Chapter 1

Symmetrization

1.1 The Decreasing Rearrangement

Schwarz symmetrization is a particular kind of rearrangement of functions defined on a domain $\Omega \subset \mathbb{R}^N$. Given a real valued function on such a domain, we construct an associated function, on the ball centered at the origin and of the same measure as Ω, assuming the same range of values and having special properties. In particular, we wish that this new function be radial and radially decreasing. In order to define this, we first construct the unidimensional decreasing rearrangement of the given function, which we now proceed to do.

First of all, we need some notation. Given a (Lebesgue) measurable subset $E \subset \mathbb{R}^N$, we denote its N - dimensional (Lebesgue) measure by $|E|$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded measurable set. Let $u : \Omega \to \mathbb{R}$ be a measurable function. For $t \in \mathbb{R}$, the level set $\{u > t\}$ is defined as

$$\{u > t\} = \{x \in \Omega \mid u(x) > t\}.$$

The sets $\{u < t\}, \{u \geq t\}, \{u = t\}$ and so on are defined by analogy. Then the **distribution function** of u is given by

$$\mu_u(t) = |\{u > t\}|.$$

This function is a monotonically decreasing function of t and for $t \geq$ ess. $\sup(u)$, we have $\mu_u(t) = 0$, while for $t \leq$ ess. $\inf(u)$, we have $\mu_u(t) = |\Omega|$. Thus the range of μ_u is the interval $[0, |\Omega|]$.

Definition 1.1.1 Let $\Omega \subset \mathbb{R}^N$ be bounded and let $u : \Omega \to \mathbb{R}$ be a measurable function. Then the **(unidimensional) decreasing rearrange-**

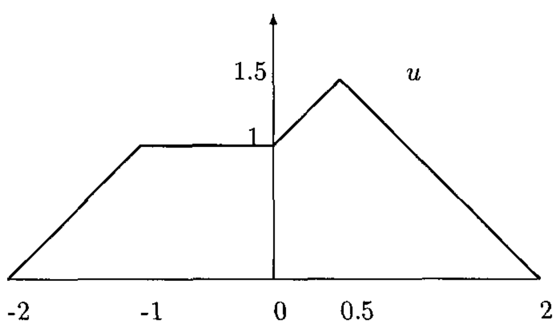
ment of u, denoted $u^{\#}$, is defined on $[0, |\Omega|]$ by

$$\left.\begin{array}{l} u^{\#}(0) = \text{ess. sup}(u) \\ u^{\#}(s) = \inf\{t \mid \mu_u(t) < s\}, \ s > 0. \end{array}\right\} \tag{1.1.1}$$

Remark 1.1.1 Essentially, $u^{\#}$ is just the inverse function of the distribution function μ_u of u. However, since $\mu_u(t)$ is just monotonically decreasing, it can have jump discontinuities. If t is a point of discontinuity, then the above definition fixes the value of $u^{\#}$ in the interval $[\mu_u(t+), \mu_u(t-)]$ as t. ■

Remark 1.1.2 In classical texts (cf. [Hardy, Littlewood, and Pólya (1952)] or [Pólya and Szegö (1951)]), the distribution function is usually defined as $\mu_u(t) = |\{|u| > t\}|$. Consequently, the definition of the rearrangement would correspond to, in our notation, that of $|u|^{\#}$. Throughout this text, we will, however, deal with the 'rearrangement with sign' *i.e.* our definition of the rearrangement will be based on the distribution function which takes the sign of the function into account. ■

Example 1.1.1 [Kawohl (1986)]



Let $\Omega = (-2, 2) \subset \mathbb{R}$. Define $u : \Omega \to \mathbb{R}$ as follows.

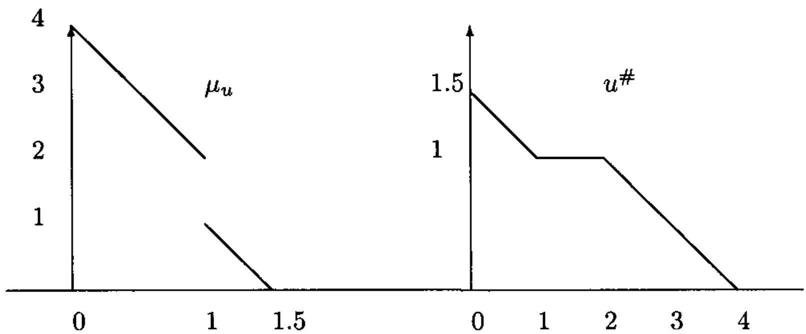
$$u(y) = \begin{cases} 2 + y, & -2 \le y \le -1 \\ 1, & -1 \le y \le 0 \\ 1 + y, & 0 \le y \le 0.5 \\ 2 - y, & 0.5 \le y \le 2. \end{cases}$$

Then, it is easy to check that

$$\mu_u(t) = \begin{cases} 4-2t, 0 \leq t < 1 \\ 3-2t, 1 \leq t \leq 1.5 \\ 0, t \geq 1.5 \end{cases}$$

and that

$$u^{\#}(s) = \begin{cases} (3-s)/2, 0 \leq s \leq 1 \\ 1, 1 \leq s \leq 2 \\ (4-s)/2, 2 \leq s \leq 4. \end{cases}$$



The following properties of the decreasing rearrangement are immediate from its definition.

