Chapter 1

The C_p Index

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1.1 Process precision and the C_p index

Process capability indices, which establish the relationships between the actual process performance and the manufacturing specifications, have been a focus of research in quality assurance and process capability analysis for the last 20 years. Capability indices that qualify process potential and process performance are practical tools for successful quality improvement activities and quality program implementation. Apparently, the first process capability index to appear in the literature is the precision index C_p , defined by Kane (1986) as:

$$C_p = \frac{USL - LSL}{6\sigma},$$

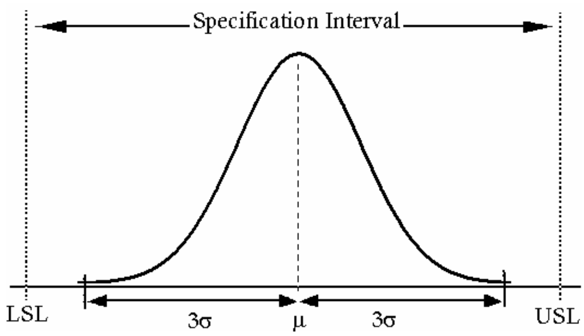

Figure 1.1. Specification interval vs. process spread for the normal distribution.

where USL is the upper specification limit, LSL is the lower specification limit, and σ is the process standard deviation (see Figure. 1.1.).To avoid confusion with the C_p-Mallow statistic the index is occasionally denoted by S_p . The numerator of C_p provides the range over which the process measurements are acceptable. The denominator gives the range over which the process is actually varying. The index C_p was designed to measure the magnitude of the overall process variation relative to the manufacturing tolerance, which is to be used for processes based on data that are normal, independent, and in the statistical control. Obviously, it is desirable to have a C_p as large as possible; small values of C_p (particularly those less than 1.00) are not acceptable because this would indicate that the natural range of variation of the process does not fit (probably exceeds) within the tolerance band. Finley (1992) refers to this index as CPI, which stands for Capability Potential or Capability Potential Index; Montgomery (1996) uses the term PCR, for Process Capability Ratio. Clearly, the index measures only the potential of a process to provide an acceptable product and does not take into account whether the process is centered or not.

Historically, the use of the capability indices was first explored within the automotive industry. Ford Motor Company (1984) initially used C_p to keep track of the process performance.

More recently, manufacturing industries have been making an extensive effort to implement statistical process control (SPC) in their plants and supply bases. Capability indices derived from SPC have received increasing usage not only in capability assessments, but also in evaluation of purchasing decisions (Kane (1986)). Capability indices are rapidly becoming a standard tool for quality reporting, especially at the management level around the world. Proper understanding and accurate estimation of the capability index is essential and crucial for a company to maintain a capable supplier.

C_P index and percentage of non-conforming ($\%NC$)

For processes with two-sided specification limits, the percentage of non-conforming items ($\%NC$) can be calculated as $1 - F(USL) + F(LSL)$, where $F(\cdot)$ is the cumulative distribution function of the process characteristic X. On the assumption of normality of the process, $\%NC$ can be expressed as:

$$\%NC = 1 - \Phi\left(\frac{USL - \mu}{\sigma}\right) + \Phi\left(\frac{LSL - \mu}{\sigma}\right).$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution (see Figure. 1.2.) and μ and σ are as above process mean and the process standard deviation, $m = (LSL + USL)/2$ is the mid-point between the lower and the upper specification limits. If the process is perfectly centered at the specification range ($\mu = m$), then the $\%NC$ can be expressed as $2\Phi(-3C_p)$. For example, $C_p = 1.00$ corresponds to $\%NC = 2700$ parts per million (ppm), and C_p =1.33 corresponds to $\%NC = 63$ ppm. However, C_p does not refer to the mean of the process, and will not provide an exact measure of percentage NC in the general case, i.e. when $\mu \neq m$. It only provides a lower bound on $\%NC$ which is $2\Phi(-3C_p)$.

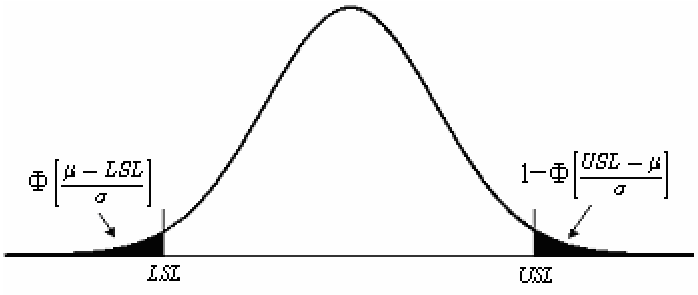

Figure 1.2. Normal distribution and proportion outside the specification limits.

1.2 Estimating and testing C_p based on a single sample

1.2.1 *Estimation of C_P*

The index C_p involves only one parameter σ to be estimated. If a single sample of size n is given as $\{x_1, x_2, \cdots, x_n\}$, a natural estimator $\hat{C}_p$ of C_p will be:

$$\hat{C}_p = \frac{USL - LSL}{6s},$$

where $s = [\sum_{i=1}^{n} (x_i - \overline{x})^2 / (n-1)]^{1/2}$ is the "conventional" estimator of the process standard deviation σ, obtained from a stable process. Under the assumption of normality, Chou and Owen (1989) obtained the probability density function (p.d.f.) of the natural estimator $\hat{C}_p$, which can be expressed as:

$$f(x) = \frac{2\left[\sqrt{(n-1)/2}\, C_p\right]^{n-1}}{\Gamma[(n-1)/2]} x^{-n} \exp\left[\frac{-(n-1)(C_p)^2}{2x^2}\right] \;, \; x > 0.$$

The distribution of s is well known under normality.

