1. Rutherford Scattering

Problem 1.1 *Using Eq.* (1.38) *calculate the approximate total cross sections for Rutherford scattering of a* 10 MeV *α-particle from a lead nucleus for impact parameters b less than* 10^{-12}, 10^{-10} *and* 10^{-8} cm. *How well do these agree with the values of πb^2?*

There are various ways of doing this problem. We will list below two very simple methods.

Method I. In general, the total cross section for Rutherford scattering is given by (see Eq. (1.38) in the text)

$$\sigma_{\mathrm{TOT}} = 8\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \int_0^1 \frac{\mathrm{d}\left(\sin\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}\right)^3}. \tag{1.1}$$

However, if the impact parameter is restricted to a finite range, say $b \leq b_0$, then we can write the total cross section as

$$\sigma_{\mathrm{TOT}}(b_0) = 8\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \int_{\theta_{b_0}}^1 \frac{\mathrm{d}\left(\sin\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}\right)^3}, \tag{1.2}$$

where θ_{b_0} is the scattering angle corresponding to the impact parameter b_0 and is given by (see Eq. (1.32) of the text)

$$b_0 = \frac{ZZ'e^2}{2E} \cot\frac{\theta_{b_0}}{2}. \tag{1.3}$$

Carrying out the integration in (1.2), we obtain

$$\begin{aligned}\sigma_{\rm TOT}(b_0) &= 8\pi\left(\frac{ZZ'e^2}{4E}\right)^2\left(-\frac{1}{2}\right)\left(1-\text{cosec}^2\frac{\theta_{b_0}}{2}\right)\\ &= 4\pi\left(\frac{ZZ'e^2}{4E}\right)^2\cot^2\frac{\theta_{b_0}}{2}\\ &= \pi\left(\frac{ZZ'e^2}{2E}\cot\frac{\theta_{b_0}}{2}\right)^2 = \pi b_0^2, \end{aligned}\tag{1.4}$$

where we have used the identification in (1.3). It follows, therefore, that

b_0 (cm)	$\sigma_{\rm TOT}(b_0) = \pi b_0^2$ (cm^2)
10^{-12}	3.2×10^{-24}
10^{-10}	3.2×10^{-20}
10^{-8}	3.2×10^{-16}

Method II. An alternative method to obtain the same result is to note that the total cross section for Rutherford scattering can be written as

$$\begin{aligned}\sigma_{\rm TOT} &= 8\pi\left(\frac{ZZ'e^2}{4E}\right)^2\int_0^1\frac{\mathrm{d}\left(\sin\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}\right)^3}\\ &= 4\pi\left(\frac{ZZ'e^2}{4E}\right)^2\int_0^1 \mathrm{d}\theta\,\cot\frac{\theta}{2}\text{cosec}^2\frac{\theta}{2}.\end{aligned}\tag{1.5}$$

This can be converted into an integral over the impact parameters using the defining relationship (see Eqs. (1.32) and (1.36))

$$b = \frac{ZZ'e^2}{2E}\cot\frac{\theta}{2},\quad \frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{ZZ'e^2}{4E}\text{cosec}^2\frac{\theta}{2},\tag{1.6}$$

so that we can write

$$\begin{aligned}\sigma_{\rm TOT} &= 4\pi\left(\frac{ZZ'e^2}{4E}\right)^2\int_0^\infty \mathrm{d}b\left(\frac{ZZ'e^2}{4E}\right)^{-1} b\left(\frac{ZZ'e^2}{2E}\right)^{-1}\\ &= 2\pi\int_0^\infty \mathrm{d}b\; b.\end{aligned}\tag{1.7}$$

This is true in general and can also be deduced from the definition of the cross section in Eq. (1.33) or (1.34) of the text. If impact parameters are smaller than some fixed value, say b_0, then the total cross section takes the form

$$\sigma_{\mathrm{TOT}}(b_0) = 2\pi \int_0^{b_0} \mathrm{d}b\, b = \pi b_0^2, \tag{1.8}$$

which is the same result as derived earlier.

Problem 1.2 *Prove that Eq.* (1.55) *follows from the relations in Eqs.* (1.53) *and* (1.54).

This problem can be solved directly from the relationship between the scattering angles in the laboratory and the CM frames. From Eq. (1.53) of the text we have

$$\cos\theta_{\mathrm{Lab}} = \frac{\cos\theta_{\mathrm{CM}} + \zeta}{(1 + 2\zeta\cos\theta_{\mathrm{CM}} + \zeta^2)^{1/2}}. \tag{1.9}$$

Through direct differentiation, it follows that

$$\begin{aligned}\frac{\mathrm{d}\cos\theta_{\mathrm{Lab}}}{\mathrm{d}\cos\theta_{\mathrm{CM}}} &= \frac{1}{(1 + 2\zeta\cos\theta_{\mathrm{CM}} + \zeta^2)^{1/2}} - \frac{\zeta(\cos\theta_{\mathrm{CM}} + \zeta)}{(1 + 2\zeta\cos\theta_{\mathrm{CM}} + \zeta^2)^{3/2}}\\ &= \frac{1 + \zeta\cos\theta_{\mathrm{CM}}}{(1 + 2\zeta\cos\theta_{\mathrm{CM}} + \zeta^2)^{3/2}},\end{aligned} \tag{1.10}$$

which leads to

$$\frac{\mathrm{d}\cos\theta_{\mathrm{CM}}}{\mathrm{d}\cos\theta_{\mathrm{Lab}}} = \frac{(1 + 2\zeta\cos\theta_{\mathrm{CM}} + \zeta^2)^{3/2}}{1 + \zeta\cos\theta_{\mathrm{CM}}}. \tag{1.11}$$

Let us note that

$$(1 + 2\zeta\cos\theta_{\mathrm{CM}} + \zeta^2) = (1 + \zeta\cos\theta_{\mathrm{CM}})^2 + \zeta^2\sin^2\theta_{\mathrm{CM}} > 0. \tag{1.12}$$

Since the differential cross sections in the two frames must be positive, the Jacobian connecting the two must also be positive. From

Eq. (1.54) of the text, this leads to the relationship (using the absolute value of the Jacobian)

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{Lab}}}(\theta_{\mathrm{Lab}}) &= \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{CM}}}(\theta_{\mathrm{CM}})\left|\frac{\mathrm{d}\cos\theta_{\mathrm{CM}}}{\mathrm{d}\cos\theta_{\mathrm{Lab}}}\right| \\ &= \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{CM}}}(\theta_{\mathrm{CM}})\,\frac{(1+2\zeta\cos\theta_{\mathrm{CM}}+\zeta^2)^{3/2}}{|1+\zeta\cos\theta_{\mathrm{CM}}|}, \qquad (1.13)\end{aligned}$$

which is the desired result (see Eq. (1.55) of the text).

Problem 1.3 *Sketch* $\cos\theta_{\mathrm{Lab}}$ *as a function of* $\cos\theta_{\mathrm{CM}}$ *for the nonrelativistic elastic scattering of particles of unequal mass, for the cases when* $\zeta = 0.05$ *and* $\zeta = 20$ *in Eqs.* (1.52) *and* (1.53).

