

Chapter 1

Quantum Mechanics: Perspectives

Quantum mechanics essentially deals with geometric and statistical concepts, involving information and measurement theories. Geometric concepts are used to build up quantum-state space and its supporting or base manifold¹, with its corresponding metric and algebra, examples are phase space, Hilbert space, etc. Because of its statistical nature, quantum states are no longer represented by a determinate point in phase space in contrast with classical mechanics. Canonical dynamical variables are represented by operators, such operators as the Hamiltonian, position operator, momentum operator, phase-space point projectors, and phase-space line projectors, whose eigenvalues correspond to classical quantities. What is perhaps even more remarkable is that a general quantum-state Hilbert-space vector is represented as a superposition of a complete set of Hilbert-space eigenvectors, basically encompassing all classical possibilities at a given instant of time. Canonically conjugate dynamical variables can no longer be measured simultaneously due to the quantum-mechanical uncertainty principle².

On the other hand, the informatics aspects of physics is common to classical and quantum physics, and pertains to abstract concepts of the basic guiding principles. Thus, the idea that ‘information is physical’ is still a subject of debate. Information theory deals with the quantification of uncertainty, indeterminism, competing possibilities, and chance, as well as the *abstract* characterization of a collection of different ‘forms’ (or *different* configurations) through the concept of *entropy*. However, the concept of measurement does not play a mathematically formal role in classical mechanics. The reason for this is that classical mechanics deals directly with measurable, deterministic, or realizable quantities, corresponding to the eigenvalues in quantum mechanics. In quantum mechanics, operator algebra of measurement is an integral part and occupies a central formal operational concept in terms of quantum operators which operate on suitable abstract space. According to Schwinger, quantum mechanics is a mathematical formalism of physical measurements.

In view of its informatics aspects, classical mechanics is a prerequisite for understanding the foundation of quantum mechanics. In fact, the abstract informatics aspect of classical mechanics, such as the maximum entropy principle, ergodicity,

¹One familiar example is the space-time manifold.

²Mathematically, their respective quantum states are related by a unitary transformations, specifically by Fourier transformations.

the principle of least action, causality, symmetry and invariance guiding principle³, are valuable theoretical tools in quantum mechanics. It is not possible to have a good working appreciation of the principles of quantum mechanics without a familiarity of the informatics aspect of classical mechanics. More recently, research in quantum computing and quantum entanglement calls for a deeper re-examination of classical information theory. Indeed, quantum information theory is a very active field of theoretical and experimental investigations and is expected to play an important role in the scientific advances of the 21st century.

The beginning of the 20th century was timely for the formal development of quantum mechanics. The Hamilton and Lagrange formulation of classical mechanics provided the mathematical abstraction of classical mechanics in terms of generalized coordinates and canonical dynamical variables⁴. Lagrange's equations looks the same in any coordinate system. Indeed, the classical Hamiltonian (expressed in terms of generalized coordinates and momentum) is the Legendre transform of the classical Lagrangian (expressed in terms of generalized coordinates and velocities). In 1746 Maupertuis first formulated the *Principle of Least Action* (PLA), further developed by the three great mathematicians, Euler, Lagrange, and Hamilton. PLA brought the concept of optimization, considered as *the intelligence* or *supreme guiding principle* to classical mechanics. Indeed, the PLA is considered one of the greatest generalizations in physical science, useful in the development of quantum mechanics. PLA describes the tendency of physical changes and processes to take the *optimum* path if left alone.

Almost the whole of physics obeys the geodetic form. Water seeks the steepest descent, and distributes itself so that its surface is as low as possible, the water then has the minimum potential energy in the earth's gravitational field. Light finds the quickest trajectory through an optical system (Fermat's principle of Least Time). The path of a body in a gravitational field (i.e. free fall in space time) is a geodesic. Feynman's formulation of quantum mechanics is based on a PLA, using path integrals. Schwinger's formulation of quantum field theory makes use of the PLA. Maxwell's equations can be derived as conditions of least action. Newton's mechanics is derived from Hamilton's principle of least action, and also Gauss's principle of least constraint. Thomson's theorem states that electrically charged particles configure so as to have the least energy. The Second Law of Thermodynamics requires that thermal systems change along a sequence of configurations, each having a higher probability of occurrence than the preceding configuration. In fact optimization in all aspect of physical, life, and socioeconomic dynamics has been seen to be the rule. Quantum mechanics also embodies these 'informatics' elements of classical mechanics, generalized probability theory, and information theory.

³This guiding principle was notably first used by Einstein in constructing his theory of relativity. Indeed, symmetry considerations for establishing the Lagrangian coupled with PLA has been the *modus operandi* of major advances in modern theoretical physics.

⁴In going over to quantum mechanics, the classical canonical variables becomes noncommuting operators for bosons (whose eigenvalues are c-numbers) and non-anticommuting operators for fermions (whose eigenvalues obey Grassmann algebra). Complex-valued canonical variables readily appeared in classical harmonic oscillator system and its quantization leads to the concept of ladder operators or creation and annihilation operators for bosons.

The central idea inherent in quantum mechanics lies in the mathematical concept of measurement, defined as an application of a *measurement operator* to an abstract quantum state of a physical dynamical system. In quantum mechanics, the state of a physical system is identified with an abstract vector in a complex Hilbert space (H-space), without any reference to specific chosen basis states. Each abstract vector in H-space is called a “ket”, and written as $|\Psi\rangle$, and every ket has a dual bra, written as $\langle\Psi|$ in the dual H-space. For conservative systems, the bra is simply the conjugate transpose (also called the Hermitian conjugate) of the ket. Each bra corresponds to exactly one ket, and vice versa, hence the use of the same labeling. One may also interpret $|\Psi\rangle$ as a column vector under any arbitrary chosen basis states, while $\langle\Psi|$ is the corresponding row vector under the same basis states with components as complex conjugate of the components of $|\Psi\rangle$. The inner product $\langle\Phi|\Psi\rangle$ is the mapping from H-space to complex numbers, whereas $\langle\Psi|\Psi\rangle$ is a special case, called the norm of $|\Psi\rangle$, and maps to real numbers only. $|\Psi\rangle$ is in general a superposition of elements from a set of eigenstates which forms complete basis states in H-space,

$$\begin{aligned} |\Psi\rangle &= \sum_q \psi(q) |q\rangle, \\ &= \sum_q \langle q|\Psi\rangle |q\rangle, \end{aligned} \quad (1.1)$$

where q is the quantum label of a basis state considered and $\psi(q) = \langle q|\Psi\rangle$ is the *complex component* or projection of $|\Psi\rangle$ along the basis state $|q\rangle$, the so-called wavefunction or probability wavefunction. The absolute value of $\langle q|\Psi\rangle$, which is given by $|\psi(q)|^2$ is the probability of finding the quantum state in the state $|q\rangle$.