1.2.2 *The r-th moment of $\hat{C}_p$*

The r-th Moment of $\hat{C}_p$, therefore can be calculated using the properties of the χ^2 distribution as:

$$E(\hat{C}_p^r) = \frac{\Gamma\left(\frac{n-r-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\left[\frac{(n-1)}{2}\right]^{\frac{r}{2}} C_p^r \,.$$

Hence the first two moments as well as the variance may be obtained as (see, *e.g.,* Chou and Owen (1989), Pearn, *et al.* Johnson (1992), and Kotz and Johnson (1993)):

$$E(\hat{C}_p) = \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\sqrt{\frac{n-1}{2}}\,C_p \,,$$

$$E(\hat{C}_p^2) = \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\frac{(n-1)C_p^2}{2} = \left(\frac{n-1}{n-3}\right)C_p^2 \,,$$

$$Var(\hat{C}_p) = \left\{\frac{n-1}{n-3} - \frac{n-1}{2}\left[\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2\right\} C_p^2 \,.$$

It well known from the properties of the gamma function $\Gamma(k) = \int_0^\infty t^{k-1}\, \mathrm{e}^{-t}\, dt$ that the coefficient in $E(\hat{C}_p)$, is larger than 1 for all values of n . For $n \geq 15$, this coefficient can accurately be approximated by $(4n-7)/(4n-4)$. Therefore, the natural estimator $\hat{C}_p$ is biased, and overestimates the actual value of C_p . Easy calculations show that for the percentage bias to be less than one percent, i.e. $\mid E(\hat{C}_p) - C_p \mid / \hat{C}_p \leq 0.01$, it is required that $n > 80$.

1.2.3 *Statistical properties of the estimated C_P*

Pearn *et al.* (1998) obtained an unbiased estimator $\tilde{C}_p = b_{n-1}\,\hat{C}_p$ where the correction factor b_{n-1} is defined as follows:

$$b_{n-1} = \sqrt{\frac{2}{n-1}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)}.$$

(*cf.* the expression of $E(\hat{C}_P)$).

Pearn *et al.* (1998) also showed that the unbiased estimator $\tilde{C}_p$ is the uniformly minimum variance unbiased estimator (UMVUE) of C_p, which is consistent, asymptotically efficient, and moreover $n^{1/2}(\tilde{C}_p - C_p)$ converges to $N(0, C_p^2/2)$ in distribution.

1.2.4 *Confidence interval for C_P*

Due to the sampling variation introduced by estimation, it is important to construct a confidence interval providing a range which includes the true C_p with a high probability. Thus, the $100(1-\alpha)\%$ (equal tails) confidence interval for C_p may be expressed as (Chou *et al.* (1990), Pearn *et al.* (1998))

$$\left[\frac{\sqrt{\chi^2_{n-1,1-\alpha/2}}}{\sqrt{n-1}}\hat{C}_p,\ \frac{\sqrt{\chi^2_{n-1,\alpha/2}}}{\sqrt{n-1}}\hat{C}_p\right]$$

$$\text{or}\ \left[\frac{\sqrt{\chi^2_{n-1,1-\alpha/2}}}{\sqrt{n-1}\,b_{n-1}}\tilde{C}_p,\ \frac{\sqrt{\chi^2_{n-1,\alpha/2}}}{\sqrt{n-1}\,b_{n-1}}\tilde{C}_p\right],$$

where $\chi^2_{n-1,\alpha/2}$ and $\chi^2_{n-1,1-\alpha/2}$ are the upper $\alpha/2$ and $1-\alpha/2$ quantiles of chi-squared distribution with $n-1$ degrees of freedom, respectively. Thus, the $100(1-\alpha)\%$ lower confidence limit (C_p^L) of C_p can be obtained as:

$$C_p^L = \sqrt{\frac{\chi^2_{n-1,\,1-\alpha}}{n-1}}\frac{\tilde{C}_p}{b_{n-1}} = \sqrt{\frac{\chi^2_{n-1,\,1-\alpha}}{n-1}}\hat{C}_p.$$

Percentage points of the chi-squared distribution are often approximate it by formulae. The two commonly used well known approximations are

$$\chi_{v,\,1-\alpha} \cong (v - \tfrac{1}{2})^{1/2} + \frac{z_{1-\alpha}}{\sqrt{2}} \qquad \text{(Fisher)}$$

$$\chi_{v,\,1-\alpha} \cong v^{1/2}\left[1 - \frac{2}{9v} + z_{1-\alpha}\left(\frac{2}{9v}\right)^{1/2}\right]^{3/2}. \qquad \text{(Wilson-Hilferty)}$$

Here z_α is the upper quintile of the standard normal distribution.