Proposition 1.1.1 *Let $u : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^N$ is bounded. Then $u^{\#}$ is a non-increasing and left-continuous function.*

Proof: (i) Let $s_1 < s_2$. Then $|\{u > t\}| < s_1$ implies that $|\{u > t\}| < s_2$. Thus,

$$\{t \mid \mu_u(t) < s_1\} \subset \{t \mid \mu_u(t) < s_2\}.$$

Thus, by definition, it follows that $u^{\#}(s_1) \geq u^{\#}(s_2)$.
(ii) Let $s \in (0, |\Omega|)$. By definition of $u^{\#}$, given $\varepsilon > 0$, there exists a t such that $u^{\#}(s) \leq t \leq u^{\#}(s) + \varepsilon$ and $\mu_u(t) < s$. Choose $h > 0$ such that $\mu_u(t) < s - h < s$. Then, for all $0 < h' \leq h$, we have $\mu_u(t) < s - h' < s$ and so $u^{\#}(s) \leq u^{\#}(s - h') \leq t < u^{\#}(s) + \varepsilon$. This proves that $u^{\#}$ is left-continuous. ■

Proposition 1.1.2 *The mapping $u \mapsto u^{\#}$ is non-decreasing, i.e. if $u \leq v$, where u and v are real valued functions on Ω, then $u^{\#} \leq v^{\#}$.*

Proof: Since $\{u > t\} \subset \{v > t\}$, we have that

$$\{t \mid |\{v > t\}| < s\} \ \subset \{t \mid |\{u > t\}| < s\}$$

and the result follows from the definition. ∎

Definition 1.1.2 Two real valued functions (with possibly different domains of definition) are said to be **equimeasurable** if they have the same distribution function. Equimeasurable functions are said to be **rearrangements** of each other.

We now show that $u^\#$ is indeed a rearrangement of u in this sense.

Proposition 1.1.3 *The functions $u : \Omega \to \mathbb{R}$ and $u^\# : [0, |\Omega|] \to \mathbb{R}$ are equimeasurable, i.e., for all t,*

$$|\{u > t\}| \ = \ |\{u^\# > t\}|. \tag{1.1.2}$$

Proof: If $u^\#(s) > t$, then, by definition, it follows that $|\{u > t\}| \geq s$. Thus,

$$\{s \mid u^\#(s) > t\} \ \subset \ \{s \mid |\{u > t\}| \geq s\}.$$

Since $u^\#$ is non-increasing, we have

$$|\{u^\# > t\}| \ = \ \sup\{s \mid u^\#(s) > t\} \ \leq \ |\{u > t\}|. \tag{1.1.3}$$

On the other hand, let $|\{u^\# \geq t\}| = s$. By the left-continuity and the non-increasing nature of $u^\#$, it follows that $u^\#(s) = t$. Then, by definition, $|\{u > t\}| \leq s$. Thus,

$$|\{u > t\}| \ \leq \ |\{u^\# \geq t\}|. \tag{1.1.4}$$

Applying (1.1.3) and (1.1.4) for $t + h$ instead of t, we get

$$|\{u^\# > t + h\}| \ \leq \ |\{u > t + h\}| \ \leq \ |\{u^\# \geq t + h\}|.$$

Passing to the limit as $h \downarrow 0$, we get

$$|\{u^\# > t\}| \ \leq \ |\{u > t\}| \ \leq \ |\{u^\# > t\}|$$

which proves (1.1.2). ∎

Corollary 1.1.1 *With the preceding notations, we have*

$$\left.\begin{array}{l} |\{u > t\}| = |\{u^\# > t\}|. \\ |\{u \geq t\}| = |\{u^\# \geq t\}|. \\ |\{u < t\}| = |\{u^\# < t\}|. \\ |\{u \leq t\}| = |\{u^\# \leq t\}|. \end{array}\right\} \tag{1.1.5}$$

Proof: The first relation of (1.1.5) has already been proved. The rest follow easily by complementation and suitable limiting arguments. ■

Remark 1.1.3 The above proposition, in the light of Definition 1.1.2, explains why $u^\#$ is called a rearrangement of u. The decreasing rearrangement is just one example of a wide variey of such constructions. The reader is referred to the book of [Kawohl (1985)] for examples of different kinds of rearrangements. ■

Corollary 1.1.2 *If* $u \geq 0$, *and if* $u \in L^p(\Omega)$ *for* $1 \leq p \leq \infty$, *then* $u^\# \in L^p((0, |\Omega|))$ *and*

$$\|u\|_{p,\Omega} \ = \ \|u^\#\|_{p,(0,|\Omega|)}.$$

(Here the norms are the corresponding L^p *- norms.)*

Proof: If $p = \infty$, then the result is contained in the definition of the rearrangement. Let $1 \leq p < \infty$. By equimeasurability, both u and $u^\#$ have the same distribution function μ and as both functions are non-negative, it follows (by a simple application of Fubini's theorem) that

$$\|u\|_{p,\Omega}^p \ = \ p \int_0^\infty t^{p-1}\mu(t)dt \ = \ \|u^\#\|_{p,(0,|\Omega|)}.$$

■

This result is also true without the non-negativity condition. In fact, as a consequence of the equimeasurability, we have the following general and powerful result.