For nonrelativistic scattering, we know from Eq. (1.53) of the text that

$$\cos\theta_{\mathrm{Lab}} = \frac{\cos\theta_{\mathrm{CM}}+\zeta}{(1+2\zeta\cos\theta_{\mathrm{CM}}+\zeta^2)^{1/2}}, \qquad \zeta = \frac{m_1}{m_2}, \qquad (1.14)$$

where m_1, m_2 represent respectively the masses of the projectile and the target in the laboratory frame. The first case that we want to consider, namely,

$$\zeta = \frac{m_1}{m_2} = 0.05 \quad \text{or} \quad m_2 = 20m_1, \qquad (1.15)$$

corresponds to the scattering of a light projectile from a heavy target, while the second case

$$\zeta = \frac{m_1}{m_2} = 20 \quad \text{or} \quad m_1 = 20m_2, \qquad (1.16)$$

describes the opposite, where a heavy projectile is scattered from a light target. The two cases, therefore, represent inverse scenarios. We will treat them separately. Furthermore, the solutions for the two cases can be worked out in two simple but equivalent ways as follows.

$\zeta = 0.05$

Method I. Here we simply evaluate the angles numerically from the formula

$$\cos\theta_{\rm Lab} = \frac{\cos\theta_{\rm CM} + 0.05}{(1 + 0.1\cos\theta_{\rm CM} + 0.0025)^{1/2}}. \tag{1.17}$$

Keeping terms up to three-digit accuracy, we have

$\cos\theta_{\rm CM}$	$\cos\theta_{\rm Lab}$
−1.00	−1.000
−0.70	−0.673
−0.50	−0.461
−0.05	0.000
0.00	0.050
0.50	0.536
0.70	0.724
1.00	1.000

This is plotted in Fig. 1.1, and it is clear that for this case of a light projectile scattering from a heavy target, the scattering angles in the laboratory and in the center-of-mass frames are approximately the same.

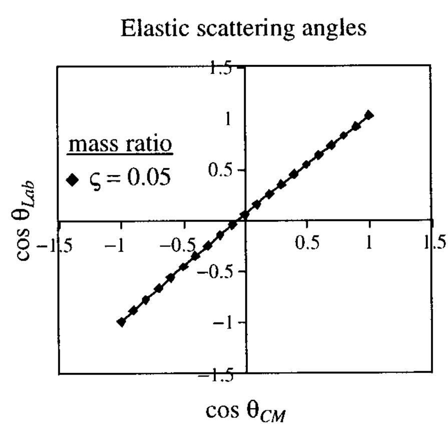

Fig. 1.1. Center-of-mass vs. laboratory scattering angles for a mass ratio $\zeta = 0.05$.

Method II. When $\zeta \ll 1$, as is true in the present case, we can write

$$\begin{aligned}
\cos\theta_{\text{Lab}} &= \frac{\cos\theta_{\text{CM}} + \zeta}{(1 + 2\zeta\cos\theta_{\text{CM}} + \zeta^2)^{1/2}} \\
&\approx \frac{\cos\theta_{\text{CM}} + \zeta}{(1 + 2\zeta\cos\theta_{\text{CM}})^{1/2}} = (\cos\theta_{\text{CM}} + \zeta)(1 + 2\zeta\cos\theta_{\text{CM}})^{-1/2} \\
&\approx (\cos\theta_{\text{CM}} + \zeta)(1 - \zeta\cos\theta_{\text{CM}}) \\
&\approx \cos\theta_{\text{CM}} + \zeta(1 - \cos^2\theta_{\text{CM}}) \\
&= \cos\theta_{\text{CM}} + \zeta\sin^2\theta_{\text{CM}} = \cos\theta_{\text{CM}} + 0.05\sin^2\theta_{\text{CM}}.
\end{aligned} \tag{1.18}$$

Here we have neglected terms of order ζ^2 and higher, which would lead to small corrections in the result. We note that since $0 \leq \sin^2\theta_{\text{CM}} \leq 1$, it follows that

$$\cos\theta_{\text{Lab}} \approx \cos\theta_{\text{CM}}, \tag{1.19}$$

as we saw from the explicit calculation.

$\underline{\zeta = 20}$

Method I. In this case, a heavy projectile scatters from a light target and we have

$$\cos\theta_{\text{Lab}} = \frac{\cos\theta_{\text{CM}} + 20}{(1 + 40\cos\theta_{\text{CM}} + 400)^{1/2}}. \tag{1.20}$$

Explicit numerical evaluation, keeping terms up to four digit accuracy, leads to:

$\cos\theta_{\text{CM}}$	$\cos\theta_{\text{Lab}}$
−1.0	1.0000
−0.7	0.9993
−0.5	0.9990
0.0	0.9987
0.5	0.9991
0.7	0.9994
1.0	1.0000

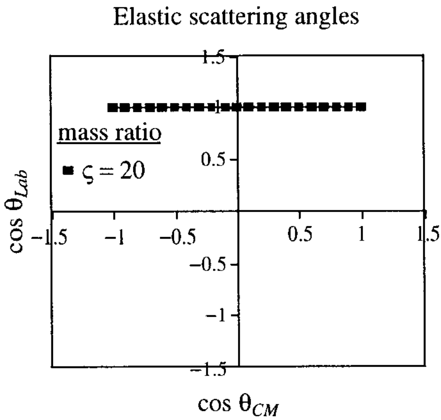

Fig. 1.2. Center-of-mass vs. laboratory scattering angles for a mass ratio $\zeta = 20$.

This is plotted in Fig. 1.2, and shows that the scattering in the laboratory frame is almost entirely in the forward direction — like a truck hitting a ping-pong ball.

Method II. When $\zeta \gg 1$, as is true in the present case, we can write

$$\begin{aligned}\cos\theta_{\text{Lab}} &= \frac{\cos\theta_{\text{CM}} + \zeta}{(1 + 2\zeta\cos\theta_{\text{CM}} + \zeta^2)^{1/2}} \\ &= \frac{1}{\zeta}(\cos\theta_{\text{CM}} + \zeta)\left(1 + \frac{2}{\zeta}\cos\theta_{\text{CM}} + \frac{1}{\zeta^2}\right)^{-1/2} \\ &\approx \frac{1}{\zeta}(\cos\theta_{\text{CM}} + \zeta)\left(1 - \frac{1}{\zeta}\cos\theta_{\text{CM}} - \frac{1}{2\zeta^2} + \frac{3}{2\zeta^2}\cos^2\theta_{\text{CM}}\right) \\ &\approx \frac{1}{\zeta}\left(\zeta - \cos\theta_{\text{CM}} + \cos\theta_{\text{CM}} - \frac{1}{\zeta}\cos^2\theta_{\text{CM}} - \frac{1}{2\zeta} + \frac{3}{2\zeta}\cos^2\theta_{\text{CM}}\right) \\ &= 1 - \frac{1}{2\zeta^2}\sin^2\theta_{\text{CM}} = 1 - \frac{1}{800}\sin^2\theta_{\text{CM}}. \qquad (1.21)\end{aligned}$$

Here we have neglected higher-order terms in $\frac{1}{\zeta}$, which are negligible. From the fact that $0 \le \sin^2\theta_{\text{CM}} \le 1$, we conclude that in this case

$$\cos\theta_{\text{Lab}} \approx 1, \qquad (1.22)$$

which is consistent with the numerical calculation. In either case (whether $\zeta = 0.05$ or $\zeta = 20$), we see directly from the tables that $\theta_{\text{Lab}} \le \theta_{\text{CM}}$.