With these approximations we would have the approximately $100(1-\alpha)\%$ confidence interval of C_p (based on Fisher's approximation) as:

$$\left[\frac{1}{(n-1)^{1/2}}\left\{\left(n - \frac{3}{2}\right)^{1/2} - \frac{z_{\alpha/2}}{\sqrt{2}}\right\}\hat{C}_p,\right.$$

$$\left.\frac{1}{(n-1)^{1/2}}\left\{\left(n - \frac{3}{2}\right)^{1/2} + \frac{z_{\alpha/2}}{\sqrt{2}}\right\}\hat{C}_p\right],$$

or based on Wilson-Hilferty's approximation

$$\left[\left\{1 - \frac{2}{9(n-1)} - z_{\alpha/2}\left(\frac{2}{9(n-1)}\right)^{1/2}\right\}^{3/2}\hat{C}_p,\right.$$

$$\left.\left\{1 - \frac{2}{9(n-1)} + z_{\alpha/2}\left(\frac{2}{9(n-1)}\right)^{1/2}\right\}^{3/2}\hat{C}_p\right].$$

Heavlin (1988) has developed alternative confidence limits for C_p, based on approximate formulas for the moments of s^{-1}. The $100(1-\alpha)\%$ confidence intervals for C_p can be calculated as:

$$\left[\left\{1-\left[\frac{1}{2(n-3)}\left(1+\frac{6}{n-1}\right)\right]^{1/2} z_{\alpha/2}\right\}\hat{C}_p,\right.$$

$$\left.\left\{1+\left[\frac{1}{2(n-3)}\left(1+\frac{6}{n-1}\right)\right]^{1/2} z_{\alpha/2}\right\}\hat{C}_p\right],$$

where as above $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution. A comparison between the three approaches should be illuminating.

1.2.5 *Sample size determination for estimation of C_P*

Franklin (1999) derived approximate sample size formulas for the process index C_p based on the Wilson-Hilferty approximation described above. The approximation has been shown to be quite accurate for a wide range of percentile points and sample sizes as small as 10. The formula suggested by Franklin (1999) for determining sample size to estimate C_p is

$$n \cong 1+\frac{2}{9}\frac{1}{\dfrac{z_\alpha}{2}+\sqrt{1+\left(\dfrac{z_\alpha}{2}\right)^2-\left(\dfrac{C_p^L}{\hat{C}_p}\right)^{2/3}}}\,. \tag{1.1}$$

It is seen that the sample size necessary to determine a desired lower confidence limit for C_p depends on the ratio of $C_p^L/\hat{C}_p$ and is an increasing function of this ratio. For example, if one wants a 95% lower confidence limit for C_p to be found as 0.80 of the value of $\hat{C}_p$ (i.e. $C_p^L/\hat{C}_p = 0.80$), the sample size can be determined by utilizing formula (1.1). Note that the sample size is always rounded up to integers.

1.2.6 *Hypothesis testing with* C_P

In the study of process capability testing, to judge if the process satisfies the preset capability requirement (i.e. being capable), we can consider the following testing hypothesis procedure with the null hypothesis H_0: $C_p \le C$ (the process is not capable), versus the alternative H_1: $C_p > C$ (the process is capable), where C is a predetermined capability requirement. For cases with a single sample, Pearn *et al.* (1998) considered the test $\phi(x) = 1$ if $\tilde{C}_p > c_0$, and $\phi(x) = 0$ otherwise. The test ϕ rejects the null hypothesis if $\tilde{C}_p > c_0$, with type I error $\alpha(c_0) = \alpha$ (the chance of incorrectly judging an incapable process as capable) and the critical value c_0 can be obtained. Pearn *et al.* (1998) showed that the test ϕ is the uniformly most powerful (UMP) test of α level, possessing the minimal type II error among all the unbiased tests. The appropriate critical value is given by:

$$c_0 = \frac{\sqrt{n-1}\, b_{n-1}}{\sqrt{\chi^2_{n-1,\,1-\alpha}}} C \, .$$

Thus, if $\tilde{C}_p > c_0$, $\phi(x) = 1$ and we reject the null hypothesis H_0, concluding that the process meets the capability requirement ($C_p > C$).

1.3 Estimating and testing C_p based on multiple samples

1.3.1 *Estimation of* C_P *and its properties*

For cases where data are collected in the form multiple samples, Kirmani *et al.* (1991) consider m samples each of size n and suggest the following estimator of C_p (where $\overline{x}_i$ is the i-th sample mean, and s_i is the i-th sample standard deviation):

$$\hat{C}_p^* = \frac{(USL - LSL)\, d_p}{6} \, ,$$

Here $$d_p = \sqrt{\frac{m(n-1)-1}{m(n-1)}}\frac{\varepsilon_{m(n-1)-1}}{s_p},$$

$$\varepsilon_{m(n\text{-}1)\text{-}1} = E\left[\frac{\chi_{m(n\text{-}1)\text{-}1}}{\sqrt{m(n\text{ - }1)\text{ - }1}}\right]$$
$$= \sqrt{\frac{2}{m(n\text{ - }1)\text{ - }1}}\Gamma\left(\frac{m(n-1)}{2}\right)\left[\Gamma\left(\frac{m(n-1)-1}{2}\right)\right]^{-1},$$

and $$s_p^2 = \frac{1}{m(n-1)}\sum_{i=1}^{m}(n-1)s_i^2 = \frac{1}{m}\sum_{i=1}^{m}s_i^2 .$$