Theorem 1.1.1 *Let* $u : \Omega \to \mathbb{R}$ *be measurable. Let* $F : \mathbb{R} \to \mathbb{R}$ *be a non-negative Borel measurable function. Then*

$$\int_\Omega F(u(x))dx \ = \ \int_0^{|\Omega|} F(u^\#(s))ds. \tag{1.1.6}$$

Proof: Let $E = [t, \infty]$ and set $F(\xi) = \chi_E(\xi)$, where χ_E is the indicator (or characteristic) function of E. Then

$$\int_\Omega F(u(x))dx \;=\; |\{u \geq t\}| \;=\; |\{u^\# \geq t\}| \;=\; \int_0^{|\Omega|} F(u^\#(s))ds.$$

Similarly, the result holds for $F = \chi_E$ where E is any interval and hence if E is any open set and, again, if E is any Borel set, by standard arguments. Hence the result is true for any non-negative simple function F. If F is any non-negative Borel function, it can be expressed as the limit of an increasing sequence $\{F_n\}$ of non-negative simple functions. Thus, for each n we have

$$\int_\Omega F_n(u(x))dx \;=\; \int_0^{|\Omega|} F_n(u^\#(s))ds. \tag{1.1.7}$$

We can pass to the limit as $n \to \infty$ to get (1.1.6) using the monotone convergence theorem. ■

Corollary 1.1.3 *Let $F : \mathbb{R} \to \mathbb{R}$ be a Borel function and let $u : \Omega \to \mathbb{R}$ be such that $F(u) \in L^1(\Omega)$. Then $F(u^\#) \in L^1((0, |\Omega|))$ and (1.1.6) is still valid.*

Proof: We write $F = F^+ - F^-$ and both F^+ and F^- are non-negative Borel functions and so (1.1.6) holds for each of them in place of F. If $F(u) \in L^1(\Omega)$, then both $\int_\Omega F^+(u(x))dx$ and $\int_\Omega F^-(u(x))dx$ are finite and we can subtract the relation for F^- from that of F^+ to get (1.1.6). ■

Corollary 1.1.4 *Let $u \in L^p(\Omega)$ for $1 \leq p < \infty$. Then $u^\# \in L^p((0, |\Omega|))$ and the corresponding L^p norms are equal.*

Proof: Take $F(t) = |t|^p$ in the preceding theorem. ■

Remark 1.1.4 The result remains true for $p = \infty$ since $u^\#(0) =$ ess. sup(u) and $u^\#(|\Omega|) =$ ess. inf(u). ■

Remark 1.1.5 Since the proofs of Theorem 1.1.1. and its consequences depended only on the equimeasurability, these results also hold for other types of rearrangements. ■

We now prove another important property of the (decreasing) rearrangement which is a consequence of Theorem 1.1.1.

Lemma 1.1.1 *Let $u : [0, l] \to \mathbb{R}$ be non-increasing. Then $u = u^\#$* a.e.

Proof: If $t < u(s)$, then, since u is non-increasing, $|\{u > t\}| \geq s$. Hence, $u^\#(s) \geq t$, by definition. This implies that

$$u^\#(s) \;\geq\; u(s) \tag{1.1.8}$$

for all $s \in [0, l]$. Now, let s be a point of continuity for u. Since u is non-increasing,

$$|\{u > u(s-h)\}| \;\leq\; s - h \;<\; s$$

for $h > 0$. Thus, by definition, $u^\#(s) \leq u(s-h)$. Hence, as $h \downarrow 0$, we get, using the continuity of u at s, that

$$u^\#(s) \;\leq\; u(s). \tag{1.1.9}$$

Thus, by (1.1.8) and (1.1.9), we get that $u^\#(s) = u(s)$ at all points of continuity of u. The result stands proved since u, being monotonic, admits at most only a countable number of discontinuities. ■

Proposition 1.1.4 *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Let $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$ is bounded. Then*

$$\psi(u^\#) \;=\; (\psi(u))^\#, \text{ a.e.} \tag{1.1.10}$$

Proof: Step 1. If $v, w : [0, l] \to \mathbb{R}$ are equimeasurable and non-increasing, then $v = w$ a.e. For, as they are non-increasing, by the preceding lemma, $v = v^\#$ and $w = w^\#$ a.e. As they are equimeasurable, by definition, $v^\# = w^\#$.