Problem 1.4 *What would be the approximate counting rate observed in the Rutherford scattering of* 10 MeV *α-particles off lead foil at an angle of $\theta = \frac{\pi}{2}$ in the laboratory? Assume an incident flux of 10^6 α-particles per second on the foil, a foil* 0.1 cm *thick, and a detector of transverse area* 1 cm × 1 cm *placed* 100 cm *from the interaction point, and density of lead of* $11.3\,\mathrm{g/cm^3}$*. What would be the counting rate at $\theta = 5°$? By about how much would your answers change if the above angles were specified for the center-of-mass — be quantitative, but use approximations where necessary. (Why don't you have to know the area of the foil?)*

From Eq. (1.40) of the text, the counting rate is given by

$$\mathrm{d}n(\theta) = N_0 \frac{\rho t A_0}{A} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta)\mathrm{d}\Omega. \tag{1.23}$$

In the present problem of the scattering of α particles from a foil of lead ($^{208}\mathrm{Pb}^{82}$), we are given

$$\begin{aligned} N_0 &= \text{incident flux/foil area} = 10^6\,\mathrm{sec}^{-1}/\text{foil area}, \\ A_0 &= \text{Avogadro's number} = 6 \times 10^{23}/\text{mole}, \\ \rho &= \text{density of the foil} = 11.3\,\mathrm{g/cm^3}, \\ t &= \text{thickness of the foil} = 0.1\,\mathrm{cm}, \\ A &= \text{Atomic weight of lead} = 208, \\ E &= \text{energy of the incident } \alpha \text{ particle} = 10\,\mathrm{MeV}. \end{aligned} \tag{1.24}$$

Furthermore, we are also given that the detector has an area

$$\mathrm{d}s = 1\,\mathrm{cm} \times 1\,\mathrm{cm} = 1\,\mathrm{cm}^2 \tag{1.25}$$

and is located at a distance

$$R = 100\,\mathrm{cm}, \tag{1.26}$$

from the target (foil). Therefore, the solid angle subtended by the detector at the scattering center is given by

$$\mathrm{d}\Omega = \frac{\mathrm{d}s}{R^2} = 10^{-4}\,\mathrm{sr}. \tag{1.27}$$

Finally, we note that for the scattering of α particles from lead ($^{208}\mathrm{Pb}^{82}$), we have

$$Z = 2, \quad Z' = 82, \tag{1.28}$$

so that the Rutherford scattering cross section takes the form

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta) &= \left(\frac{ZZ'e^2}{4E}\right)^2 \left(\frac{1}{\sin\frac{\theta}{2}}\right)^4 (\mathrm{sr})^{-1} \\ &= \left(2 \times 82 \times \frac{\hbar c}{4 \times 10\,\mathrm{MeV}} \times \frac{e^2}{\hbar c}\right)^2 \operatorname{cosec}^4 \frac{\theta}{2} (\mathrm{sr})^{-1}, \end{aligned} \tag{1.29}$$

where we note that

$$\hbar c \approx 197\ \mathrm{MeV-F}, \quad \frac{e^2}{\hbar c} = \text{fine structure constant} = \frac{1}{137}, \tag{1.30}$$

and 1 F = 1 fm or 1 Fermi = 10^{-13} cm. Using these values, the differential cross section at any angle takes the form

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta) &= \left(164 \times \frac{197\ \mathrm{MeV-F}}{40\ \mathrm{MeV}} \times \frac{1}{137}\right)^2 \operatorname{cosec}^4 \frac{\theta}{2} (\mathrm{sr})^{-1} \\ &\approx 0.4 \times 10^{-24} \operatorname{cosec}^4 \frac{\theta}{2}\ \mathrm{cm}^2/\mathrm{sr}. \end{aligned} \tag{1.31}$$

Using all of these results, we can calculate the counting rate at any angle from (1.23) as:

$$\begin{aligned}\mathrm{d}n(\theta) &= 10^6/\mathrm{sec} \times \frac{6 \times 10^{23} \times 11.3 \times 0.1/\mathrm{cm}^2}{208} \\ &\quad \times 0.4 \times 10^{-24} \operatorname{cosec}^4 \frac{\theta}{2}\ \mathrm{cm}^2/\mathrm{sr} \times 10^{-4}\ \mathrm{sr} \\ &\approx 0.13 \operatorname{cosec}^4 \frac{\theta}{2}\ \mathrm{counts/sec}. \end{aligned} \tag{1.32}$$

It follows now that

$$\boxed{\mathrm{d}n(\theta = \tfrac{\pi}{2}) \approx 0.13 \times 4\,\mathrm{counts/sec} \approx 0.5\,\mathrm{counts/sec},}$$

$$\boxed{\mathrm{d}n(\theta = 5^\circ = \tfrac{\pi}{36}) \approx 0.13 \times 28 \times 10^4\ \mathrm{counts/sec} \approx 3.6 \times 10^4\ \mathrm{counts/sec}.}$$

For α particles scattering from lead ($^{208}\text{Pb}^{82}$), we have $\zeta = \frac{m_1}{m_2} = \frac{4}{208} \approx 0.02$. As a result, using (1.18) we obtain

$$\cos\theta_L = \cos\theta_{\text{CM}} + \zeta \sin^2\theta_{\text{CM}} = \cos\theta_{\text{CM}} + 0.02\sin^2\theta_{\text{CM}}, \qquad (1.33)$$

so that for $\theta_{\text{CM}} = \frac{\pi}{2}$, we obtain

$$\cos\theta_{\text{Lab}} = 0.02$$
$$\text{or} \quad \theta_{\text{Lab}} \approx \frac{\pi}{2} - 0.02,$$

which leads to

$$\Delta\theta|_{\theta_{\text{CM}}=\frac{\pi}{2}} = \theta_{\text{CM}} - \theta_{\text{Lab}} \approx 0.02. \qquad (1.34)$$