Recall that under the normality assumption the statistic s_p/σ is distributed as $\chi_{m(n-1)-1}/[m(n-1)-1]^{1/2}$. Therefore, the estimator $\hat{C}_p^*$ is distributed as:

$$\hat{C}_p^* \sim \frac{\sqrt{m(n-1)-1}\,\varepsilon_{m(n-1)-1}}{\sqrt{\chi^2_{m(n-1)}}}C_p .$$

Note that for $m=1$, the above estimator reduces to the ordinary estimator (UMVUE) of C_P . The estimator $\hat{C}_p^*$ is unbiased, and p.d.f., $g(y)$ say, for $y>0$, can be obtained using the p.d.f. of the χ^2 distribution and denoting $h \equiv [m(n-1)-1]\,\varepsilon^2_{m(n-1)-1}C_p^2$, (which can be expressed as a function of C_p) becomes:

$$g(y) = \frac{2h^{m(n-1)/2}}{2^{m(n-1)/2}\Gamma[m(n-1)/2]}y^{-[m(n-1)+1]}\exp\left[-\frac{h}{2}\left(\frac{1}{y^2}\right)\right].$$

More recently, Pearn and Yang (2003) investigated some statistical properties of $\hat{C}_p^*$ and showed that $\hat{C}_p^*$ is asymptotically efficient and moreover $(mn)^{1/2}(\tilde{C}_p^* - C_p)$ converges to $N(0, C_p^2/2)$ in distribution. The variance of $\hat{C}_p^*$ can be calculated following Kirmani *et al.* (1991) as:

$$Var(\hat{C}_p^*) = E[(\hat{C}_p^*)^2] - [E(\hat{C}_p^*)]^2$$
$$= (USL - LSL)^2 \varepsilon^2_{m(n-1)-1} \frac{[m(n-1)-1]}{36m(n-1)} E\left(\frac{1}{s_P^2}\right) - C_p^2$$
$$= C_p^2 \left\{ \left[\frac{m(n-1)-1}{m(n-1)-2} \right] \varepsilon^2_{m(n-1)-1} - 1 \right\} = C_p^2 \left\{ \frac{1}{\varepsilon^2_{m(n-1)-2}} - 1 \right\}.$$

Recall that $\varepsilon_{m(n-1)-2}$ is a constant involving Γ functions in the argument $m(n-1)/2$.

In addition to being unbiased, it can be shown that $\hat{C}_p^*$ is consistent for C_p . For multiple samples with variable sample sizes, we can easily show that the generalized estimator $\hat{C}_p^*$ obtained from m samples each of size n_i, $i = 1, \cdots, m$, is given by:

$$\hat{C}_p^* = b_{\sum_{i=1}^{m}(n_i-1)} \times \frac{USL - LSL}{6s_p}, \text{ where } s_p^2 = \frac{\sum_{i=1}^{m}(n_i-1)s_i^2}{\sum_{i=1}^{m}(n_i-1)}, \text{ and}$$

$$b_{\sum_{i=1}^{m}(n_i-1)} = \sqrt{\frac{2}{\sum_{i=1}^{m}(n_i-1)}}\,\Gamma\left(\frac{\sum_{i=1}^{m}(n_i-1)}{2}\right)\left[\Gamma\left(\frac{\sum_{i=1}^{m}(n_i-1)-1}{2}\right)\right]^{-1}.$$

This generalized estimator is unbiased.

1.3.2 *Lower confidence bound on C_P*

Since the estimator $\hat{C}_p^*$ is subject to a sampling error, it is desirable to construct a confidence interval to provide a range which contains the true C_p with a desired high probability. For cases where multiple samples are available due to sampling at various times, Kirmani *et al.* (1991) used the lower confidence bound corresponding to the prescribed minimum value of the capability (the precision requirement). A process is considered to be capable, if the precision requirement is greater than the lower confidence bound. The $100(1-\alpha)$ percent lower confidence bound can be expressed as:

$$C_L^* = \hat{C}_p^* \sqrt{\frac{\chi^2_{m(n-1),1-\alpha}}{[m(n-1)-1]\varepsilon^2_{m(n-1)-1}}}.$$

1.3.3 *Hypothesis testing with C_P*

To test whether the process meets the precision requirement, we shall consider the following testing hypothesis procedure with H_0: $C_p \le C$ (the process is incapable), versus the alternative H_1: $C_p > C$ (the process is capable). Thus, we may consider the test of the form $\phi'(x) = 1$ if $\hat{C}_p^* > c_0'$, and $\phi'(x) = 0$, otherwise. The test ϕ' rejects the null hypothesis if $\hat{C}_p^* > c_0'$, with the type I error $\alpha(c_0') = \alpha$, (the chance of incorrectly judging an incapable process as capable one). The critical value c_0' can be obtained by finding the value satisfying the equation:

$$P\left(\hat{C}_p^* > c'_0 \middle| H_0 : C_p \le C\right)$$
$$= P\left(\chi^2_{m(n-1)} < \frac{[m(n-1)-1]\varepsilon^2_{m(n-1)-1}}{c_0'^2} C^2\right) = \alpha.$$