An eigenstate is defined as the state in H-space which yield deterministic value of the measurement operator,

$$Q|q\rangle = q|q\rangle,$$

where Q is the mathematical symbol for a measurement operator and q is the corresponding deterministic eigenvalue or classical quantity. The projection operator for the state $|q\rangle$ in H-space is defined by $|q\rangle\langle q|$. The completeness of the eigenstate vectors as basis states in H-space is expressed by

$$\sum_q |q\rangle\langle q| = 1.$$

Hermitian operators corresponds to observables such as position, momentum, and energy. The eigenstates of a Hermitian operator are orthonormal,

$$\langle q'|q\rangle = \delta_{q',q}.$$

Thus the measurement operator Q may be written as

$$Q = \sum_{q'} q' |q'\rangle\langle q'|.$$

Any arbitrary operator other than Q may be expanded in terms of the q -eigenstates as

$$P = \sum_{q''q'} \langle q'' | P | q' \rangle | q'' \rangle \langle q' |.$$

In this sense measurement operators are projectors in H-space. The role played by the position operator Q and momentum operator P typifies the role of other canonical operators, Hermitian or non-Hermitian, having either real or complex-valued eigenvalues.

The combined state vector of two or more quantum mechanical systems are defined by the tensor product of their respective state vectors, the result can thus be either in ‘entangled’ and/or ‘separable product’ quantum state. Quantum entanglement is the functional basis of quantum computers consisting of quantum bits or ‘qubits’ which has found explosive development in the last decade. In later chapters, measurement operators are generalized to quantum operation on the subsystem of a composite system where the total state is described by quantum-state tensors instead of quantum-state vector.

In view of the general description of a quantum state as superposition of eigenstates, quantum mechanics deals with the totality of all possibilities that the whole physical system may evolve and can be realized at a given time. Equation of motion in quantum dynamics deals with the wavefunctions (or Wigner distribution function⁵) in the Schrödinger picture, or deals with the quantum operators in the Heisenberg picture, not with the canonical classical variables directly in stark contrast with the determinism of classical mechanics. For example, if the eigenstate $|q\rangle$ represent the eigenstate of the position operator, then $\psi(q)$ in Eq. (1.1) represents a complex wavefunction whose value, $|\psi(q)|^2$, measure the probability density in real space. This is clearly a generalization⁶ of positive-valued measure of classical probability theory, a generalization which allows for quantum interference effects in its dynamics.

Any measurement process which is intended to gain definite location of the system then collapses the state of the physical system to a specific position eigenstate, i.e., a specific location in real space. This has come to be known as the ‘wavepacket collapse’ in the quantum physics community. This collapse however entails complete lack of knowledge of the canonically conjugate momentum, i.e., somehow a decrease of entropy with respect to the position results in maximum entropy with respect to the momentum since subsequent measurements of momentum in this collapse state will only yield completely unpredictable random values of the momentum. This is another expression of the quantum mechanical uncertainty principle.

⁵We shall see later that the Wigner distribution function represents the component of the density operator expanded in terms of the complete set of phase-space point projectors, often denoted by $\Delta(p, q)$.

⁶This is reminiscent of the generalization of positive real numbers to complex numbers.

1.1 Wave Mechanics of Particles: Schrödinger Wave Function

The state of the particle in classical mechanics is described by a point in momentum (p)-position (q) space or phase space. The passage to quantum mechanics is to abandon the classical phase space point as representing the state of the particle. The momentum of the particle must now be measured by applying the momentum operator P on the eigenstates of momentum, $|p\rangle$, in Hilbert space. Likewise the position of the particle must be measured by applying the position operator Q on the eigenstate of position, $|q\rangle$, in Hilbert space. In quantum mechanics these two ‘measurements’ do not commute, since the measurement of position abandon any information about the momentum and vice versa. This can be illustrated by simulating the measurement in the limit of zero error, as in classical mechanics, using the position operator Q whose eigenvalue is q and abstract eigenket $|q\rangle$. In this limiting case, the corresponding eigenstate in Hilbert space is the Dirac-delta function in the coordinate representation given by $|x, q\rangle = \langle x | q\rangle = \delta(x - q)$ centered at position q . Thus, we $Q |x, q\rangle = q |x, q\rangle$. Since the set of position eigenfunctions, $\{|q\rangle\}$, forms a complete basis states, we can expand the momentum eigenfunction in terms of the position eigenfunctions of all q values. Thus, in the coordinate representation we can write

$$|x, p\rangle = \langle x | p\rangle = \sum_q \langle q | p\rangle |x, q\rangle.$$

By virtue of the completeness of the momentum eigenfunctions, the inverse transform then leads to

$$|x, q\rangle = \sum_p \langle p | q\rangle |x, p\rangle.$$

Thus the transformation function is the Fourier transformation function,

$$\langle q | p\rangle = \exp\left(-\frac{i}{\hbar}p \cdot q\right), \quad (1.2)$$

with $|x, p\rangle = C \exp\left(\frac{i}{\hbar}p \cdot x\right)$, where C is the normalization constant, usually taken to be $\frac{1}{\sqrt{V}}$, where V is the volume. Clearly, if the state of the particle after measurement collapses to the position eigenstates, $\delta(x - q)$, then all the momentum values are *equally probable* in this collapse state and hence all information of the momentum is lost and the subsequent application of the momentum operator will yield an indeterminate or *random* values for the momentum. This embodies the statement that the operators Q and P do not commute.