Step 2. The result will be proved if we can show that $\psi(u^\#)$ and $(\psi(u))^\#$ are both equimeasurable and non-increasing on $[0, |\Omega|]$. That they are non-increasing follows from the definition of the rearrangement and the fact that ψ is non-decreasing. Now,

$$\begin{aligned} |\{\psi(u^\#) > t\}| = \int_0^{|\Omega|} \chi_{\{\psi(u^\#)>t\}}(s)ds \;&=\; \int_\Omega \chi_{\{\psi(u)>t\}}(x)dx \\ = |\{\psi(u) > t\}| \;&=\; |\{(\psi(u))^\# > t\}|. \end{aligned}$$

The equality of the two integrals above is a consequence of Theorem 1.1.1 since the function $s \mapsto \chi_{\{\psi>t\}}(s)$ is a non-negative Borel function. The last equality is a consequence of the equimeasurability of a function and its rearrangement. ■

Corollary 1.1.5 *For $u : \Omega \to \mathbb{R}$, we have $(u^+)^\# = (u^\#)^+$.*

Exercise 1.1.1 Show by means of an example that, in general, given two functions $v, w : \Omega \to \mathbb{R}$, $(v+w)^\#$ and $v^\# + w^\#$ are not equal. However, if $c \in \mathbb{R}$, show that

$$(v+c)^\# \;=\; v^\# + c.$$

Exercise 1.1.2 Let Ω be the annulus

$$\{x \in \mathbb{R}^2 \mid 0 < R_1 < |x| < R_0\}.$$

Let $u(x) = (R_0^2 - |x|^2)/2$. Show that

$$u^\#(s) \;=\; \frac{1}{2}\left(R_0^2 - R_1^2 - \frac{s}{\pi}\right).$$

1.2 Some Rearrangement Inequalities

We begin by proving the L^p continuity of the rearrangement map.

Proposition 1.2.1 *Let $p = 1$ or ∞. Then for $f, g \in L^p(\Omega)$,*

$$\|f^\# - g^\#\|_{p,(0,|\Omega|)} \;\leq\; \|f - g\|_{p,\Omega}. \tag{1.2.1}$$

Proof: Let $p = \infty$. Then for almost all $x \in \Omega$, we have

$$|f(x) - g(x)| \;\leq\; \|f - g\|_{\infty,\Omega}.$$

Thus,

$$f(x) - \|f - g\|_{\infty,\Omega} \;\leq\; g(x) \;\leq\; f(x) + \|f - g\|_{\infty,\Omega}.$$

By the monotonicity of the rearrangement map (cf. Proposition 1.1.2) and by Exercise 1.1.1, we deduce that

$$f^\#(s) - \|f - g\|_{\infty,\Omega} \;\leq\; g^\#(s) \;\leq\; f^\#(s) + \|f - g\|_{\infty,\Omega}$$

which immediately gives (1.2.1) for $p = \infty$.

Let $p = 1$. Set $h = \max\{f, g\}$. Then as $f \leq h$ and $g \leq h$, we have that $f^\# \leq h^\#$ and $g^\# \leq h^\#$. Now,

$$|f^\# - g^\#| \;\leq\; |f^\# - h^\#| + |h^\# - g^\#| \;=\; 2h^\# - f^\# - g^\#.$$

Thus,

$$\int_0^{|\Omega|} |f^\#(s) - g^\#(s)|ds \le \int_0^{|\Omega|}(2h^\#(s) - f^\#(s) - g^\#(s))ds$$
$$= \int_\Omega (2h(x) - f(x) - g(x))dx$$
$$= \int_\Omega |f(x) - g(x)|dx$$

which proves the result for $p = 1$. ■

Theorem 1.2.1 *Let $1 \le p \le \infty$. The mapping $u \mapsto u^\#$ is continuous from $L^p(\Omega)$ into $L^p((0, |\Omega|))$.*

Proof: If $p = 1$ or $p = \infty$, the result follows from the preceding proposition. Let $1 < p < \infty$. Let $u_n \to u$ in $L^p(\Omega)$. Since Ω is bounded, it follows that $u_n \to u$ in $L^1(\Omega)$ as well and so $u_n^\# \to u^\#$ in $L^1((0, |\Omega|))$. Hence, for a subsequence, $u_{n_k}^\# \to u^\#$ a.e. Further,

$$\|u_{n_k}^\#\|_{p,(0,|\Omega|)} \;=\; \|u_{n_k}\|_{p,\Omega} \;\to \|u\|_{p,\Omega} \;=\; \|u^\#\|_{p,(0,|\Omega|)}.$$

Hence it follows that $u_{n_k}^\# \to u^\#$ in $L^p((0, |\Omega|))$ as well. In fact, as the limit is independent of the subsequence, it follows that the entire sequence $\{u_n^\#\}$ converges to $u^\#$ in $L^p((0, |\Omega|))$ and this completes the proof. ■

We saw in the previous section that the rearrangement preserves the integral of a function over the domain (take $F(t) = t$ in Theorem 1.1.1). We now consider integrals over proper subsets.