Hence, given the value of precision requirement C, the critical value c_0' of the α-level significance can be obtained by solving the equation

$$1 - \int_0^{c_0'} \frac{2k^{m(n-1)/2}}{2^{m(n-1)/2}\Gamma[m(n-1)/2]} y^{-[m(n-1)+1]} \exp\left[-\frac{k}{2}\left(\frac{1}{y^2}\right)\right] dy = \alpha,$$

where $k = [m(n\text{ - }1)\text{ - }1]\,\varepsilon^2_{m(n\text{-}1)\text{-}1}\,C_p^2$, or alternatively using Kirmani *et al.* (1991), to arrive at:

$$c_0' = C\sqrt{\frac{[\,m\,(n-1)-1\,]\varepsilon^2_{m(n-1)-1}}{\chi^2_{m(n-1),1-\alpha}}}\,.$$

Under the same conditions, the p-value (the risk of wrongly rejecting the null hypothesis H_0: $C_p \le C$) corresponding to $\hat{C}_p^*$, a specific value obtained via the sample data, denoted by $\hat{c}_p^*$ is:

$$P\left(\hat{C}_p^* > \hat{c}_p^* \,|\, C_p = C\right)$$

$$= 1 - \int_0^{\hat{c}_p^*} \frac{2k^{m(n\text{-}1)/2}}{2^{m(n\text{-}1)/2}\,\Gamma[\,m(n\text{ - }1)/2\,]}\, y^{-[m(n\text{-}1)+1]} \exp\left[-\frac{k}{2}\left(\frac{1}{y^2}\right)\right] dy$$

$$= \mathrm{P}\left\{\chi^2_{m(n\text{-}1)}\,\frac{[m(n\text{ - }1)\text{ - }1]\,\varepsilon^2_{m(n\text{-}1)\text{-}1}\,C^2}{\hat{c}_p^*}\right\} = P\left\{\chi^2_{m(n\text{-}1)} \le \frac{k}{\hat{c}_p^*}\right\}.$$

Pearn and Yang (2003) developed an efficient test for C_p for the cases with multiple samples, and have shown that the proposed test is indeed the UMP test. The power of the UMP test (i.e. probability of correctly rejecting the null hypothesis $C_p \le C$ when $C_p > C$ is true), can be computed for the alternative hypothesis, H_1: $C_p = C_1 > C$. The power of the test, denoted as $\pi(C_p)$ is obtained by calculating

$$\pi(C_p) = P\left(\hat{C}_p > c^* \mid C_p = C_1\right)$$

$$= P\left\{\chi^2_{m(n\text{-}1)} \le \frac{[m(n\text{ - }1)\text{ - }1]\varepsilon^2_{m(n\text{-}1)\text{-}1}C_1^2}{c^{*2}} \mid C_p = C_1\right\}.$$

1.4 Estimating and testing C_p based on $(\bar{X},\ R)$ control chart samples

For applications where the data are obtained in the form of a single sample, this problem has been discussed in an earlier work of Kane (1986). Chou *et al.* (1990) provide tables for lower confidence limit on C_p when σ estimated by the sample standard deviation S. In this case, Pearn *et al.* (1998) introduced an unbiased estimator of C_p and showed that the unbiased estimator is also UMVUE. These authors also proposed an efficient test for C_p based on a single sample and showed that the test is a UMP (uniformly most powerful) test. Kirmani *et al.* (1991) considered the estimation of σ and the precision index C_p using the data in the form of multiple samples. When σ estimated by the sample range d_2, Li *et al.* (1990) provide tables for the lower confidence limit on C_p . Pearn and Yang (2003) propose an unbiased estimator of C_p for multiple samples when σ is estimated by pooled sample variance, and have shown that the unbiased estimator is the UMVUE of C_p, which is asymptotically efficient. Pearn and Yang (2003) also developed an efficient test for C_p when multiple samples are used and showed that the proposed test is indeed an UMP test.

For applications where a routine-based data collection plans are implemented, a common practice for process control is to estimate the process precision by analyzing the past "in control" data. Consider m preliminary multiple samples (subgroups), each of size n taken from the control chart samples. To estimate σ we usually use either the sample standard deviation or the sample range. The control chart can be used as a monitoring device or a logbook to show the effect of changes in the process performance. Observe that a process may be in control but not necessarily operating at an acceptance level. Thus, management intervention will be required either to improve the process capability or to change the manufacturing requirements ensuring that the products meet at least the minimum acceptable level. We emphasize again that the process must be stable in order to produce a reliable estimate of process capability. If the process is out of control in the early stages of process capability analysis, it will be undesirable and unreliable to estimate process capability. The priority action is to determine and eliminate the assignable causes in order to bring the process into an in-control state.