In general, an arbitrary quantum-state vector of the particle, $|\Psi\rangle$ is a superposition of the complete basis in Hilbert space spanned by either the abstract momentum or position eigenstates. If we expand $|\Psi\rangle$ in terms of the position eigenstates, we have

$$|\Psi\rangle = \sum_q \psi(q) |q\rangle = C' \int dq \psi(q) |q\rangle.$$

In this coordinate representation, we have

$$Q |\Psi\rangle = \sum_q q \psi(q) |q\rangle = C' \int dq q \psi(q) |q\rangle$$

and

$$\begin{aligned} P |\Psi\rangle &= \sum_q \psi(q) P |q\rangle = C' \int dq \psi(q) P |q\rangle \\ &= C' \int dq \psi(q) \sum_p p \exp\left\{-\frac{i}{\hbar} p \cdot q\right\} |p\rangle \\ &= C' \int dq \psi(q) \frac{\hbar}{-i} \nabla_q \sum_p \exp\left\{-\frac{i}{\hbar} p \cdot q\right\} |p\rangle, \\ C' \int dq \psi(q) P |q\rangle &= C' \int dq \psi(q) \frac{\hbar}{-i} \nabla_q |q\rangle. \end{aligned} \quad (1.3)$$

Upon integrating by parts in the last line, we have

$$P |\Psi\rangle = C' \int dq \left[\frac{\hbar}{i} \nabla_q \psi(q) \right] |q\rangle.$$

Note that the momentum operator acting on the basis states is $i\hbar\nabla_q$, Eq. (1.3), whereas the same momentum operator acting on the probability amplitude $\psi(q)$ is $-i\hbar\nabla_q$. Now the Schrödinger wavefunction in the coordinate representation is given by $\psi(q) = \langle q | \Psi \rangle$. By virtue of the orthonormality of the basis states, it follows that the momentum and position operators for the Schrödinger wavefunction are defined by the following relations,

$$P\psi(q) \Rightarrow -i\hbar\nabla_q\psi(q), \quad (1.4)$$

$$Q\psi(q) \Rightarrow q\psi(q). \quad (1.5)$$

In terms of momentum eigenstates we also have

$$|\Psi\rangle = \sum_p \psi(p) |p\rangle = C' \int dp \psi(p) |p\rangle.$$

Thus, we also have in the momentum representation of the Schrödinger wavefunction,

$$P\psi(p) \Rightarrow p\psi(p), \quad (1.6)$$

$$Q\psi(p) \Rightarrow i\hbar\nabla_p\psi(p), \quad (1.7)$$

$$Q|p\rangle \Rightarrow -i\hbar\nabla_p|p\rangle. \quad (1.8)$$

The noncommutation relation between P and Q is therefore expressed as

$$[Q, P] = i\hbar.$$

We can also deduce the noncommutation relation between P and Q using the basis states $|q\rangle$. We have

$$P |q\rangle = i\hbar \nabla_q |q\rangle.$$

Therefore

$$P |q\rangle \langle q| = (i\hbar \nabla_q |q\rangle) \langle q|.$$

Integrating or summing over q and using the closure relation q -eigenstates, we obtain

$$\begin{aligned} \sum_q P |q\rangle \langle q| &= \sum_q (i\hbar \nabla_q |q\rangle) \langle q|, \\ P \sum_q |q\rangle \langle q| &= P = \int dq (i\hbar \nabla_q |q\rangle) \langle q| \\ P &= i\hbar \int dq \{ \nabla_q (|q\rangle \langle q|) - |q\rangle (\nabla_q \langle q|) \} \\ &= -i\hbar \int dq |q\rangle (\nabla_q \langle q|). \end{aligned}$$

The representation for the position operator Q is trivial

$$Q = \sum_q q |q\rangle \langle q|.$$

The commutation of Q and P using the basis vector now reads

$$\begin{aligned} [Q, P] &= -i\hbar \int dq q |q\rangle \langle q| \left(\int dq' |q'\rangle \nabla_{q'} \langle q'| \right) + i\hbar \int dq' |q'\rangle \nabla_{q'} \langle q'| \int dq q |q\rangle \langle q| \\ &= -i\hbar \int dq' dq q |q\rangle \langle q| |q'\rangle \nabla_{q'} \langle q'| + i\hbar \int dq' dq \nabla_{q'} \langle q'| q |q\rangle \langle q| \\ &= -i\hbar \int dq' dq q |q\rangle \delta(q - q') \nabla_{q'} \langle q'| + i\hbar \int dq' dq \nabla_{q'} \langle q'| q |q\rangle \delta(q - q') \\ &= -i\hbar \int dq q |q\rangle \nabla_q \langle q| + i\hbar \int dq \nabla_q \langle q| q |q\rangle \\ &= -i\hbar \int dq q |q\rangle \nabla_q \langle q| + i\hbar \int dq (\nabla_q \langle q|) q |q\rangle + i\hbar \nabla_{qq} \int dq |q\rangle \langle q| \\ &\quad + i\hbar \int dq q \langle q| \nabla_q |q\rangle. \end{aligned}$$

The first two terms cancel and we are left with

$$\begin{aligned} [Q, P] &= i\hbar \int dq \nabla_q q |q\rangle \langle q| + i\hbar \int dq q \langle q| \nabla_q |q\rangle \\ &= i\hbar + i\hbar \int dq q \langle q| \nabla_q |q\rangle, \end{aligned}$$

where we have used the relation, $\nabla_{q_i} q_j = \delta_{ij}$. We can show that the last term is zero by writing

$$\begin{aligned} \int dq q \langle q | \nabla_q | q \rangle &= \int dq q \langle q | i\hbar \nabla_q \int dp e^{-\frac{i}{\hbar} p \cdot q} | p \rangle \\ &= \int dq dp q \langle q | e^{-\frac{i}{\hbar} p \cdot q} p | p \rangle \\ &= \int dq dp q e^{-\frac{i}{\hbar} p \cdot q} p \langle q | p \rangle \\ &= \int dq dp q e^{-\frac{i}{\hbar} p \cdot q} p e^{\frac{i}{\hbar} p \cdot q} \\ &= \int dq q \int dp p \\ &= 0, \end{aligned}$$

since both remaining integrals have odd integrand.

The noncommutativity of Q and P signifies that their bases states are *mutually unbiased*, in the sense that the complete knowledge the eigenstates of P renders all the eigenstates of Q to be equally probable, and vice versa. Note that we have used the same basis states to represent Q and P to evaluate the commutation relation.