On the other hand, for $\theta_{\text{CM}} = 5° = \frac{\pi}{36}$, we have

$$\cos\theta_{\text{Lab}} = \cos\theta_{\text{CM}} + 0.02\sin^2\theta_{\text{CM}} \approx 1 - \frac{1}{2}\left(\frac{\pi}{36}\right)^2 + 0.02\left(\frac{\pi}{36}\right)^2$$
$$= 1 - \frac{1}{2} \times 0.96\left(\frac{\pi}{36}\right)^2 \approx 1 - \frac{1}{2}\left(0.98 \times \frac{\pi}{36}\right)^2$$
$$\text{or} \quad \theta_{\text{Lab}} \approx 0.98 \times \frac{\pi}{36},$$

which leads to

$$\Delta\theta|_{\theta_{\text{CM}}=5°=\frac{\pi}{36}} = \theta_{\text{CM}} - \theta_{\text{Lab}} \approx 0.02 \times \frac{\pi}{36} \approx 0.002. \qquad (1.35)$$

From Eq. (1.32), the relative change in counting rate arising from this difference in angle is given approximately by:

$$\frac{|\Delta(\text{d}n)|}{\text{d}n}(\theta) = \frac{4\frac{\Delta\theta}{2}\cos\frac{\theta}{2}}{(\sin\frac{\theta}{2})^5} \Big/ \frac{1}{(\sin\frac{\theta}{2})^4} = 2\Delta\theta\cot\frac{\theta}{2}. \qquad (1.36)$$

This leads to

$$\boxed{\frac{|\Delta(\text{d}n)|}{\text{d}n}(\theta = \tfrac{\pi}{2}) \approx 2 \times 0.02 \times 1 = 0.04 = 4\%,}$$

$$\boxed{\frac{|\Delta(\text{d}n)|}{\text{d}n}(\theta = \tfrac{\pi}{36}) \approx 2 \times 0.002 \times 23 \approx 0.092 = 9.2\%.}$$

Problem 1.5 *Sketch the cross section in the laboratory frame as a function of* $\cos\theta_{\rm Lab}$ *for the elastic scattering of equal-mass particles when* $\frac{d\sigma}{d\Omega_{\rm CM}}$ *is isotropic and equal to* 100 mb/sr. *What would be your result for* $\zeta = 0.05$ *in Eq.* (1.52)? (*You may use approximations where necessary.*)

We know from Eqs. (1.53) and (1.55) of the text that

$$\cos\theta_{\rm Lab} = \frac{\cos\theta_{\rm CM} + \zeta}{(1 + 2\zeta\cos\theta_{\rm CM} + \zeta^2)^{1/2}},$$
$$\frac{d\sigma}{d\Omega_{\rm Lab}}(\theta_{\rm Lab}) = \frac{d\sigma}{d\Omega_{\rm CM}}\frac{(1 + 2\zeta\cos\theta_{\rm CM} + \zeta^2)^{3/2}}{|1 + \zeta\cos\theta_{\rm CM}|}, \qquad (1.37)$$

where $\zeta = \frac{m_1}{m_2}$.

When $m_1 = m_2$, namely, the particle masses are equal, we have $\zeta = 1$, and we obtain from the first relationship

$$\cos\theta_{\rm Lab} = \frac{1 + \cos\theta_{\rm CM}}{\sqrt{2}(1 + \cos\theta_{\rm CM})^{1/2}} = \left(\frac{1 + \cos\theta_{\rm CM}}{2}\right)^{1/2},$$

which leads to

$$\begin{aligned}\frac{(1 + 2\zeta\cos\theta_{\rm CM} + \zeta^2)^{3/2}}{|1 + \zeta\cos\theta_{\rm CM}|} &= \frac{2^{3/2}(1 + \cos\theta_{\rm CM})^{3/2}}{(1 + \cos\theta_{\rm CM})} \\ &= 2^{3/2}\,(1 + \cos\theta_{\rm CM})^{1/2} \\ &= 4\left(\frac{1 + \cos\theta_{\rm CM}}{2}\right)^{1/2} \\ &= 4\cos\theta_{\rm Lab}. \end{aligned} \qquad (1.38)$$

Thus, for $\zeta = 1$ (equal masses), we can write

$$\frac{d\sigma}{d\Omega_{\rm Lab}}(\theta_{\rm Lab}) = 4\frac{d\sigma}{d\Omega_{\rm CM}}(\theta_{\rm CM})\cos\theta_{\rm Lab}. \qquad (1.39)$$

For an isotropic cross section in the center-of-mass frame of value

$$\frac{d\sigma}{d\Omega_{\rm CM}}(\theta_{\rm CM}) = 100\,{\rm mb/sr}, \qquad (1.40)$$

we obtain

$$\frac{d\sigma}{d\Omega_{\rm Lab}}(\theta_{\rm Lab}) = 400\cos\theta_{\rm Lab}\,{\rm mb/sr}, \qquad (1.41)$$

which leads to a straight line with zero intercept for the laboratory cross section plotted against $\cos\theta_{\rm Lab}$ (see Fig. 1.3).

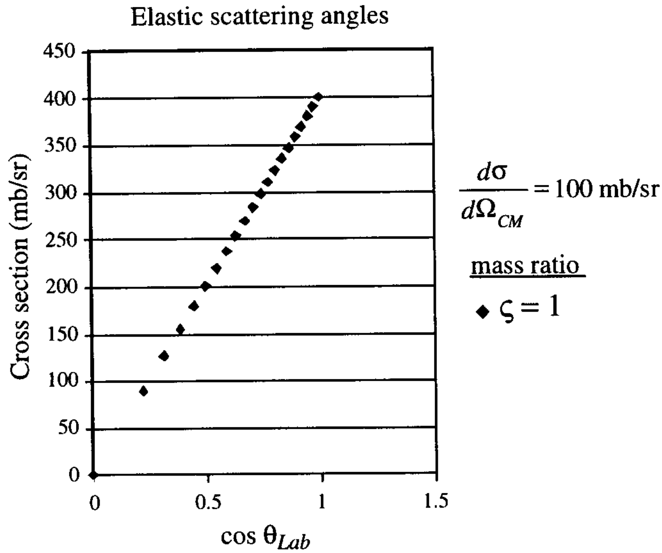

Fig. 1.3. Differential cross section vs. laboratory angle for mass ratio $\zeta = 1.0$.