1.4.1 *Estimation of C_p based on $(\bar{X}, R)$ samples*

Let m samples each of size n, from a $(\bar{X}, R)$ control chart be available, and $R_{i,n}$ be the range of a sample of size n ($i = 1, \cdots, m$) and $\bar{R}_{m,n}$ be the average range in m samples of each size n. Then the mean and variance of the relative standardized range $\bar{R}_{m,n}/\sigma$ are given by

$$E(\bar{R}_{m,n}/\sigma) = E(R_{1,n}/\sigma) = d_2 \tag{1.2}$$

$$Var(\bar{R}_{m,n}/\sigma) = \frac{Var(R_{1,n})}{m\sigma^2} = \frac{d_3^2}{m}, \tag{1.3}$$

where d_2 and d_3 are functions of n, widely tabulated in quality control books and literature based on the assumption of normality and independence (originally presented in Pearson's Table A (1932)). Thus, the estimated process capability precision using the range method can be expressed as:

$$\hat{C}_{p(R)} = \frac{USL - LSL}{6\hat{\sigma}_R}, \text{ where } \hat{\sigma}_R = \frac{\bar{R}_{m,n}}{d_2}.$$

If $m = 1$, the c.d.f. of the range from a standard normal distribution is

$$F(x) = P\left(\frac{R_{1,n}}{\sigma} \le x\right)$$

$$= n\int_{-\infty}^{\infty} [\Phi(x+t) - \Phi(t)]^{n-1}\,\phi(t)\,dt \text{ , for } t > 0 \text{ ,}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the c.d.f. and the p.d.f. of the standard normal distribution $N(0,1)$. Using the first two moments of the average range, Patnaik (1950), over 50 years ago, has shown that $\overline{R}_{m,n} / \sigma$ is distributed approximately as $c\chi_v / \sqrt{v}$, where χ_v^2 is the chi-square distribution with v degree of freedom and c and v are constants which are functions of the first two moments of the sample average range. Specifically

$$E(\overline{R}_{m,n} / \sigma) = \frac{c}{\sqrt{v}} \sqrt{2}\,\Gamma(\frac{v+1}{2}) \Big/ \Gamma(\frac{v}{2}) \text{ ,} \tag{1.4}$$

and $$Var(\overline{R}_{m,n} / \sigma) = \frac{c^2}{v}\left\{ v - 2\left[\Gamma(\frac{v+1}{2}) \Big/ \Gamma(\frac{v}{2})\right]^2 \right\} \text{ .} \tag{1.5}$$

From the coefficients of the mean and variance of average range, and the values of d_2 and d_3, the values of the mean (expected value) and variance in (1.2) and (1.3) are available. Equating (1.4) to (1.2) and (1.5) to (1.3), we can easily obtain the values of c and v, as unique solutions to the system of equations above.

In the early days of control chart usage (the pre-computer era), the range method of estimating σ was widely employed to simplify the arithmetic calculations associated with control chart operations. With the availability of modern computer software and efficient hand-held calculators for control chart operations, numerical calculations are not an issue any more, and other methods could be used. If the sample size is relatively small $n \le 8$, the range method results in almost as good an estimator of variance σ^2 as does the usual sample variance s^2, which possesses attractive statistical properties. The relative efficiency (R.E.) of the range method to s^2 is shown in Table 1.1 for various sample sizes (Montgomery (2001)). For moderate and large values of n, say $n \geq 10$, the range method rapidly loses its efficiency, since it ignores all the information presented in the sample between the largest and the smallest values. Nevertheless we emphasize that for the small sample sizes, which are often employed in various control charts (n = 4, 5, or 6), it is quite satisfactory (under approximate normality).

Table 1.1. The relative efficiency of the range method to the sample variance S^2

n	2	3	4	5	6	10
R.E.	1.000	0.992	0.975	0.955	0.930	0.850

1.4.2 *Hypothesis testing for C_p based on $(\bar{X}, R)$ samples*

When estimating the process capability precision by means of the range method from the $(\bar{X}, R)$ control chart samples, the critical value $C_{0(R)}$ can be obtained by finding an appropriate value satisfying the following equation:

$$P(\hat{C}_{p(R)} \geq C_{0(R)} \mid C_p = C) = \alpha$$

$$= P\left(\frac{d}{3\hat{\sigma}_R} \geq C_{0(R)}\right) = P\left(\frac{\bar{R}_{m,n}}{\sigma} \geq \frac{d_2}{C_{0(R)}}\frac{d}{3\sigma}\right)$$

$$\simeq P\left(\chi_v^2 \leq \frac{\sqrt{v}d_2}{cC_{0(R)}}C\right).$$

Note that $d = (USL - LSL)/2$ is unrelated to d_2 and d_3. In fact, the critical value $C_{0(R)}$ for the range (R) method can be expressed as

$$C_{0(R)} = \frac{\sqrt{v}d_2}{c\sqrt{\chi^2_{v,\alpha}}} C ,$$

where $\chi^2_{v,\alpha}$ is the lower α -th percentile of the chi-square distribution with v degree of freedom.

Under the same conditions, the p-value corresponding to $\hat{c}_{p(R)}$, a specific value obtained from the sample data, can be calculated as:

$$p\text{ -value} = P(\hat{C}_{p(R)} \geq \hat{c}_{p(R)} \mid C_p = C)$$

$$= P\left(\frac{\overline{R}_{m,n}}{\sigma} \geq \frac{d_2}{\hat{c}_{p(R)}} \frac{d}{3\sigma}\right) \simeq P\left(\chi^2_v \leq \frac{\sqrt{v}d_2}{c\,\hat{c}_{p(R)}} C\right) = G\left(\left[\frac{\sqrt{v}d_2}{c\,\hat{c}_{p(R)}} C\right]^2\right).$$

where $G(\cdot)$ is the c.d.f. of the chi-square distribution with v degree of freedom.