1.1.1 Some Algebraic Relations of Q and P

From this commutation relation one can easily prove by mathematical induction the following algebraic relations involving powers of Q and P ,

$$[Q, P^m] = i\hbar m P^{m-1} = i\hbar \frac{\partial}{\partial P} P^m, \quad (1.9)$$

$$[P, Q^m] = -i\hbar m Q^{m-1} = -i\hbar \frac{\partial}{\partial Q} Q^m. \quad (1.10)$$

From these relations, it also follows that if $F(P)$ and $G(Q)$ are functions that may be expanded in power series, then

$$[Q, F(P)] = i\hbar \frac{\partial}{\partial P} F(P), \quad (1.11)$$

$$[P, G(Q)] = -i\hbar \frac{\partial}{\partial Q} G(Q). \quad (1.12)$$

For a function of P and Q , namely, $F(P, Q)$, we have

$$[Q, F(P, Q)] = i\hbar \frac{\partial}{\partial P} F(P, Q), \quad (1.13)$$

$$[P, F(P, Q)] = -i\hbar \frac{\partial}{\partial Q} F(P, Q), \quad (1.14)$$

where in the last two relations, the order of the factors involving Q and P in $F(P, Q)$ must be preserved.

1.1.2 Deterministic Schrödinger Wave Equation

The particle Hamiltonian operator, \mathcal{H} , acting on the state $|\Psi\rangle$ now reads, using the position eigenfunction expansion of $|\Psi\rangle$,

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \mathcal{H} |\Psi\rangle = C' \int dq \left[\left\{ -\frac{\hbar^2}{2m} \nabla_q^2 + V(q) \right\} \psi(q, t) \right] |q\rangle.$$

Since \mathcal{H} is Hermitian, the presence of $i\hbar$ renders the time evolution as a *unitary* evolution of the quantum states. Upon multiplying by $\langle q'|$ on both side of the equation and using the orthonormality of the position eigenfunction, we obtain the Schrödinger equation in the coordinate representation,

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi(q, t) &= \mathcal{H} \psi(q, t) \\ &= \left\{ -\frac{\hbar^2}{2m} \nabla_q^2 + V(q) \right\} \psi(q, t), \end{aligned} \quad (1.15)$$

where $V(q)$ is the external potential seen by the particle. The above equation can readily be separated into temporal and spatial parts using separation of variables, by writing

$$\psi(q, t) = \psi(q) \Phi(t),$$

which yields

$$\left\{ -\frac{\hbar^2}{2m} \nabla_q^2 + V(q) \right\} \psi(q) = E \psi(q) \quad (1.16)$$

or

$$\mathcal{H} \psi(q) = E \psi(q),$$

where

$$\Phi(t) = e^{-\frac{i}{\hbar} E t}.$$

Similarly, in the momentum representation the Schrödinger equation is given by

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi(p, t) &= \mathcal{H} \psi(p, t) \\ &= \left\{ \frac{p^2}{2m} + V(i\hbar \nabla_p) \right\} \psi(p, t). \end{aligned}$$

Since in general $V(q)$ is a complicated function of q , the momentum representation of the Schrödinger equation is rarely employed and solved directly. However, it should be pointed out that in energy-band dynamics of Bloch electrons in solid, where the energy bands are *nondegenerate*, each energy band could be a complicated

function of the crystal momentum, $E_\lambda(p)$, so that the *effective* Hamiltonian for each band λ , to be used in Eq. (1.16), becomes⁷

$$\mathcal{H}_{eff} = E_\lambda(-i\hbar\nabla_q) + V_\lambda(q). \quad (1.17)$$

From the Schrödinger equation, Eq. (1.15), the expression for the expectation (average) value of energy or classical energy value is given by

$$H = \langle E \rangle = \int dq \psi^\dagger(q, t) \left\{ -\frac{\hbar^2}{2m} \nabla_q^2 + V(q) \right\} \psi(q, t), \quad (1.18)$$

where the time-dependence drops out for a conservative system. In the above expression, the order of $\psi^\dagger(q, t)$ relative to $\psi(q, t)$ is fixed, and no ambiguity should arise since these are classical quantities for now.

What we have done so far is to treat the quantum statistical aspects of the momentum and position measurements. The Schrödinger complex wavefunction $\psi(q, t)$ is called the probability amplitude since it is the coefficient in the expansion of a quantum state in terms of the position-eigenfunction basis states. The dynamical aspect is given by the time-dependent Schrödinger equation, which should be viewed as derivable from the Least Action Principle. Indeed, $i\hbar\psi^\dagger(q)$ and $\psi(q)$ acquire the status of a canonically conjugate variables. In the so-called *second quantization* scheme to treat *many-particle systems*, $\psi(q)$ and $\psi^\dagger(q)$ become annihilation and creation operators, respectively, at point q .

1.1.3 Isotopic Wavefunction and Many-Body Wavefunction

‘Isotopic’ or multi-component wavefunction is often associated with a particle or system under consideration which can have more than one internal degrees of freedom. This component wavefunction has the form, $\psi_\alpha(x)$, $\alpha = 1, 2, \dots, n$, where α is an internal parameter, such as spin for electron, for energy level of two-level atoms, or for a system and an environment which are accessible to a quasiparticle. The total wavefunction is written as,

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \dots \\ \psi_n(x) \end{pmatrix}.$$

On the other hand, state-tensor or *direct-product* wavefunction is often associated with many-body problems, such as a combined system of spin- $\frac{1}{2}$ electron and electromagnetic-field photons or lattice vibrational-mode phonons, where the wavefunction for each part must be specified simultaneously.

⁷For narrow-gap semiconductors and semimetals, the effective Hamiltonian is a matrix in the energy-band indices. When the quantum dynamics is limited to the energy-band edge, as in most cases using the $k \cdot p$ method, a two-band Hamiltonian matrix is often similar to the Hamiltonian of the relativistic Dirac electrons, such as the $k \cdot p$ Hamiltonian for bismuth-antimony alloys. A ‘Foldy-Woutheyesen type’ of energy-band decoupling procedure would also endow each decoupled energy-band dynamics with effective *g-factor* for the spin degree of freedom.

1.1.3.1 Decoupling of Isotopic Degrees of Freedom

The reduction of a simple⁸ isotopic problem, described by a multi-component wavefunction, to a single-component Schrödinger wave equation involves a modification of the diagonal Hamiltonian by the addition of an embedding potential or ‘self-energy’ terms. For example, for *time-independent* Schrödinger equation with two-component wavefunction,

$$\begin{pmatrix} H_{SS} & H_{SE} \\ H_{ES} & H_{EE} \end{pmatrix} \begin{pmatrix} \psi_S \\ \psi_E \end{pmatrix} = E \begin{pmatrix} \psi_S \\ \psi_E \end{pmatrix}$$

or

$$\begin{pmatrix} E - H_{SS} & -H_{SE} \\ -H_{ES} & E - H_{EE} \end{pmatrix} \begin{pmatrix} \psi_S \\ \psi_E \end{pmatrix} = 0,$$

can be written in terms of Schrödinger equation for ψ_S alone

$$\left(H_{SS} + H_{SE} (E - H_{EE})^{-1} H_{ES} \right) \psi_S = E \psi_S.$$

Thus, the elimination of ψ_E results in the inclusion of an extra energy-dependent term, which we denote by $\Sigma(E)$,

$$\Sigma(E) = H_{SE} (E - H_{EE})^{-1} H_{ES},$$

and the *self-consistent* ψ_S is an eigenfunction of the effective Hamiltonian, $\mathcal{H}_{eff} = H_{SS} + H_{SE} (E - H_{EE})^{-1} H_{ES}$.

