and some of its consequences. We prove this theorem in the form which is most suitable for our purposes and discuss various other versions as well as generalizations. This chapter also includes a table in which we list a number of one-parameter families of frequently encountered t-norms, together with their properties. Chapters 3 and 4 are devoted to functional equations and inequalities that involve associative functions. Many of the results presented here stem from problems that arose in connection with further developments of the theory.

The book concludes with two appendices and an extensive bibliography. In Appendix A, we list examples and counterexamples that illuminate the text. In Appendix B, we present a series of open problems for further research.

This book is intended to be primarily a reference work. Nevertheless, it is also suitable for use as a text for a one-semester advanced undergraduate or beginning graduate course on functional equations. True, we deal primarily with one class of functions and a restricted class of functional equations. However, in the process, we employ most of the standard techniques – and some new ones – for solving functional equations and many of the basic equations and inequalities are encountered in the process. Thus, it can be argued that the fact that this book is focused on one central theme is an advantage rather than a hindrance.

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