CHAPTER 4

New Assumptions

Certain old assumptions die hard, particularly those having to do with circularity of motion. The circle is a sacrosanct figure for many cultures and, though it may be refuted as the actual description of a natural phenomenon, it is never completely abandoned as the ideal for which nature somehow strives. Bohr's circular orbits would give way to other models but the role of the circle itself continued to enlarge. Here we are going to explore, not quantum theory itself but the assumptions on which it is based. These include the circle, or more accurately the sphere, in a prominent role.

The prospect of an electron disappearing from one orbit in order to reappear in another was sufficiently disturbing to prompt a review of what an electron might actually be. In fact, the question of what it *is* rapidly gave way to the question of what properties it might *have* that one could observe. Presumably it had location, and some locations were more likely for it to be in than others. Anything that had position could have velocity and acceleration too. And it had mass, which meant it also would have momentum.

Also it possessed energy, and in discrete frequencies, as observed in the Balmer experiment. As we saw in an earlier chapter, this would mean that in some respects it behaved more like a wave than a particle. Specifically, a "particle", imagined perhaps as a billiard ball, can have any speed, any momentum, any acceleration. A continuum of possibilities exists for the billiard ball, or the particle. An old-fashioned wave, although it can have any amplitude, has to decide what its frequency will be. Of course waves can be superimposed to make new waves, but one can describe what frequencies will be present in the sum. And hydrogen, in its emission of beautiful red light during the Balmer experiment, always chooses the exact same frequency to emit. In this choice, hydrogen behaves more like a wave than like a particle.

Now, as soon as you have an object that resists classification as matter or energy or wave or whatever, it is arguably natural to think in terms of observable properties the object might have rather than actual attributes of itself. You might think these are the same, but they most certainly are not. In the case of hydrogen, it is very profitable to think of the electron as having, not a particular position and momentum, but rather a set of probabilities that describe the likelihood of finding it in a particular region of space at any given time.

A *probability distribution* is nothing more than a function, as shown in Fig. 4.1, that describes the likelihood of the electron being in a certain region. This probability is expressed as the area under the graph of the function and above the region in question. By fiat, the total area is one. These distributions take the place of what was formerly envisioned as the *location* of the electron. Such a distribution could equally well describe either the average time spent by the objectified electron in a particular region, or some other function related to the energy of the electron (according to the laws of physics) which could play the mathematical role of position without all the annoying ideological baggage that goes along with the concept.

It is already hard to imagine an electron as something which simply *might turn up* here or there, and whose average behavior is all one can determine and describe as a probability distribution. This bit of imagination is necessary for our story. But one can even go further, although not out of strict necessity. One could suspend disbelief and attempt to think of an electron, formerly an actual object with a home of its own in one's psyche, not just as an object with characteristic probabilities for being found in various proximities to the nucleus, but as being the actual *probability distribution* itself. If you can do this, you might actually be one step ahead of the average physicist. Or one step behind, depending on how you look at it. It is an ancient doctrine that, at the most basic level of existence, the world is formed out of *number*. This is not science, this is mysticism at its finest. Nonetheless, the world is getting to look more and more that way. If you can imagine all of the properties of the electron, and then mentally *throw away* the actual electron as unnecessary to the many calculations that follow, then you can justifiably say that science is no longer merely science. You would have become a Pythagorean.

Figure 4.1 is a picture of a probability distribution, which you might recognize as the usual bell curve describing grades in a class. If an object moved around on the real axis according to this distribution, it would spend most of its time between $X = -1$ and $X = 1$ and the remainder outside. If you were to reach between two lines drawn at these points and grab it, you would have a chance of catching it there exactly equaling the area under the curve and between the two lines. The total area under the curve is one, which makes the chance of it being somewhere at all also one. This is a very fancy way of asserting the existence of an electron. In Chapter 10, we'll apply this same formalism to the hydrogen atom.