When multiple samples taken from the ($\overline{X}$, R) control chart at various times are available, using Patnaik's (approximate) distribution of the average range, the 100 $(1-\alpha)$ % lower confidence bound $C_{L(R)}$ can be constructed (see *e.g.* Li *et al.* (1990) and Pearn *et al.* (2004). The lower confidence bound satisfies

$$P(\hat{C}_p \geq C_{L(R)}) = 1 - \alpha$$

$$= P\left(\frac{\hat{\sigma}_R}{\sigma} \geq \frac{C_{L(R)}}{\hat{C}_{p(R)}}\right) = P\left(\frac{\overline{R}_{m,n}}{\sigma} \geq \frac{d_2 C_{L(R)}}{\hat{C}_{p(R)}}\right)$$

$$\simeq P\left(\chi^2_v \geq \frac{\sqrt{v}d_2 C_{L(R)}}{c\,\hat{C}_{p(R)}}\right).$$

Consequently, we have that:

$$\frac{\sqrt{v}d_2 C_{L(R)}}{c\hat{C}_{p(R)}} = \chi^2_{v,\alpha}\text{, or the ratio } \frac{C_{L(R)}}{\hat{C}_{p(R)}} = \frac{c}{\sqrt{v}d_2}\sqrt{\chi^2_{v,\alpha}}\,,$$

Observe that the ratio values of $C_{L(R)}/\hat{C}_{p(R)}$ depends on v, c, d_2, and α. The values of v, c, d_2 are determined from m and n. We shall call this ratio $C_{L(R)}/\hat{C}_{p(R)}$ a lower confidence factor.

1.5 Estimating and testing C_p based on $(\bar{X}, S)$ control chart samples

1.5.1 *Estimation of C_p based on $(\bar{X}, S)$ samples*

If m samples each of size n from a ($\bar{X}$, S) control chart are available, Kirmani *et al.* (1991) suggests measuring the process precision index C_p, by its natural estimator $\hat{C}_p$ defined, analogously to the case $m = 1$, as follows:

$$\hat{C}_{p(S)} = \frac{USL - LSL}{6\hat{\sigma}_S}, \quad \hat{\sigma}_S = \frac{1}{\varepsilon_{n-1}}\bar{S},$$

where $$\bar{S} = \frac{1}{m}\sum_{i=1}^{m} s_i, \quad s_i = \left[\frac{1}{n-1}\sum_{j=1}^{n}(x_{ij} - \bar{x}_i)^2\right]^{1/2}$$

and the correction factor $$\varepsilon_{n-1} = \sqrt{\frac{2}{n-1}}\frac{\Gamma[n/2]}{\Gamma[(n-1)/2]}.$$

Here $\bar{X}_i$ and S_i represent the sample mean and the sample standard deviation of the i-th sample, and the correction ε_{n-1} is usually denoted in the general quality control literature by c_4. Kirmani *et al.* (1991) showed that under the normality assumption, the statistic $\bar{S}$ is approximately distributed according to the normal distribution. Hence with a reasonable approximation:

$$\frac{\overline{S} - \sqrt{n-1}\,\varepsilon_{n-1}}{\sqrt{\frac{(n-1)(1-\varepsilon_{n-1}^2)}{m}}} \sim N(0,1)\cdot$$

This approximation is particularly appropriate in the situations where adequately tight control of the process variability is desirable so that at least moderately large subgroups ($n > 10$) are available. In this case, the S-chart is shown to be preferred over the R-chart. We note that the expressions for the distribution of $\hat{C}_{p(S)}$ presented in Kirmani *et al.* (1991), Kocherlakota (1992), and Kotz and Lovelace (1998) ought to be adjusted. In fact, these authors assume the distribution of $\hat{\sigma}_S$ to be:

$$\hat{\sigma}_S \sim N\left(\sigma, \frac{\sigma^2}{m}\frac{1-\varepsilon_{n-1}^2}{\varepsilon_{n-1}^2}\right).$$

Consequently,

$$\hat{C}_{p(S)} \sim \left[1 + N\left(0, \frac{1-\varepsilon_{n-1}^2}{m\,\varepsilon_{n-1}^2}\right)\right]^{-1} C_p\,.$$

The estimator $\hat{C}_{p(S)}$ is biased. Its p.d.f., $g(x)$ say, can be expressed as follows:

$$g(x) = \frac{C_p}{\sqrt{2\pi}k}\, x^{-2} \quad \exp\left[-\frac{(C_p\,/\,x - 1)^2}{2k^2}\right] \quad \text{for } x > 0\,,$$

$g(x)$ is a function of the constant value C_p and a variant the inverse chi-square p.d.f..

Here

$$k = \sqrt{\frac{1-\varepsilon_{n\text{-}1}^2}{m\,\varepsilon_{n\text{-}1}^2}} \quad \text{and} \quad \varepsilon_{n-1} = \sqrt{\frac{2}{n-1}}\frac{\Gamma(n/2)}{\Gamma[(n\text{ - }1)/2]} \text{ as above.}$$

For the cases when $m > 1$ multiple samples of size n are available in view of sampling from the ($\bar{\mathrm{X}}$, S) control charts at various time points. Kirmani *et al.* (1991) have also constructed the $100(1-\alpha)\%$ lower confidence bound $C_{L(s)}$ to be

$$C_{L(S)} = \hat{C}_{p(S)}\left[1+z_\alpha\sqrt{\frac{1-\varepsilon_{n-1}^2}{m\,\varepsilon_{n-1}^2}}\right],$$

where as above z_α is the upper α quantile of the standard normal distribution.