Proposition 1.2.2 *Let $\Omega \subset \mathbb{R}^N$ be bounded and let $u : \Omega \to \mathbb{R}$ be an integrable function. Let $E \subset \Omega$ be a measurable subset. Then*

$$\int_E u(x)dx \;\le\; \int_0^{|E|} u^\#(s)ds. \tag{1.2.2}$$

Equality holds in (1.2.2) if, and only if,

$$(u|_E)^\# \;=\; u^\#|_{[0,|E|]}, \text{ a.e.}$$

Proof: Let $v = u|_E$. If $s \in [0, |E|]$, and if $|\{u > t\}| < s$, then

$$|\{v > t\}| \;=\; |\{u > t\} \cap E| \;<\; s.$$

Thus

$$\{t \mid |\{u > t\}| < s\} \;\subset\; \{t \mid, |\{v > t\}| < s\}$$

and so $v^\#(s) \leq u^\#(s)$. Thus

$$\int_E u(x)dx \;=\; \int_0^{|E|} v^\#(s)ds \;\leq\; \int_0^{|E|} u^\#(s)ds, \qquad (1.2.3)$$

which proves (1.2.2). If equality holds in (1.2.2), then we have equality throughout in (1.2.3) and this is possible if, and only if, $v^\# = u^\#$ a.e. in E and the result is proved. ■

Lemma 1.2.1 *Let $u : \Omega \to \mathbb{R}$ and let $t \in \mathbb{R}$. Define*

$$E_t = \{x \in \Omega \mid u(x) > t\}$$
$$F_t = \{x \in \Omega \mid u(x) \leq t\} \;= \Omega \backslash E_t.$$

Define $b : \mathbb{R} \times \Omega \to \mathbb{R}$ by

$$b(t,x) = \begin{cases} \chi_{E_t}(x) & \text{if } t \geq 0 \\ -\chi_{F_t}(x) & \text{if } t < 0. \end{cases}$$

Then

$$u(x) \;=\; \int_{-\infty}^{+\infty} b(t,x)dt. \qquad (1.2.4)$$

Proof: If $u(x) \geq 0$,

$$\int_{-\infty}^{+\infty} b(t,x)dt \;=\; \int_0^{u(x)} dt \;=\; u(x).$$

If $u(x) < 0$, then

$$\int_{-\infty}^{+\infty} b(t,x)dt \;=\; -\int_{u(x)}^{0} dt \;=\; u(x).$$

■

Lemma 1.2.2 *Let $f, g : \Omega \to \mathbb{R}$ with g integrable over Ω. Let $a \leq f \leq b \leq +\infty$ with $a \in \mathbb{R}$. Then*

$$\int_\Omega f(x)g(x)dx \;=\; a\int_\Omega g(x)dx + \int_a^b \left(\int_{\{f>t\}} g(x)dx\right) dt. \qquad (1.2.5)$$

Proof: Assume that $a \geq 0$ (the other case can be similarly treated). Setting $E_t = \{f > t\}$, we have, by the preceding lemma,

$$f(x) \;=\; \int_0^b \chi_{E_t}(x)dt.$$

Thus, by Fubini's theorem,

$$\int_\Omega f(x)g(x)dx = \int_\Omega g(x)\int_0^b \chi_{E_t}(x)dtdx = \int_0^b\int_\Omega g(x)\chi_{E_t}(x)dxdt$$

which gives

$$\int_\Omega f(x)g(x)dx = \int_0^a\int_\Omega g(x)dxdt + \int_a^b\int_{E_t} g(x)dxdt$$

from which (1.2.5) follows immediately. ∎

Exercise 1.2.1 In the above proposition, if $b \in \mathbb{R}$ and if $-\infty \le a \le f \le b$, show that

$$\int_\Omega f(x)g(x)dx = b\int_\Omega g(x)dx - \int_a^b\left(\int_{\{f\le t\}} g(x)dx\right)dt.$$

We are now in a position to prove an important rearrangement inequality.

Theorem 1.2.2 *(Hardy - Littlewood) Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where $(1/p)+(1/q)=1,\ 1\le p,q\le\infty$. Then*

$$\int_\Omega f(x)g(x)dx \le \int_0^{|\Omega|} f^\#(s)g^\#(s)ds. \tag{1.2.6}$$

Proof: Assume first that $f \in L^\infty(\Omega)\cap L^p(\Omega)$. Let a and b be real numbers such that $a \le f \le b$. Then, by the preceding lemma,

$$\begin{aligned}\textstyle\int_\Omega f(x)g(x)dx &= a\textstyle\int_\Omega g(x)dx + \int_a^b\int_{\{f>t\}} g(x)dxdt\\ &= a\textstyle\int_0^{|\Omega|} g^\#(s)ds + \int_a^b\int_{\{f>t\}} g(x)dxdt\\ &\le a\textstyle\int_0^{|\Omega|} g^\#(s)ds + \int_a^b\int_0^{|\{f>t\}|} g^\#(s)dsdt\\ &= a\textstyle\int_0^{|\Omega|} g^\#(s)ds + \int_a^b\int_0^{|\{f^\#>t\}|} g^\#(s)dsdt\\ &= \textstyle\int_0^{|\Omega|} f^\#(s)g^\#(s)ds\end{aligned}$$

where the last equality comes from applying (1.2.5) to $f^\#$ and $g^\#$.