For the case of a light projectile with $\zeta = 0.05$, we can use our earlier result (1.18) to write

$$\cos\theta_{\mathrm{Lab}} \approx \cos\theta_{\mathrm{CM}} + 0.05\sin^2\theta_{\mathrm{CM}}, \tag{1.42}$$

which can be inverted to give

$$\cos\theta_{\mathrm{CM}} \approx \cos\theta_{\mathrm{Lab}} - 0.05. \tag{1.43}$$

Using this, we obtain

$$\begin{aligned} \frac{(1 + 2\zeta\cos\theta_{\mathrm{CM}} + \zeta^2)^{3/2}}{|1 + \zeta\cos\theta_{\mathrm{CM}}|} &\approx \frac{(1 + 0.1\cos\theta_{\mathrm{CM}})^{3/2}}{(1 + 0.05\cos\theta_{\mathrm{CM}})} \\ &\approx (1 + 0.15\cos\theta_{\mathrm{CM}})(1 - 0.05\cos\theta_{\mathrm{CM}}) \\ &\approx 1 + 0.1\cos\theta_{\mathrm{CM}} \\ &\approx 1 + 0.1(\cos\theta_{\mathrm{Lab}} - 0.05) \\ &\approx 1 + 0.1\cos\theta_{\mathrm{Lab}}. \end{aligned} \tag{1.44}$$

Therefore, for an isotropic cross section in the center-of-mass given by (1.40), the cross section in the laboratory takes the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{Lab}}}(\theta_{\mathrm{Lab}}) \approx \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{CM}}}(\theta)(1 + 0.1\cos\theta_{\mathrm{Lab}}) = 100(1 + 0.1\cos\theta_{\mathrm{Lab}})$$
$$= (100 + 10\cos\theta_{\mathrm{Lab}})\,\mathrm{mb/sr}. \tag{1.45}$$

In this case, the laboratory cross section is again linear with $\cos\theta_{\mathrm{Lab}}$, but has a much smaller slope and a finite intercept, as shown in Fig. 1.4.

Note that $\int_{\mathrm{CM}} d\sigma = \int_{\mathrm{Lab}} d\sigma$!

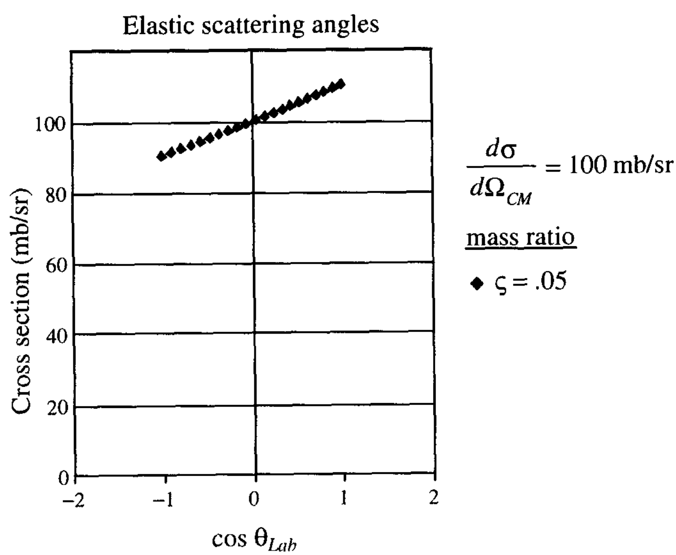

Fig. 1.4. Differential cross section vs. laboratory angle for mass ratio $\zeta = 0.05$.

Problem 1.6 *Certain radioactive nuclei emit α particles. If the kinetic energy of these α particles is* 4 MeV, *what is their velocity if you assume them to be nonrelativistic? How large an error do you make in neglecting special relativity in the calculation of v? What is the closest that such an α particle can get to the center of a Au nucleus?*

The α particle has a rest mass given approximately by

$$M = 4 \times 10^3 \,\text{MeV}/c^2. \tag{1.46}$$

If the α particle has a kinetic energy

$$T = 4\,\text{MeV}, \tag{1.47}$$

and if we treat it nonrelativistically, then we have

$$\begin{aligned}
T = \frac{1}{2}Mv_{\text{NR}}^2 = \frac{1}{2}Mc^2 \times \left(\frac{v_{\text{NR}}}{c}\right)^2 &= 4 \text{ MeV} \\
\text{or} \quad \frac{1}{2} \times 4 \times 10^3 \text{ MeV} \times \left(\frac{v_{\text{NR}}}{c}\right)^2 &= 4 \text{ MeV} \\
\text{or} \quad \left(\frac{v_{\text{NR}}}{c}\right)^2 &= 2 \times 10^{-3} \\
\text{or} \quad \frac{v_{\text{NR}}}{c} &= \sqrt{20} \times 10^{-2} \approx 0.045.
\end{aligned} \tag{1.48}$$

Here v_{NR} represents the magnitude of the velocity of the nonrelativistic particle.

On the other hand, if we treat the α particle as relativistic, we can then use the relativistic relationships from Eqs. (A.7) and (A.10) of Appendix A of the text to write

$$\begin{aligned}
E &= \gamma Mc^2, \\
T &= E - Mc^2 = (\gamma - 1)Mc^2, \\
cP &= \sqrt{T^2 + 2Mc^2T} = \sqrt{\gamma^2 - 1}Mc^2,
\end{aligned} \tag{1.49}$$

where P is the magnitude of the momentum. The relativistic velocity now follows using Eq. (A.8) of the text

$$\frac{v_{\text{R}}}{c} = \frac{cP}{E} = \frac{\sqrt{(\gamma+1)(\gamma-1)}}{\gamma}. \tag{1.50}$$

From the fact that the α particle has kinetic energy

$$T = 4\,\text{MeV}, \tag{1.51}$$

we can determine the Lorentz factor using (1.49)

$$\begin{aligned}
T = (\gamma - 1)Mc^2 &= 4\,\text{MeV} \\
\text{or} \quad (\gamma - 1) \times 4 \times 10^3\,\text{MeV} &= 4\,\text{MeV} \\
\text{or} \quad \gamma &= 1 + 10^{-3}.
\end{aligned} \tag{1.52}$$

Using this in (1.50), we can determine the relativistic velocity

$$\begin{aligned}\frac{v_{\mathrm{R}}}{c} &= \frac{\sqrt{(2+10^{-3})10^{-3}}}{1+10^{-3}} \\ &\approx \sqrt{20}\times 10^{-2}(1+0.5\times 10^{-3})^{1/2}(1-10^{-3}) \\ &\approx \sqrt{20}\times 10^{-2}(1+0.025\times 10^{-3})(1-10^{-3}) \\ &\approx \sqrt{20}\times 10^{-2}(1-0.00075). \end{aligned} \tag{1.53}$$

Using (1.48) we see that we can write the relativistic velocity in Eq. (1.53) as

$$v_{\mathrm{R}} = v_{\mathrm{NR}}(1-0.00075). \tag{1.54}$$

Consequently, we can define the relative error in neglecting relativity as

$$\frac{|\Delta v|}{v_{\mathrm{NR}}} = \frac{|v_{\mathrm{R}}-v_{\mathrm{NR}}|}{v_{\mathrm{NR}}} \approx 0.00075 = 0.07\%. \tag{1.55}$$

For the scattering of such an α particle from gold (Au), the distance of closest approach can be dertermined as follows. First we note that the distance of closest approach is attained when the impact parameter vanishes (for head on collisions). From Eq. (1.25) of the text, we see that the distance of closest in this case ($b=0$) is given by

$$r_0 = \frac{ZZ'e^2}{E}, \tag{1.56}$$

where, for scattering of α particles from gold (Au), we have

$$Z=2, \quad Z'=79. \tag{1.57}$$

If we treat the α particle nonrelativistically, we have

$$E = T = 4\,\mathrm{MeV}. \tag{1.58}$$

Using all of these, we determine

$$\begin{aligned} r_0 &= ZZ'\times\frac{\hbar c}{E}\times\frac{e^2}{\hbar c} \\ &= 2\times 79\times\frac{197\ \mathrm{MeV}-\mathrm{F}}{4\ \mathrm{MeV}}\times\frac{1}{137} \\ &\approx 56\ \mathrm{F} = 5.6\times 10^{-12}\,\mathrm{cm}. \end{aligned} \tag{1.59}$$