For *time-dependent* problem, we have

$$\frac{i\hbar\partial}{\partial t} \begin{pmatrix} \psi_S \\ \psi_E \end{pmatrix} = \begin{pmatrix} H_{SS} & H_{SE} \\ H_{ES} & H_{EE} \end{pmatrix} \begin{pmatrix} \psi_S \\ \psi_E \end{pmatrix}$$

or

$$\frac{i\hbar\partial}{\partial t} \psi_S = H_{SS}\psi_S + H_{SE}\psi_E, \quad (1.19)$$

$$\left(\frac{i\hbar\partial}{\partial t} - H_{EE} \right) \psi_E = H_{ES}\psi_S. \quad (1.20)$$

We can formally solve for ψ_E of Eq. (1.20) using the retarded Green function, $\mathcal{G}(t-t')$, which satisfies

$$\left(\frac{i\hbar\partial}{\partial t} - H_{EE} \right) \mathcal{G}(t-t') = \hat{I}_{EE} \delta(t-t').$$

With $H_{ES} \psi_S$ as a source function in Eq. (1.20), we readily obtain

$$\psi_E(t) = \int_{-\infty}^t \mathcal{G}(t-t') H_{ES}\psi_S(t') dt'.$$

⁸For the Dirac relativistic Hamiltonian, the decoupling scheme is more involved through the so-called Foldy-Woutheyens transformation, which endow spin for the fermion quasiparticles.

Upon substituting in Eq. (1.19), we have in terms of ψ_S alone,

$$\frac{i\hbar\partial}{\partial t}\psi_S = H_{SS}\psi_S + \int_{-\infty}^t H_{SE}\mathcal{G}(t-t')H_{ES}\psi_S(t')dt'.$$

Note that the energy-dependence of the ‘self-energy’ of the time-independent problem has now emerge as a *memory* kernel of the time-dependent problem.

The decoupling schemes in the above simple examples are pretty straightforward, whereas the decoupling scheme for Dirac relativistic electrons and multi-band dynamics in crystalline solids makes use of the Foldy-Woutheyesen type of transformations. Moreover, we shall see in Part 4 of this book that the multicomponent superfield theory of real-time nonequilibrium Green’s function technique is much more involved. Indeed, the theory of nonequilibrium quantum physics involves both multicomponent and many-body quantum states.

1.1.3.2 Phenomenological ‘Decoupling’ or Reduction of Many-Body Problems

The reduction of direct-product problem, or many-body systems, has acquired great importance in the area of quantum information and computing. The entanglement of qubit subsystem with the environment, and the need to perform quantum operations on the qubit subsystem have focused attention to phenomenological superoperator and generalized quantum operations. In particular, this has led to generalization of the Von Neumann measurements with the so-called Kraus representation of a generalized quantum operation. These will be further discussed in Part 7 and 8 of this book.

1.2 Generator of Position Eigenstates

From Eq. (1.3), we have $P|q\rangle = i\hbar\nabla_q|q\rangle$. Using Fourier series expansion, one can generate coordinate representation of state $|q\rangle$, namely $\langle x|q\rangle = \delta(x-q)$ by a displacement operator $T(q)$,

$$\begin{aligned}\langle x|q\rangle &= \delta(x-q) \\ &= T(q)\delta(x) \\ &= \exp\{-q\cdot\nabla_x\}\delta(x) \\ &= \exp\{q\cdot\nabla_{q'}\}\delta(x-q')|_{q'\rightarrow 0} \\ &= \exp\left\{\frac{1}{i\hbar}q\cdot(i\hbar\nabla_{q'})\right\}\delta(x-q')|_{q'\rightarrow 0}.\end{aligned}$$

Therefore, we have

$$|q\rangle = \exp\left\{-\frac{i}{\hbar}q\cdot P\right\}|0\rangle, \quad (1.21)$$

where the operator $P = i\hbar\nabla_q$ is operating on the basis eigenstates $|q\rangle$, refer to Eq. (1.3), since a negative displacement in the components correspond to positive

displacement of the eigenstate basis. We also note from Eq. (1.11) that

$$\begin{aligned} \left[Q, \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} \right] &= i\hbar \frac{\partial}{\partial P} \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} \\ &= q \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\}. \end{aligned} \quad (1.22)$$

Therefore,

$$Q \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} = \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} Q + q \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\}, \quad (1.23)$$

$$\exp \left\{ \frac{i}{\hbar} q \cdot P \right\} Q \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} = Q + q. \quad (1.24)$$

Multiplying both sides of Eq. (1.23) by the eigenket with component $\langle x | q' \rangle = \delta(x - q')$ and taking the limit $q' \Rightarrow 0$, we have

$$Q \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} |0\rangle = \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} Q |0\rangle + q \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} |0\rangle,$$

this reduces to

$$Q \left[\exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} |0\rangle \right] = q \left[\exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} |0\rangle \right],$$

which verifies that the state $[\exp \{ -\frac{i}{\hbar} q \cdot P \} |0\rangle] = |q\rangle$ is an eigenstate of the position operator with displaced eigenvalue by q . However, if the limit $q' \Rightarrow 0$ is not taken then the state $[\exp \{ -\frac{i}{\hbar} q \cdot P \} |q'\rangle] = |q' + q\rangle$ is an eigenstate of the position operator with eigenvalue $q' + q$.