If $1 \le p < \infty$, the general case can be completed by a density argument since we know that the mapping $u \mapsto u^\#$ is continuous from $L^p(\Omega)$ into

$L^p((0, |\Omega|))$. ■

Exercise 1.2.2 Let $f, g \in L^2(\Omega)$. Using (1.2.6), show that

$$\|f^\# - g^\#\|_{2,(0,|\Omega|)} \leq \|f - g\|_{2,\Omega}.$$

Exercise 1.2.3 Let $\psi : \mathbb{R} \to \mathbb{R}$ be non-decreasing. Let $f \in L^p(\Omega)$ and $\psi(g) \in L^q(\Omega)$, where $(1/p) + (1/q) = 1,\ 1 \leq p, q \leq \infty$. Show that

$$\int_\Omega f(x)\psi(g(x))dx \leq \int_0^{|\Omega|} f^\#(s)\psi(g^\#(s))ds.$$

Exercise 1.2.4 (a) Let $u : \Omega \to \mathbb{R}$. Show that $(\chi_{\{u>t\}})^\# = \chi_{\{u^\#>t\}}$.
(b) Show that, if $u, v : \Omega \to \mathbb{R}$, then

$$\int_\Omega u(x)\chi_{\{v\leq t\}}(x)dx \geq \int_0^{|\Omega|} u^\#(s)\chi_{\{v^\#\leq t\}}(s)ds.$$

We saw that the rearrangement mapping is non-expansive between the relevant L^p spaces when $p = 1, 2$ or ∞. In fact, it is true for all $1 \leq p \leq \infty$ as the following theorem shows.

Theorem 1.2.3 *Let $f, g \in L^p(\Omega)$, where $1 < p < \infty$. Then*

$$\|f^\# - g^\#\|_{p,(0,|\Omega|)} \leq \|f - g\|_{p,\Omega}. \tag{1.2.7}$$

Proof: Set $J(t) = |t|^p$. Define

$$J_+(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ |t|^p & \text{if } t > 0 \end{cases}$$

and

$$J_-(t) = \begin{cases} |t|^p & \text{if } t \leq 0 \\ 0 & \text{if } t > 0 \end{cases}$$

so that $J = J_+ + J_-$. Both J_+ and J_- are convex and differentiable functions. Thus,

$$J_+(f(x)-g(x)) = \int_{g(x)}^{f(x)} J_+'(f(x)-t)dt = \int_{-\infty}^{+\infty} J_+'(f(x)-t)\chi_{\{g\leq t\}}(x)dt.$$

Hence, after an application of Fubini's theorem, we get

$$\int_\Omega J_+(f(x) - g(x))dx = \int_{-\infty}^{+\infty}\int_\Omega J_+'(f(x) - t)\chi_{\{g\leq t\}}(x)dxdt \tag{1.2.8}$$

and, similarly,

$$\int_0^{|\Omega|} J_+(f^\#(s) - g^\#(s))ds = \int_{-\infty}^{+\infty}\int_0^{|\Omega|} J'_+(f^\#(s) - t)\chi_{\{g^\# \le t\}}(s)dsdt. \tag{1.2.9}$$

Now, since J_+ is convex, we have that J'_+ is non-decreasing. Thus, by Proposition 1.1.4 and Exercise 1.1.1, we get

$$(J'_+(f(x) - t))^\#(s) = J'_+(f^\#(s) - t).$$

Hence, by virtue of Exercise 1.2.4, we deduce that

$$\int_\Omega J'_+(f(x) - t)\chi_{\{g \le t\}}(x)dx \ge \int_0^{|\Omega|} J'_+(f^\#(s) - t)\chi_{\{g^\# \le t\}}(s)ds.$$

Thus, using (1.2.8) and (1.2.9), we deduce that

$$\int_\Omega J_+(f(x) - g(x))dx \ge \int_0^{|\Omega|} J_+(f^\#(s) - g^\#(s))ds.$$

We can prove, in the same way, the same inequality with J_- in place of J_+. Adding, we get (1.2.7). ■.

1.3 Schwarz Symmetrization

Henceforth, given a measurable subset $E \subset \mathbb{R}^N$ of finite measure, we will denote by E^*, the open ball centered at the origin and having the same measure as E, *i.e.* $|E^*| = |E|$. Given a vector $x \in \mathbb{R}^N$, we denote its Euclidean norm by $|x|$. Finally, we will denote by ω_N, the volume of the unit ball in $\mathbb{R}^N$. Notice that

$$\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2} + 1)},$$

where $\Gamma(s)$ is the usual gamma function.