Problem 1.7 *An electron of momentum* 0.511 MeV/c *is observed in the laboratory. What are its* $\beta = \frac{v}{c}$, $\gamma = (1-\beta^2)^{-1/2}$, *kinetic energy, and total energy?*

The rest mass of an electron is

$$m = 0.511\,\text{MeV}/c^2. \tag{1.60}$$

If the electron has a momentum

$$p = 0.511\,\text{MeV}/c, \tag{1.61}$$

then it is fairly relativistic. Using Einstein's relationship we have the total energy of the electron

$$E = \sqrt{p^2c^2 + m^2c^4} = \sqrt{2}\,mc^2 = \gamma mc^2, \tag{1.62}$$

where we have used (1.49). This determines the Lorentz factor

$$\gamma = \sqrt{2}. \tag{1.63}$$

From the definition of the Lorentz factor

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}}, \quad \beta = \frac{v}{c}, \tag{1.64}$$

we obtain

$$\beta = \left(1 - \frac{1}{\gamma^2}\right)^{1/2} = \left(1 - \frac{1}{2}\right)^{1/2} = \frac{1}{\sqrt{2}}. \tag{1.65}$$

The value of the total energy is obtained from (1.62)

$$E = \sqrt{2}mc^2 = \sqrt{2} \times 0.511\ \text{MeV} \approx 0.722\,\text{MeV}. \tag{1.66}$$

The kinetic energy follows from (1.49)

$$T = (\gamma - 1)mc^2 = (\sqrt{2} - 1) \times 0.511\ \text{MeV} \approx 0.211\,\text{MeV}. \tag{1.67}$$

Problem 1.8 *What are the approximate values of the kinetic energy for the recoiling lead nucleus and the momentum transfers (in* eV *units) at the cutoffs specified in Problem* 1.1*?*

We note from Eq. (1.1) of the text that in a scattering involving α particles, conservation of momentum determines the recoil velocity of the target

$$\vec{v}_t = \frac{m_\alpha}{m_t}(\vec{v}_0 - \vec{v}_\alpha), \tag{1.68}$$

where m_α, m_t represent the mass of the α particle and target, respectively, and $\vec{v}_0$, $\vec{v}_\alpha$ denote respectively the initial and final velocities of the α particle in the laboratory. It follows that the recoil kinetic energy of the target in the laboratory is given by

$$\begin{aligned}\text{Recoil energy} &= \frac{1}{2}m_t v_t^2 = \frac{m_t}{2} \times \frac{m_\alpha^2}{m_t^2}(\vec{v}_0 - \vec{v}_\alpha)^2 \\ &= \frac{1}{2m_t}(\vec{p}_0 - \vec{p}_\alpha)^2 \\ &= \frac{1}{2m_t}\left(p_0^2 + p_\alpha^2 - 2\vec{p}_0 \cdot \vec{p}_\alpha\right). \end{aligned} \tag{1.69}$$

For elastic scattering, because m_α is far smaller than m_t, the magnitude of the initial and final momentum of the α particle is essentially the same. (The nucleus absorbs essentially no energy, but just momentum.) We can therefore write

$$\begin{aligned}\text{Recoil energy} &\approx \frac{1}{m_t}p_0^2(1 - \cos\theta) \\ &= \frac{2m_\alpha}{m_t}E(1-\cos\theta) = \frac{4m_\alpha}{m_t}E\sin^2\frac{\theta}{2} \\ &= \frac{4m_\alpha}{m_t}\,\frac{E}{1 + b^2\left(\frac{2E}{ZZ'e^2}\right)^2}, \end{aligned} \tag{1.70}$$

where E represents the energy of the incident α particle, and we used the relationship between the scattering angle and the impact parameter given in Eq. (1.32) of the text. For the scattering of the α particle from lead ($^{208}Pb^{82}$), as in Eq. (1.31), we can calculate

$$\begin{aligned}\left(\frac{ZZ'e^2}{2E}\right)^2 &= \left(ZZ' \times \frac{\hbar c}{2E} \times \frac{e^2}{\hbar c}\right)^2 \\ &= \left(2 \times 82 \times \frac{197\text{ MeV} - \text{F}}{2 \times 10\text{ MeV}} \times \frac{1}{137}\right)^2 \\ &\approx 1.4 \times 10^{-24}\,\text{cm}^2. \end{aligned} \tag{1.71}$$

Using this, we can write the recoil energy as a function of impact parameter as

$$\begin{aligned}\text{Recoil energy} &= \frac{4m_\alpha}{m_t}\,\frac{E}{1+b^2\left(\frac{2E}{ZZ'e^2}\right)^2}\\ &\approx \frac{4\times 4}{208}\,\frac{10\ \text{MeV}}{1+\frac{b^2}{1.4\times 10^{-24}\ \text{cm}^2}}\\ &= \frac{0.8\,\text{MeV}}{1+0.7\times 10^{24}\ b^2/\text{cm}^2}.\end{aligned} \tag{1.72}$$

Using Eq. (1.72), we tabulate below the values of recoil energy for different cutoffs on impact parameter:

b (cm)	Recoil energy (MeV)
10^{-12}	0.5
10^{-10}	1.1×10^{-4}
10^{-8}	1.1×10^{-8}

Since $T_{\text{Pb}} \ll T_\alpha$, our assumption that the initial and final energies of the α particle are the same holds well. Note that, for impact parameters of 10^{-8} cm, energy transfers to the nucleus are vanishingly small 10^{-2} eV. At these "enormous" distances, there must be some shielding of nuclear charge by the external electrons, and so the calculation cannot be valid. Also, for transferring energy to bound electrons, rather than to nuclei, the electrons cannot absorb arbitrary amounts since they are located in quantized orbits.

Problem 1.9 *Taking the ultrarelativistic limit of Eq. (1.71), find an approximate expression for θ_{Lab} at $\theta_{\text{CM}} = \frac{\pi}{2}$, and evaluate θ_{Lab} for $\gamma_{\text{CM}} = 10$ and $\gamma_{\text{CM}} = 100$. Does the approximation hold best for particles with small or large mass values?*

Equation (1.71) in the text relates the laboratory and CM angles as follows:

$$\tan\theta_{\text{Lab}} = \frac{1}{\gamma_{\text{CM}}}\,\frac{\tilde{\beta}\sin\theta_{\text{CM}}}{\tilde{\beta}\cos\theta_{\text{CM}}+\beta_{\text{CM}}} \approx \frac{1}{\gamma_{\text{CM}}}\,\frac{\sin\theta_{\text{CM}}}{\cos\theta_{\text{CM}}+1} \approx \frac{1}{\gamma_{\text{CM}}} \tag{1.73}$$

where the next-to-last term is for highly relativistic scattering, and the final term is for $\theta_{CM} = \frac{\pi}{2}$.