These are the new objects of physics, replacing the objectified electron in the mind of the quantum mechanist. Furthermore, all the laws will be expressed as if the wave-like properties of the electron were all that mattered. Much has been lost contextually in the passing to such a system, especially certain comforting analogies to the macroscopic world. Much, however, is gained in syntax. To have an object that can be considered equally conveniently as wave or particle or probability distribution, depending on the attributes it presents in a given situation, represents an enormous gain in syntax.

To be specific, treating the electron in this manner places it in a class of mathematical objects which can be acted upon by the tools of partial differential equations, functional analysis, abstract algebra and representation theory, all of which topics we will explore later. Mathematically, the electron behaves not at all like a planet but much more like a weight on a spring. The linguistic capabilities of the mathematics become much stronger in this new situation. Generally speaking, this gain in syntax is the very reason abstraction is valued in mathematics and physics. Often

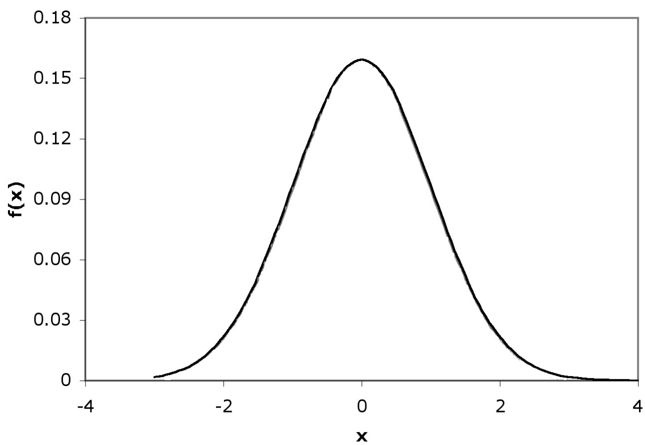

Figure 4.1. The Gaussian distribution function.

abstraction is the first step in the change of viewpoint that increases one's mathematical options in a certain situation. In our situation, the change of viewpoint as to what constitutes an electron resulted in an entire field of study, namely quantum mechanics.

This is half of the story. If the old assumption about the electron made it an object that *moves in a circle*, quantum mechanics replaces the "object" with its observable properties, described in the form of mathematical functions. There is no longer an "object", except as lodged in the conservative mind of the physicist. Whatever we are looking at here doesn't "move", either, in any detectable sense. It remains to ask, what concept arises to take the place of the "circle", and how must it relate to the new notions of what an electron ought to be?

Let us return to the experiment of Balmer that gave us hydrogen's line spectrum. The radiation of light from, say, a light bulb, is such a commonplace occurrence now that it is difficult to find novelty in the observation that the photon emitted by the electron was the same frequency when viewed from all directions. This may seem most unsurprising, as how should the electron be able to know which direction to send its energy? Yet, this simple observation provides the atom with the desired spherical symmetry. That is, whatever laws describe the atom's energy spectrum must also respect the spherical symmetry of the spatial variables. Rotate the three-dimensional space variables through any angle, along any axis, and the laws of physics must remain unchanged. The same color light must be emitted. In fact, this observation yields a whole set of symmetries for the atom, as big as the set of isometries of the sphere. An *isometry* is a rigid motion of an object that sets it down back upon itself. In this case, the isometries of the sphere consist of rotations about any diameter of it.

Here we have circles, and in abundance. The collection of isometries of the sphere is an object with rich mathematical structure. It has a beautiful interaction with the laws of physics as described by Schrödinger's equation. The mathematics brought to bear on the problem is of a depth and flavor attained only in this century. It represents the combined efforts of both mathematicians and physicists, spawning much of the current activity in both fields. The story of the hydrogen atom in this century exemplifies also what has happened to much of the physical sciences, as mathematical "laws", which are theorems, and mathematical "objects", which are sometimes complete abstractions, grew into the conceptual roles formerly filled by familiar physical objects, such as planets, billiard balls, and the like.