Definition 1.3.1 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $u : \Omega \to \mathbb{R}$ be a measurable function. Then, its **Schwarz symmetrization** or, the **spherically symmetric and decreasing rearrangement** is the function $u^* : \Omega^* \to \mathbb{R}$ defined by

$$u^*(x) = u^\#(\omega_N|x|^N),\ x \in \Omega^*.$$

Observe that, if R is the radius of Ω^*, then

$$\begin{aligned}\int_{\Omega^*} u^*(x)dx = \int_{\Omega^*} u^\#(\omega_N|x|^N)dx &= \int_0^R u^\#(\omega_N r^N) N\omega_N r^{N-1}dr \\ = \quad \int_0^{|\Omega^*|} u^\#(s)ds \quad &= \int_0^{|\Omega|} u^\#(s)ds.\end{aligned}$$

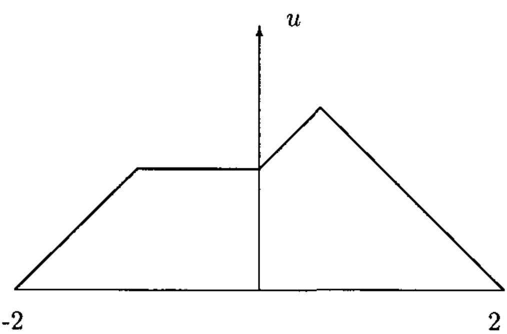
We can translate all the results obtained in the preceding sections for the decreasing unidimensional rearrangement to get the corresponding results for the Schwarz symmetrization. In particular, we easily deduce the following:

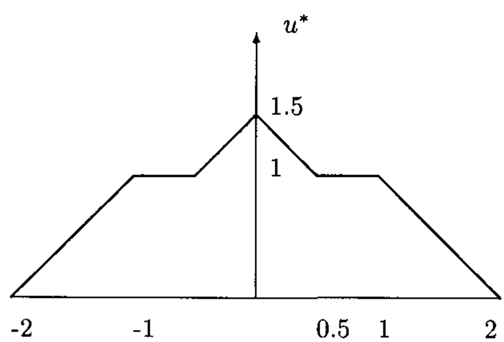
- u^* is radially symmetric and decreasing.
- $u, u^\#$ and u^* are all equimeasurable.
- If $F : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that either $F \geq 0$ or $F(u) \in L^1(\Omega)$, then
$$\int_{\Omega^*} F(u^*(x))dx \;=\; \int_\Omega F(u(x))dx. \tag{1.3.1}$$
In particular, u and u^* have the same L^p norms and $\int_\Omega u(x)dx = \int_{\Omega^*} u^*(x)dx$ when u is integrable over Ω.
- If $\psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, then
$$(\psi(u))^* \;=\; \psi(u^*).$$
- The mapping $u \mapsto u^*$ is a non-expansive mapping from $L^p(\Omega)$ into $L^p(\Omega^*)$ for $1 \leq p \leq \infty$.
- If $E \subset \Omega$ is a measurable subset, then,
$$\int_E u(x)dx \;\leq\; \int_0^{|E|} u^\#(s)ds \;=\; \int_{E^*} u^*(x)dx. \tag{1.3.2}$$
Equality occurs if, and only if, $(u|_E)^* = u^*|_{E^*}$.
- (Hardy - Littlewood) If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where $(1/p) + (1/q) = 1$, then,
$$\int_\Omega f(x)g(x)dx \;\leq\; \int_0^{|\Omega|} f^\#(s)g^\#(s)ds \;=\; \int_{\Omega^*} f^*(x)g^*(x)dx. \tag{1.3.3}$$

We conclude this section by computing the Schwarz symmetrization of the function in Example 1.1.1.

Example 1.3.1 Let $\Omega = (-2,2) \subset \mathbb{R}$ and let $u : \Omega \to \mathbb{R}$ be as in Example 1.1.1. Then $\Omega^* = (-2,2)$ as well and u^* is given by

$$u^*(-x) = u^*(x) = \begin{cases} \frac{3}{2} - x, \; 0 \le x \le 0.5 \\ 1, \; 0.5 \le x \le 1 \\ 2 - x, \; 1 \le x \le 2. \end{cases}$$





1.4 Variations on the Theme

In this section, we briefly describe some variants of the notions discussed previously. Just as we defined the unidimensional decreasing rearrangement, we can define the increasing rearrangement as well.

Definition 1.4.1 Let $\Omega \subset \mathbb{R}^N$ be bounded and let $u : \Omega \to \mathbb{R}$ be a measurable function. The (unidimensional) **increasing rearrangement** of u, denoted $u_{\#}$, is defined on $[0, |\Omega|]$ by

$$\begin{aligned} u_{\#}(|\Omega|) &= \text{ess. sup}(u) \\ u_{\#}(s) &= \inf\{t \mid |\{u < t\}| > s\}, \ s \in [0, |\Omega|). \end{aligned} \tag{1.4.1}$$

As before, $u_{\#}$ is essentially the inverse of the function $m(t) = |\{u < t\}|$. It is non-decreasing, and equimeasurable with u. If $u \leq v$, where u and v are measurable functions defined on Ω, then $u_{\#} \leq v_{\#}$.