Thus $\theta_{Lab} = \arctan(\frac{1}{\gamma_{CM}})$. For $\gamma_{CM} = 10$ and 100, this corresponds to $\arctan(0.1) = 5.7^\circ$ and $\arctan(0.01) = 0.57^\circ$, respectively.

Clearly, such approximations hold best at high energies when both β_{CM} and the $\tilde{\beta}$ of the produced particle are large. That is, when the particle mass in $\tilde{\beta}$ can be ignored. It should be recognized that high-energy scattering does not necessarily imply that the mass of any produced particle can be assumed to be negligible.

Problem 1.10 *What is the minimum impact parameter needed to deflect* 7.7 MeV *α-particles from gold nuclei by at least* 1°*? What about by at least* 30°*? What is the ratio of probabilities for deflections of* $\theta > 1^\circ$ *relative to* $\theta > 30^\circ$*? (See the CRC Handbook for the density of gold.)*

For the scattering of a 7.7 MeV α-particle from gold, we have

$$Z = 2, \quad Z' = 79, \quad E = 7.7\,\text{MeV}, \tag{1.74}$$

so that we obtain

$$\begin{aligned}\frac{ZZ'e^2}{2E} &= ZZ' \times \frac{\hbar c}{2E} \times \frac{e^2}{\hbar c} \\ &= 2 \times 79 \times \frac{197\,\text{MeV} - \text{F}}{2 \times 7.7\,\text{MeV}} \times \frac{1}{137} \\ &\approx 14.5 \times 10^{-13}\,\text{cm} \approx 1.4 \times 10^{-12}\,\text{cm}.\end{aligned} \tag{1.75}$$

We know from Eq. (1.32) of the text that

$$b = \frac{ZZ'e^2}{2E} \cot\frac{\theta}{2}, \tag{1.76}$$

which leads to

$$b(\theta) \approx 1.4 \times 10^{-12} \cot\frac{\theta}{2}\,\text{cm}. \tag{1.77}$$

We note that for

$$\theta = 1^\circ = \frac{\pi}{180} \approx \frac{1}{60} \ll 1, \tag{1.78}$$

we have

$$\tan\frac{\theta}{2} \approx \frac{\theta}{2} \approx \frac{1}{120}, \quad \cot\frac{\theta}{2} = \frac{1}{\tan\frac{\theta}{2}} \approx 120. \tag{1.79}$$

Similarly, for

$$\theta = 30^\circ = \frac{\pi}{6} \approx \frac{1}{2} \tag{1.80}$$

we have

$$\tan\frac{\theta}{2} \approx \frac{\theta}{2} \approx \frac{1}{4}, \quad \cot\frac{\theta}{2} = \frac{1}{\tan\frac{\theta}{2}} \approx 4. \tag{1.81}$$

Using these values in (1.77), we obtain

$$\begin{aligned} b(\theta = 1^\circ) &\approx 1.4 \times 10^{-12} \times 120\,\text{cm} = 1.7 \times 10^{-10}\,\text{cm}, \\ b(\theta = 30^\circ) &\approx 1.4 \times 10^{-12} \times 4\,\text{cm} = 5.6 \times 10^{-12}\,\text{cm}. \end{aligned} \tag{1.82}$$

As we have already seen in Problem 1.1, the probability of scattering for angles greater than θ_b goes as the area πb^2. Therefore, using (1.82) we have

$$\begin{aligned} \frac{\sigma(\theta > 1^\circ)}{\sigma(\theta > 30^\circ)} &= \frac{b^2(\theta = 1^\circ)}{b^2(\theta = 30^\circ)} \\ &\approx \left(\frac{1.7 \times 10^{-10}}{5.6 \times 10^{-12}}\right)^2 \approx 900. \end{aligned} \tag{1.83}$$

In other words, there will be approximately 900 more particle collisions for $\theta > 1^\circ$ than for $\theta > 30^\circ$.

Problem 1.11 *Consider a collimated source of* 8 MeV *α-particles that provides* 10^4 α/sec *that impinge on a* 0.1 mm *gold foil. What counting rate would you expect in a detector that subtends an annular cone of* $\Delta\theta = 0.05$ *rad, at a scattering angle of* $\theta = 90^\circ$*? Compare this to the rate at* $\theta = 5^\circ$*. Is there a problem? Is it serious (see Problem* 1.12*). (Hint: You can use the small-angle approximation where appropriate, and find the density of gold in the CRC Handbook.)*

For scattering of 8 MeV α particles from gold, we have

$$\begin{aligned}
Z &= 2, \quad Z' = 79, \quad E = 8\,\text{MeV},\\
\rho &= \text{density of gold} \approx 19.3\,\text{g/cm}^3,\\
t &= \text{Thickness of gold foil} = 0.1\,\text{mm} = 10^{-2}\,\text{cm},\\
N_0 &= \text{Incident flux} = 10^4/\text{sec},\\
A &= \text{Atomic weight of gold} = 197,\\
A_0 &= \text{Avogadro's number} \approx 6 \times 10^{23}/\text{mole},\\
\Delta\theta &= \text{Angle subtended by the detector} = 0.05\,\text{rad}.
\end{aligned} \tag{1.84}$$

We can therefore calculate

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta) &= \left(\frac{ZZ'e^2}{4E}\right)^2 \operatorname{cosec}^4\frac{\theta}{2}(\text{sr})^{-1}\\
&= \left(ZZ' \times \frac{\hbar c}{4E} \times \frac{e^2}{\hbar c}\right)^2 \operatorname{cosec}^4\frac{\theta}{2}(\text{sr})^{-1}\\
&= \left(2 \times 79 \times \frac{197\ \text{MeV} - \text{F}}{4 \times 8\ \text{MeV}} \times \frac{1}{137}\right)^2 \operatorname{cosec}^4\frac{\theta}{2}(\text{sr})^{-1}\\
&\approx 0.48 \times 10^{-24} \operatorname{cosec}^4\frac{\theta}{2}\,\text{cm}^2/\text{sr}.
\end{aligned} \tag{1.85}$$

Similarly, we have

$$\begin{aligned}
\frac{A_0\rho t}{A} &\approx \frac{6 \times 10^{23} \times 19.3 \times 10^{-2}/\text{cm}^2}{197}\\
&\approx 6 \times 10^{20}/\text{cm}^2.
\end{aligned} \tag{1.86}$$

From Eq. (1.40) of the text we therefore obtain the counting rate

$$\begin{aligned}
\mathrm{d}n(\theta) &= N_0\frac{A_0\rho t}{A}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta)\,\mathrm{d}\Omega\\
&\approx 10^4/\text{sec} \times 6\times10^{20}/\text{cm}^2 \times 0.48 \times 10^{-24}\operatorname{cosec}^4\frac{\theta}{2}\,\text{cm}^2/\text{sr} \times \mathrm{d}\Omega\\
&= 2.88\operatorname{cosec}^4\frac{\theta}{2}\,\mathrm{d}\Omega(\text{sec} - \text{sr})^{-1}.
\end{aligned} \tag{1.87}$$