We can symmetrize the translation operator by inserting $\exp \{ \frac{i}{\hbar} p \cdot Q \}$ in front of $|0\rangle$ which effectively insert unity. We thus have

$$|q\rangle = \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} |0\rangle.$$

By the use of the Campbell-Baker-Hausdorff operator identity, namely,

$$\exp(A + B) = \exp(A) \exp(B) \exp\left(-\frac{[A, B]}{2}\right),$$

provided that

$$[A, [A, B]] = 0 = [B, [A, B]],$$

a condition that is easily satisfied concerning P and Q since $[Q, P] = i\hbar$, a c -number. We have $A = -\frac{i}{\hbar}q \cdot P$, $B = \frac{i}{\hbar}p \cdot Q$, and obtained

$$\begin{aligned} & \exp\left\{-\frac{i}{\hbar}q \cdot P\right\} \exp\left\{\frac{i}{\hbar}p \cdot Q\right\} \\ &= \exp\left\{-\frac{i}{\hbar}(q \cdot P - p \cdot Q)\right\} \exp\left\{\frac{\left[-\frac{i}{\hbar}q \cdot P, \frac{i}{\hbar}p \cdot Q\right]}{2}\right\} \\ &= \exp\left\{-\frac{i}{\hbar}(q \cdot P - p \cdot Q)\right\} \exp\left\{-\frac{i}{\hbar}\frac{p \cdot q}{2}\right\}. \end{aligned} \quad (1.25)$$

Therefore we have the symmetric form for the displacement operator generating the state $|x, q\rangle$ from $|x, 0\rangle$ given by

$$T(q)^{sym} = \exp\left\{-\frac{i}{\hbar}\frac{p \cdot q}{2}\right\} \exp\left\{\frac{i}{\hbar}(p \cdot Q - q \cdot P)\right\}. \quad (1.26)$$

The displacement operator may also be interpreted as an operator for the preparation of the quantum eigenstate out of the ‘vacuum’, $|0\rangle^9$, basis eigenstate. The symmetric operator

$$\begin{aligned} Y_{p,q} &= \exp\left\{\frac{i}{\hbar}(p \cdot Q - q \cdot P)\right\} \\ &= \exp\left\{\frac{i}{\hbar}\frac{p \cdot q}{2}\right\} \exp\left\{-\frac{i}{\hbar}q \cdot P\right\} \exp\left\{\frac{i}{\hbar}p \cdot Q\right\}, \end{aligned} \quad (1.27)$$

is often referred to as the generalized Pauli-matrix operator. This will be made clear when we discuss discrete phase space on finite fields.

Similarly, we have for the displacement operator of the momentum eigenket $|p\rangle$

$$\begin{aligned} \langle x | p \rangle &= C \exp\left(\frac{ip \cdot x}{\hbar}\right) \\ &= T(p) C \exp\left(\frac{ip' \cdot x}{\hbar}\right)_{p' \Rightarrow 0} \\ &= \exp\{p \cdot \nabla_{p'}\} C \exp\left(\frac{ip' \cdot x}{\hbar}\right)_{p' \Rightarrow 0} \\ &= \exp\{p \cdot \nabla_{p'}\} C \exp\left(\frac{ip' \cdot x}{\hbar}\right)_{p' \Rightarrow 0} \\ &= \exp\left\{\frac{1}{-i\hbar}p \cdot (-i\hbar \nabla_{p'})\right\} C \exp\left(\frac{ip' \cdot x}{\hbar}\right)_{p' \Rightarrow 0}, \\ |p\rangle &= \exp\left\{\frac{i}{\hbar}p \cdot Q\right\} |p' = 0\rangle. \end{aligned} \quad (1.28)$$

⁹Here, the concept of a vacuum state does not have a special meaning since $|0\rangle$ represent arbitrary reference position. It is introduced simply to bring analogy with zero-eigenvalue of non-Hermitian operators in later chapters, there the state $|0\rangle$ has a distinguished position.

We also note from Eq. (1.12) that

$$\begin{aligned} \left[P, \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} \right] &= -i\hbar \frac{\partial}{\partial Q} \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} \\ &= p \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\}. \end{aligned} \quad (1.29)$$

Therefore,

$$P \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} = \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} P + p \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\}, \quad (1.30)$$

$$\exp \left\{ -\frac{i}{\hbar} p \cdot Q \right\} P \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} = P + p. \quad (1.31)$$

Multiplying both sides of Eq. (1.30) by the eigenket with component $|p'\rangle$ and taking the limit $p' \Rightarrow 0$, we have

$$P \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} |p' = 0\rangle = \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} P |p' = 0\rangle + p \exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} |p' = 0\rangle,$$

this reduces to

$$P \left[\exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} |0\rangle \right] = p \left[\exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} |0\rangle \right],$$

which verifies that the state

$$\exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} |0\rangle = |p\rangle \quad (1.32)$$

is an eigenstate of the position operator with displaced eigenvalue by p . However, if the limit $p' \Rightarrow 0$ is not taken then the state

$$\left[\exp \left\{ \frac{i}{\hbar} p \cdot Q \right\} |p'\rangle \right] = |p' + p\rangle$$

is an eigenstate of the position operator with eigenvalue $p' + p$. This translation operator, $\exp \left\{ \frac{i}{\hbar} p \cdot Q \right\}$, on the $|p\rangle$ states is the one we have used in combination with the $|q\rangle$ -state displacement operator to form a symmetric operator.

Note that the displacement of the eigenvalues of the eigenstates are for positive displacements. By virtue of the *covariance* of the coefficient of expansion of arbitrary state in terms of the position and momentum eigenkets, the coefficient of expansion has their eigenfunction parameter shifted by negative values, since Q and P are Hermitian operators. This result is useful, for example, in determining the transformation properties of the Wigner distribution function, which as we shall see later, is the coefficient of expansion of the density operator in terms of the phase-space point projectors.

Table 1.1 Unification of Canonical Variables

| Universal Canonical Operators | Q | P |
|---|-----------|-------------------------|
| Bosons | $\phi(x)$ | $i\hbar\phi^\dagger(x)$ |
| Fermions | $\psi(x)$ | $i\hbar\psi^\dagger(x)$ |
| Mixed Universal q - p Representation: Coherent States | a_l | $i\hbar a_l^\dagger$ |

1.3 Discrete Phase Space on Finite Fields

The formalism discussed above holds *naturally* for discrete ‘lattice’ points of the eigenvalues q and p , i.e., discrete phase space over a finite field. Examples of these are (i) discrete finite number of lattice atomic position, q , in a crystalline solid, obeying the Born-von Karman boundary condition, and (ii) the simplest finite field consisting of only of two points in either q or p eigenvalues, governed by modular arithmetic. This describes the quantum mechanics of two-level systems and spin $\frac{1}{2}$ particles. This will be the topic of later chapters.