Exercise 1.4.1 Let $u : \Omega \to \mathbb{R}$ be measurable. Show that

$$u^{\#}(s) = u_{\#}(|\Omega| - s).$$

Proposition 1.4.1 *Let $u : \Omega \to \mathbb{R}$ be a measurable function. Then,*

$$u^{\#} = -(-u)_{\#} \quad a.e.. \tag{1.4.2}$$

Proof: Notice that

$$\begin{aligned} |\{u^{\#} > t\}| = \quad |\{u > t\}| \quad &= |\{-u < -t\}| \\ = |\{(-u)_{\#} < -t\}| &= |\{-(-u)_{\#} > t\}|. \end{aligned}$$

Since $u^{\#}$ and $-(-u)_{\#}$ are both non-increasing and equimeasurable, they are equal a.e. ■

Corollary 1.4.1 *(Hardy - Littlewood) Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where $(1/p) + (1/q) = 1$. Then*

$$\int_{\Omega} f(x)g(x)dx \geq \int_0^{|\Omega|} f_{\#}(s)g^{\#}(s)ds. \tag{1.4.3}$$

Proof: We have

$$-\int_{\Omega} f(x)g(x)dx \leq \int_{\Omega} (-f)^{\#}(s)g^{\#}(s)ds = -\int_{\Omega} f_{\#}(s)g^{\#}(s)ds$$

and the result follows immediately. ■

Definition 1.4.2 Let $u : \Omega \to \mathbb{R}$ be a measurable function where $\Omega \subset \mathbb{R}^N$ is bounded. Then the **spherically symmetric and increasing rearrangement** of u is defined on Ω^* by

$$u_*(x) = u_{\#}(\omega_n |x|^N). \tag{1.4.4}$$

Exercise 1.4.2 Compute $u_\#$ and u_* when u is as in the Example 1.1.1.

Exercise 1.4.3 Let $f : \Omega \to \mathbb{R}$ be a measurable function where $\Omega \subset \mathbb{R}^N$ is bounded. Show that

$$(f^+)^\#(s)(f^-)_\#(s) = 0 \text{ for all } 0 \le s \le |\Omega|.$$

The next variant we would like to discuss is the definition of the radially symmetric and decreasing rearrangement of a function defined on all of $\mathbb{R}^N$.

Definition 1.4.3 Let $f : \mathbb{R}^N \to \mathbb{R}$ be a Borel measurable function. It is said to **vanish at infinity** if, for every $t > 0$, the sets $\{|f| > t\}$ are of finite measure.

Definition 1.4.4 Let $f : \mathbb{R}^N \to \mathbb{R}$ be a Borel measurable function vanishing at infinity. Then its **spherically symmetric and decreasing rearrangement** , f^*, is defined on $\mathbb{R}^N$ by

$$f^*(x) = \int_0^\infty \chi_{\{|f|>t\}^*}(x)dt. \tag{1.4.5}$$

Since the integrand in (1.4.5) is in terms of characteristic functions of balls, clearly, f^* is radially symmetric. Also it is obvious that it is radially decreasing, *i.e.* if $x, y \in \mathbb{R}^N$ are such that $|x| \le |y|$, then $f^*(x) \ge f^*(y)$.

Assume that $f^*(x) > t$. Let, if possible, $x \notin \{|f| > t\}^*$. Then, for all $t' > t$, $x \notin \{|f| > t'\}^*$ since these are smaller concentric balls. Thus, it follows from (1.4.5) that $f^*(x) \le t$, which is a contradiction. Thus, $x \in \{|f| > t\}^*$.

Conversely, if $x \in \{|f| > t\}^*$, then, clearly, t is not the supremum of $|f|$ and so we can find a $t' > t$ such that $x \in \{|f| > t'\}^*$ which is a smaller concentric ball. Now, for all $t'' < t'$, it is evident that $x \in \{|f| > t''\}^*$. Thus, it follows from (1.4.5) that $f^*(x) \ge t' > t$. Thus, we have shown that

$$\{f^* > t\} = \{|f| > t\}^*.$$

Since the sets $\{f^* > t\}$ are all open, it follows that f^* is measurable and in fact, lower-semicontinuous. It also follows that f^* and $|f|$ are equimeasurable and the results of the preceding sections follow as a consequence. In particular, the L^p norms of f and f^* are the same. Notice that we are essentially symmetrizing $|f|$ rather than f (cf. Remark 1.1.2).

It can be shown that the map $f \mapsto f^*$ is non-expansive from $L^p(\mathbb{R}^N)$ into itself and that the Hardy - Littlewood inequality holds. Finally, we state an important property of this rearrangement in $\mathbb{R}^N$. We refer the reader to the book of [Lieb and Loss (1997)] for the proof and for generalizations.

Theorem 1.4.1 *(Riesz' inequality) Let f, g and h be non-negative Borel measurable functions on $\mathbb{R}^N$ vanishing at infinity. Define*

$$I(f,g,h) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)g(x-y)h(y)dxdy.$$

Then

$$I(f,g,h) \leq I(f^*,g^*,h^*). \tag{1.4.6}$$