Preface

The study of isoperimetric inequalities goes back to antiquity. The solution of *Dido's problem* states that, of all plane domains of given perimeter, the circular disc, and the disc alone, maximizes the enclosed area. Equivalently, of all domains of given area, the disc minimizes the perimeter. This is expressed via the *classical isoperimetric inequality*

$$L^2 \geq 4\pi A,$$

where L stands for the perimeter and A for the enclosed area of a domain in the plane. Equality is attained only for the disc. In three dimensional space, if S is the surface area of a body and V its volume, then

$$S^3 \geq 36\pi V^2$$

and equality is attained only for the sphere. Thus, of all bodies of given volume, the sphere, and the sphere alone, has least surface area. (Why are soap bubbles spherical in shape? The soap bubble will attain a position of equilibrium when the potential energy due to surface tension is minimal. This energy is proportional to the surface area of the bubble. Hence, for a given volume of air blown to form a bubble, it will take the spherical shape which has the least surface area.) These two inequalities can be generalized to all space dimensions.

In a broader sense, an isoperimetric problem tries to optimize a given domain dependent functional keeping some geometric parameter of the domain (like its volume) fixed. The study of such problems started with the conjecture of Saint Venant in 1856 regarding the optimal shape of the cross section of a prism in order to maximize its torsional rigidity. This was finally settled by Pólya in 1948. In 1894, Lord Rayleigh, in his treatise on the theory of sound, made conjectures regarding the vibrations of certain elastic bodies. He conjectured that, of all fixed membranes of given area, the circular membrane has the lowest fundamental frequency of vibration. This was proved by Faber and Krahn towards the end of the first quarter of the twentieth century. He also conjectured that of all vibrating thin clamped plates of given area, the circular plate has the lowest fundamental frequency. This conjecture resisted solution for nearly a century and was finally settled by Nadirashvili in 1992. While the Faber - Krahn result can be extended to all dimensions, the conjecture regarding plates remains unsolved in dimensions greater than three even today. Poincaré conjectured that of all bodies with given volume, the ball alone has the least electrostatic capacity and this was proved by Szegö in 1930.

The study of isoperimetric inequalities involves a fascinating interplay of analysis, geometry and the theory of partial differential equations. Several conjectures have been made and while many have been resolved, a large number still remain open.

One of the first comprehensive treatises on this subject is the book by Pólya and Szegö (1951). It has since been complemented by several review articles and also by books such as those by Bandle (1980) and Mossino (1984). Since then, several new results have been proved, and a few conjectures have been resolved, especially those concerning eigenvalues of elliptic partial differential operators.

As seen from the examples cited above, Nature often seems to choose the perfect symmetric form, *viz.* spherical symmetry, when optimizing various characteristics of bodies. One of the principal tools in the study of isoperimetric problems, especially when spherical symmetry is involved, is *Schwarz symmetrization*, which is also known as the *spherically symmetric and decreasing rearrangement* of functions. The aim of this book is to give an introduction to the theory of Schwarz symmetrization and study some of its applications. Other equally important types of symmetrization (for example, *Steiner symmetrization*) are not treated here.

The first chapter introduces the notion of Schwarz symmetrization and proves several of its properties. The principal result of the second chapter is the famous inequality of Pólya and Szegö regarding Dirichlet integrals. Since this proof depends on the classical isoperimetric inequality and the co-area formula, these are also studied. The relationship between the classical isoperimetric inequality and Sobolev's inequality is also discussed.

Usually, in their full generality, the proofs of the isoperimetric inequality, the co-area formula and the Polya - Szegö theorem involve heavy doses of geometric measure theory. Here, a simple and elementary proof of the isoperimetric inequality (that I learnt from a lecture by X. Cabre) has been presented in the case of smooth domains. Simple versions of the co-area formula that are enough to prove the Pólya - Szegö theorem are proved using techniques mainly drawn from the study of partial differential equations.

The third chapter centers around Talenti's theorem which compares the solution of a second order elliptic boundary value problem with that of a 'symmetrized problem'. Several applications of this result are studied.

The fourth chapter looks at various isoperimetric inequalities involving eigenvalues of elliptic operators. The fifth (and last) chapter deals with isoperimetric inequalities involving positive solutions of some non-linear problems and uses them to obtain the radial symmetry of such solutions when the domain is a ball.

Effort has been made to keep the exposition as simple and as self-contained as possible. Occasionally, in order that we do not get mired in technical details and thus lose the main thread of the argument, some proofs, which are long and which involve completely different techniques, have been omitted and the 'interested reader' is given appropriate references. But such instances are few and far between. A knowledge of the existence theory of weak solutions of elliptic partial differential equations in Sobolev spaces is, however, assumed. Apart from this and a general mathematical maturity at the graduate level, there are no other prerequisites. The text is peppered with several exercises.

While the bibliography is fairly extensive, it is, as is to be expected, far from exhaustive. Further references to a vast and rich literature can be found in the works cited here. At the end of each of the last three chapters, bibliographic comments indicate results and directions not treated in the text.

I have taught most of the material covered in this book, as a short introductory course, to graduate students at the Università degli Studi di Roma, La Sapienza, Rome, Italy. I wish to take this opportunity to thank the university for its hospitality. Portions of this book were also taught at instructional conferences in India organized by the Aligarh Muslim University, Aligarh, and the Tata Institute of Fundamental Research (TIFR) at its Bangalore Centre (Indian Institute of Science Campus). I thank the organizers of these conferences for giving me the opportunity to deliver those lectures.

I would like to thank the Institute of Mathematical Sciences, Chennai for the excellent facilities it provided which greatly eased the task of bringing out this volume. In particular, I wish to record my appreciation

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I learnt much about symmetrization from the works of, and through discussions with, several experts. It is a pleasure to express my thanks to Professor G. Talenti (especially for inviting me to the conference at Cortona, which turned out to be a valuable experience, and for all that he has done for me), to Professors G. Trombetti and V. Ferone (and all their colleagues) for very enjoyable and useful visits to Naples, to Professor Jacqueline Mossino for her warm hospitality at Orsay and to Professor Filomena Pacella for giving me the opportunity to teach this course in Rome and for fruitful collaboration (some of our joint work appears in this volume).

I am grateful to my family for the constant support given to me. I am also very grateful to Professor P. G. Ciarlet who has been a constant source of inspiration and encouragement. I also wish to thank Professor R. Wong, the editor of the 'Series in Analysis', and the staff of World Scientific for their cooperation in bringing out this volume.

Finally, I would like to dedicate this book to the memory of Professor Jacques-Louis Lions, who was one of my teachers and to whom I owe more than I can possibly express.

S. Kesavan