1.4 Non-Hermitian Canonical Variables

In our discussions above, we have focused on the canonical variables, p and q , whose corresponding quantum operators are P and Q , respectively. The meaning we have attached to these canonical variables, so far, are that q and p are the position and momentum variables, respectively, of a single particle. Thus the operators, P and Q , are Hermitian operators since these are observables.

However, in dealing with the quantization of classical fields and Schrödinger wave fields (also known as *second quantization*¹⁰), we have to deal with non-Hermitian canonical operators, these are the creation and annihilation field operators for bosons and fermions particles. For classical fields, these canonical variables are basically the *normal* coordinates labeled by index, l . In dealing with non-Hermitian canonical operators, the operators P and Q assume a universal meaning. Table 1.1 shows the translation of the canonical operators P and Q in the language of the boson and fermion field operators, $\{\phi(x), \phi^\dagger(x)\}$ and $\{\psi(x), \psi^\dagger(x)\}$, respectively, and of the mixed q - p representation of the coherent states formulation $\{a_l, a_l^\dagger\}$, where x and l label the field *components*,

Then our previous treatment of the Hermitian operators, P and Q , is universally applicable to these non-Hermitian canonical operators as well, the existence of zero eigenvalue, the respective *right* eigenvectors of Q and P , the respective *left* eigenvectors of Q and P , and the universal generation of eigenvectors through appropriate translation operation from the vacuum (zero eigenvalue) eigenstate. In what follows, we will retain the symbols q and p for the eigenvalues of the non-Hermitian field operators, with the caveat that for *fermions* q and p are elements of the Grassmann algebra¹¹, not ordinary c -numbers.

¹⁰Classical fields, such as vibrational fields and electromagnetic fields, naturally exhibit *discrete* (‘quantum’) oscillating modes, similar to the Schrödinger wave equation. Hence the name second quantization.

¹¹Whereas the commutation relation of boson variables readily follows from that of harmonic

1.4.1 Left and Right Eigenvectors of Non-Hermitian Operators

We have the right eigenvectors for bosons

$$\begin{aligned}\phi(x) |q(x)\rangle &= q(x) |q(x)\rangle, \\ (i\hbar\phi^\dagger(x)) |p(x)\rangle &= p(x) |p(x)\rangle,\end{aligned}$$

and for fermions

$$\begin{aligned}\psi(x) |q(x)\rangle &= q(x) |q(x)\rangle, \\ (i\hbar\psi^\dagger(x)) |p(x)\rangle &= p(x) |p(x)\rangle.\end{aligned}$$

Similarly, we have the left eigenvectors for bosons defined as

$$\begin{aligned}\langle q(x) | \phi(x) &= \langle q(x) | q(x), \\ \langle p(x) | (i\hbar\phi^\dagger(x)) &= \langle p(x) | p(x),\end{aligned}$$

and for fermions

$$\begin{aligned}\langle q(x) | \psi(x) &= \langle q(x) | q(x), \\ \langle p(x) | (i\hbar\psi^\dagger(x)) &= \langle p(x) | p(x).\end{aligned}$$

Then, considering Eq. (1.21) as a universal relation, we have

$$\begin{aligned}|q\rangle &= \exp\left\{-\frac{i}{\hbar}q \cdot P\right\} |0\rangle \\ &= \exp\left\{-\frac{i}{\hbar}q \cdot i\hbar\phi^\dagger\right\} |0\rangle \\ &= \exp\left\{q \cdot \phi^\dagger\right\} |0\rangle\end{aligned}$$

for bosons, and for fermions this becomes

$$\begin{aligned}|q\rangle &= \exp\left\{-\frac{i}{\hbar}q \cdot P\right\} |0\rangle \\ &= \exp\left\{-\frac{i}{\hbar}q \cdot i\hbar\psi^\dagger\right\} |0\rangle \\ &= \exp\left\{q \cdot \psi^\dagger\right\} |0\rangle.\end{aligned}$$

oscillator problem; the anticommutation property of fermion variables is derived from the Slater determinant formalism of many-body quantum theory of half-integral spin particles. However, we shall see that both statistical properties originate from the geometric phase concept by viewing bosons and fermions as quasiparticles in a vacuum condensate structure, similar to the Born-Oppenheimer approximation in condensed matter.

Equation (1.32) becomes for bosons

$$\begin{aligned} |p\rangle &= \exp\left\{\frac{i}{\hbar}p \cdot Q\right\} |p=0\rangle \\ &= \exp\left\{\frac{i}{\hbar}p \cdot \phi\right\} |p=0\rangle, \end{aligned}$$

and for fermions

$$\begin{aligned} |p\rangle &= \exp\left\{\frac{i}{\hbar}p \cdot Q\right\} |p=0\rangle \\ &= \exp\left\{\frac{i}{\hbar}p \cdot \psi\right\} |p=0\rangle. \end{aligned}$$

The dot products is understood to be an Einstein summation convention, e.g., $q \cdot \phi^\dagger = \sum_x q_x \phi^\dagger(x)$. It should be pointed out that for Hermitian operators the zero-eigenvalue state does not have a distinguished position among all the eigenstates, it is just an *arbitrary* position. However, for non-Hermitian operators, the zero-eigenvalue state have a *distinguished* position and is often referred to as the *vacuum* state.

We note that the *left* and *right* eigenvectors form a complete *biorthogonal* basis states,

$$\begin{aligned} Q|q\rangle &= q|q\rangle, \\ \langle q'|Q|q\rangle &= q\langle q'|q\rangle, \\ q'\langle q'|q\rangle &= q\langle q'|q\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} q'\langle q'|q\rangle - q\langle q'|q\rangle &= 0, \\ (q' - q)\langle q'|q\rangle &= 0, \end{aligned}$$

hence, if $q' \neq q$, then we conclude that $\langle q'|q\rangle = 0$. Similarly, $\langle p'|p\rangle = 0$ if $p' \neq p$. These are also complete bases, $\int |q\rangle\langle q|dq = I$ and $\int |p\rangle\langle p|dp = I$. The transformation function, Eq. (1.2), is now translated into the dot product of *complex* eigenvalues q and p , by virtue of the non-Hermitian character of the field operators, so that

$$|p\rangle = C \sum_q \exp\left(-\frac{i}{\hbar}p \cdot q\right) |q\rangle,$$

where C is a normalization constant and the summation sign may stand for the *functional integral* over the space of complex functions, q . We will not discuss here the case for fermions since the Grassmann algebra for the eigenvalues, q and p , complicates the notations¹².