For scattering with azimuthal symmetry, we can write

$$\mathrm{d}\Omega = 2\pi\sin\theta\mathrm{d}\theta, \tag{1.88}$$

and if we identify $\mathrm{d}\theta \approx \Delta\theta = 0.05\,\text{rad}$, we get

$$\mathrm{d}\Omega \approx 2\pi\sin\theta \times 0.05\,\text{sr} \approx 0.3\sin\theta\,\text{sr}. \tag{1.89}$$

Putting this back into (1.87), we obtain

$$\begin{aligned} dn(\theta) &\approx 2.88 \text{ cosec}^4 \frac{\theta}{2} (\text{sec} - \text{sr})^{-1} \times 0.3 \sin\theta \text{ sr} \\ &\approx 0.86 \sin\theta \text{ cosec}^4 \frac{\theta}{2} (\text{sec})^{-1}. \end{aligned} \tag{1.90}$$

It follows that

$$dn\left(\theta = \frac{\pi}{2}\right) \approx 0.86 \times 1 \times (\sqrt{2})^4 \approx 3.4/\text{sec}. \tag{1.91}$$

On the other hand, for $\theta = 5° = \frac{\pi}{36} \approx \frac{1}{12} \ll 1$, we have

$$\sin\theta \approx \theta \approx \frac{1}{12}, \quad \text{cosec}^4 \frac{\theta}{2} \approx \left(\frac{2}{\theta}\right)^4 \approx (24)^4, \tag{1.92}$$

and we obtain

$$dn(\theta = 5°) \approx 0.86 \times \frac{1}{12} \times (24)^4/\text{sec} \approx 2.4 \times 10^4/\text{sec}. \tag{1.93}$$

This is, in fact, larger than the incident flux, and, if this were true, conservation of probability (particle number) would be violated, which is a serious problem! For one thing, we note that the approximation

$$d\theta \approx \Delta\theta, \tag{1.94}$$

is meaningful only when

$$\frac{\Delta\theta}{\theta} \ll 1, \tag{1.95}$$

which is clearly violated when $\Delta\theta = 0.05$ rad and $\theta = 5° \approx \frac{1}{12} \approx 0.08$ rad. This is one of the sources of the difficulty. For other sources of this error, we turn to the solution of the next problem.

Problem 1.12 *Consider the expression Eq.* (1.41) *for Rutherford Scattering of α-particles from gold nuclei. Integrate this over all angles to obtain n. In principle, n cannot exceed N_0, the number of incident particles. Why? What cutoff value for θ would be required in the integral, that is, some $\theta = \theta_0 > 0$, to assure that n does not exceed N_0 in Problem* 1.4? (*Hint: After integrating, use the small-angle approximation to simplify the calculation.*) *Using the Heisenberg uncertainty principle $\Delta p_x \Delta x \approx \hbar$, where Δx is some* *transverse distance corresponding to a change in transverse momentum of* $\Delta p_x = p_{in}\theta_0 \approx \sqrt{2mE}\theta_0$*, calculate the distances* Δx *to which you have to restrict the description of the scattering. Are these distances sufficiently restrictive? Explain!*

Both Eqs. (1.40) and (1.41) of the text are equivalent and give the counting rate at a scattering angle θ. Let us look at Eq. (1.40) of the text

$$\frac{\mathrm{d}n}{\mathrm{d}\Omega}(\theta) = N_0 \frac{A_0 \rho t}{A} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta). \tag{1.96}$$

Since N_0, A_0, ρ, t, A are constants independent of scattering angle, integrating the above relationship over all angles (above a certain cutoff value, corresponding to some cutoff in impact parameter, as was discussed in Problem 1.1), we obtain

$$\begin{aligned} n(b > b_0) &= N_0 \frac{A_0 \rho t}{A}\, \sigma(b > b_0) = N_0 \frac{A_0 \rho t}{A} \times \pi b_0^2 \\ \text{or} \quad \frac{n(b > b_0)}{N_0} &= \frac{A_0 \rho t}{A} \times \pi b_0^2. \end{aligned} \tag{1.97}$$

Clearly, the total number of particles scattered per second, $n(b > b_0)$, cannot exceed the total incident flux, namely, $n(b > b_0) \leq N_0$ for conservation of probability (particle number). This leads to the inequality

$$\begin{aligned} &\frac{A_0 \rho t}{A} \times \pi b_0^2 \leq 1 \\ \text{or} \quad & b_0 \leq \left(\frac{1}{\pi} \frac{A}{A_0 \rho t}\right)^{1/2}. \end{aligned} \tag{1.98}$$

For the case of the α-particle scattering from gold discussed in Problem 1.11, we can use (1.86) to obtain

$$b_0 \leq \left(\frac{1}{\pi} \frac{1}{6 \times 10^{20}}\right)^{1/2} \mathrm{cm} \approx 2.2 \times 10^{-11}\, \mathrm{cm}. \tag{1.99}$$

On the other hand, from Eq. (1.32) of the text

$$b = \frac{ZZ'e^2}{2E} \cot \frac{\theta_b}{2}, \tag{1.100}$$

we can calculate

$$\begin{aligned}\cot\frac{\theta_{b_0}}{2} &= \frac{2E}{ZZ'e^2}\, b_0 = \frac{1}{ZZ'} \times \frac{2E}{\hbar c} \times \frac{\hbar c}{e^2}\, b_0 \\ &\leq \frac{1}{2\times 79} \times \frac{2\times 8\ \text{MeV}}{197\ \text{MeV} - \text{F}} \\ &\quad \times 137 \times 2.2 \times 10^{-11}\ \text{cm} \approx 16 \end{aligned} \tag{1.101}$$

$$\text{or} \quad \cot\frac{\theta_{b_0}}{2} \approx \frac{2}{\theta_{b_0}} \leq 16$$

$$\text{or} \quad \theta_{b_0} \geq \frac{2}{16} = \frac{1}{8} \approx \frac{\pi}{24} = 7.5^\circ.$$

For scattering angles below this bound, namely $\theta < 7.5^\circ$, conservation of probability will be violated, which is what we saw explicitly in the solution of Problem 1.11 for $\theta = 5^\circ$. Although derived from very physical considerations, this bound on the scattering angle seems artificial. After all, a particle can scatter at any angle, and should not be subject to any such bound. In fact, all of this is an artifact of our formalism, and can be seen as follows. First, we note that the bound on the impact parameter (1.99) is close to the size of the nucleus, and consequently imposing such a cutoff may affect the validity of the calculation. In fact, at these low energies the probability of scattering is very high, but in our derivation we assumed it to be low (no second scatter ...). Our formula therefore works fine for large, but not for small angles.