¹²A good discussion of Grassmann algebra for fermions is given by J. Schwinger in his book, "Quantum Kinematics and Dynamics" (Addison-Wesley, 1991). See also the construction of path integrals for fermion many-body problems given by F. A. Buoat, Phys. Rev. A **33**, 2544 (1986).

1.5 Coherent State Formulation as a Mixed q - p Representation

We will focus on bosons, in particular the coherent formulation of harmonic oscillators (to be discussed in more details in later chapters), which as we shall see is really a mixed q - p formulation in the *universal/generalized* sense. The canonical non-Hermitian operators are the creation and annihilation operators, \hat{a} and \hat{a}^\dagger , respectively. Coherent state formulation is based on the *right* eigenstate of \hat{a} , $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ coupled with the *left* eigenvector of \hat{a}^\dagger , $\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^\dagger$. Thus, α and α^\dagger correspond to the eigenvalues $q \Rightarrow \alpha$ and $p \Rightarrow i\hbar\alpha^\dagger$ respectively. Therefore, the transformation function corresponding to Eq. (1.2) becomes

$$\begin{aligned} \langle\alpha|\alpha\rangle &= \exp\left(-\frac{i}{\hbar}p \cdot q\right) \\ &= \exp\left(-\frac{i}{\hbar}i\hbar\alpha^\dagger \cdot \alpha\right) \\ &= \exp|\alpha|^2, \end{aligned} \tag{1.33}$$

which is sometimes written in the formulation as $\langle\alpha|\alpha\rangle = c_o \exp|\alpha|^2$ where c_o is chosen depending on the common phase transformation of the q states relative to the p states, usually $c_o = (2\pi\hbar)^{-d}$, here d is the dimension or number of canonical variables, q or p . The generation of the eigenstates from the vacuum follows from the prescription of the generalized Pauli operator of Eq. (1.27)

$$\begin{aligned} |\alpha\rangle &= \exp\left\{\frac{i}{\hbar}p \cdot q\right\} \exp\left\{-\frac{i}{\hbar}q \cdot P\right\} \exp\left\{\frac{i}{\hbar}p \cdot Q\right\} |0\rangle \\ &= \exp\left\{\frac{i}{\hbar}p \cdot q\right\} \exp\left\{-\frac{i}{\hbar}q \cdot P\right\} |0\rangle \\ &= \exp\left\{\frac{i}{\hbar}i\hbar\alpha^\dagger \cdot \alpha\right\} \exp\left\{-\frac{i}{\hbar}\alpha \cdot i\hbar\hat{a}^\dagger\right\} |0\rangle \\ &= \exp\left\{-\frac{|\alpha|^2}{2}\right\} \exp\{\alpha\hat{a}^\dagger\} |0\rangle. \end{aligned}$$

The completeness relation in the mixed q - p representation, by virtue of the coherent state formulation, is given by

$$\begin{aligned} \int (|q\rangle \langle q| |p\rangle \langle p|) dq \wedge dp &= \int (|q\rangle \langle p| |q\rangle \langle p|) dq \wedge dp \\ &= \frac{1}{2\pi\hbar} \int (|q\rangle \langle p|) e^{\frac{i}{\hbar}qp} dq \wedge dp \\ &\Rightarrow \frac{1}{2\pi\hbar} \int (|\alpha\rangle \langle\alpha|) e^{\frac{i}{\hbar}\alpha i\hbar\alpha^*} d\alpha \wedge d(i\hbar\alpha^*) \\ &= \frac{1}{2\pi} \int (|\alpha\rangle \langle\alpha|) e^{-|\alpha|^2} id(\alpha_r + i\alpha_i) \wedge d(\alpha_r - i\alpha_i) \\ &= \frac{1}{\pi} \int (|\alpha\rangle \langle\alpha|) e^{-|\alpha|^2} d\alpha_r d\alpha_i \\ &= 1, \end{aligned} \tag{1.34}$$

where the operation denoted by ‘ \wedge ’ is the *outer product* (cross product) of the complex quantities, dq and dp so as to yield an *incremental area* in a complex plane¹³. The last line of Eq. (1.34) is often written as

$$\frac{1}{\pi} \int (|\alpha\rangle \langle\alpha|) d^2\alpha = 1,$$

which follows by normalizing the coherent states, $|\alpha\rangle \Rightarrow e^{-\frac{|\alpha|^2}{2}} |\alpha\rangle$ [refer to Eq. (1.33)]. Based on this coherent-state normalization, the transformation function, $\langle\beta|\alpha\rangle$, becomes a non-orthogonality relation for coherent states,

$$\begin{aligned} \langle\beta|\alpha\rangle &= e^{-\frac{|\beta|^2}{2}} e^{-\frac{|\alpha|^2}{2}} \exp\left(-\frac{i}{\hbar} i\hbar\beta^\dagger \cdot \alpha\right) \\ &= e^{-|\alpha-\beta|^2}, \end{aligned} \quad (1.35)$$

which becomes approximately orthogonal as $|\alpha-\beta|^2$ increases.

It should be emphasized that through the use of *universal viewpoint* for non-Hermitian canonical variables, we have obtained above the important relations for, and *true nature* of, coherent states without the aid of occupation-number states of a harmonic oscillator, $|n\rangle$, as is usually done in almost all discussions giving the introduction of coherent states. In fact, as we shall see in this book, the *universal nature* of the canonical operators, Q and P , and their respective eigenvalues, q and p , permeates all formal aspects of quantum dynamics, either in the continuum phase space or discrete phase space on finite fields. Examples of the later are (a) finite discrete lattice points in crystalline solid obeying the Born-von Karman boundary condition, and (b) two discrete points obeying modular arithmetic for two-level quantum systems, such as a two-level atom or spin $\frac{1}{2}$ particle. These will be further discussed in later chapters.

¹³The ‘volume’ (area for simple harmonic oscillator) element in phase space between the canonically transformed phase spaces are related by

$$J (dq \times dp) = dQ \times dP$$

Therefore for canonical transformation, $J = 1$, and the so-called ‘*wedge form*’ (differential geometry terminology)

$$dq \times dp = dQ \times dP$$

is also invariant. In two variables this means the elementary ‘area’ in phase space is invariant (this also reflects the invariant way of counting of states).