1.5.2 *Hypotheses testing for C_p based on ($\bar{X}$, S) samples*

If a ($\bar{\mathrm{X}}$, S) control chart is available, then the critical value $C_{0(S)}$ can be obtained by determining the appropriate value satisfying the equation

$$P(\hat{C}_{p(S)} \geq C_{0(S)} \mid C_p = C) = \alpha$$

$$= 1 - \int_0^{c_0} \frac{C_p}{\sqrt{2\pi}k} x^{-2} \exp\left[-\frac{(C_p/x-1)^2}{2k^2}\right] dx \cdot$$

In fact, the critical value $C_{0(S)}$ can be found and expressed as follows, with z_α representing the lower 100α % percentage point of the standard normal distribution, $N(0,1)$:

$$C_{0(S)} = \frac{C}{1 \ + \ z_\alpha \sqrt{\dfrac{1-\varepsilon_{n-1}^2}{m\,\varepsilon_{n-1}^2}}},$$

where
$$\varepsilon_{n\text{-}1} = \sqrt{\frac{2}{n\text{ - }1}} \frac{\Gamma[n/2]}{\Gamma[(n\text{ - }1)/2]}.$$

Under the same conditions, the p-value corresponding to $\hat{c}_{p(S)}$, a specific value calculated from the sample data, can be expressed as:

$$\begin{aligned} p\text{-value} &= P(\hat{C}_{p(S)} > \hat{c}_{p(S)} \,\big|\, C_p = C) \\ &= 1 - \int_0^{\hat{c}_{p(S)}} \frac{C_p}{\sqrt{2\pi}k} x^{-2} \exp\left[-\frac{(C_p/x-1)^2}{2k^2}\right] dx, \end{aligned}$$

where k and ε_{n-1} are as given above.

1.6 A Bayesian approach to assessment of C_P

The majority of the research works for testing capability indices have been focused on using the traditional distributional frequency approaches. An alternative is to use the Bayesian approach, which essentially specifies a prior distribution for the parameter of interest to obtain the posterior distribution of the parameter and then infers about the parameter by only using its posterior distribution given the observations. Cheng and Spiring (1989) proposed a Bayesian procedure for assessing the capability index C_p with a non-informative prior, $\pi(\mu, \sigma) = 1/\sigma$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$, and obtained the posterior probability:

$$p = \Pr\{process\ is\ capable \mid sample\} = \Pr\{C_p > w \mid \mathbf{x}\}$$

which is equivalent to finding an one-sided credible interval for σ. Furthermore, Pearn and Wu (2003) adopt a non-informative reference prior chosen by Cheng and Spiring (1989), given a pre-specified precision level $w > 0$ (preset process capability requirement), and the posterior probability, based on the index C_p, that a process is capable is given as

$$p = \Pr\{C_p > w \mid \mathbf{x}\} = \int_{\frac{1}{t}}^{\infty} \frac{y^{\alpha-1}}{\Gamma(\alpha)} \times \exp(-y)dy$$

$$= \frac{\Gamma(\alpha, 1/t)}{\Gamma(\alpha)} = 1 - G(1/t, \alpha, 1),$$

with $$t = \frac{2\gamma}{\sum_{i=1}^{m}(n_i - 1)}\left(\frac{C_p^*}{w}\right)^2 = \frac{2\gamma}{\sum_{i=1}^{m}(n_i - 1)}\left(\frac{\hat{C}_p^*}{w\, b_{\sum_{i=1}^{m}(n_i-1)}}\right)^2,$$

and $$\gamma = \frac{\sum_{i=1}^{m}\sum_{j=1}^{ni}(x_{ij} - \overline{x}_i)^2}{\sum_{i=1}^{m}\sum_{j=1}^{ni}(x_{ij} - \overline{\overline{x}})^2} = \frac{\sum_{i=1}^{m}(n_i - 1)s_p^2}{\sum_{i=1}^{m}(n_i - 1)s_p^2 + \sum_{i=1}^{m} n_i(\overline{x}_i - \overline{\overline{x}})^2},$$

where $$b_n = [2/(n)]^{1/2}\,\Gamma[(n)/2]/\Gamma[(n-1)/2]$$

and $$s_p^2 = \frac{\sum_{i=1}^{m}\sum_{j=1}^{ni}(x_{ij} - \overline{x}_i)^2}{\sum_{i=1}^{m}(n_i - 1)}.$$

Weaknesses of C_p

Most industrial companies at the dawn of the 21-st century are no longer relying solely on C_P to quantify process capability in view of the perceived weakness of the index. The major weakness of this index lies in the fact that it measures potential capability as defined by the actual process spread and does not take into account the actual mean of the process (see Figure. 1.3.). Therefore C_P provides no indication of the actual process performance. It does not reflect the impact that shifting the process mean has on process's ability to produce product within the specifications (Chen et al. (1988, 1990) and Kane (1986)). For this reason a more refined index the C_{pk} was developed. The indices C_P and C_{pk}, when used together, provide a proper indication of the process capability with regard to both of the process spread and of the process location.

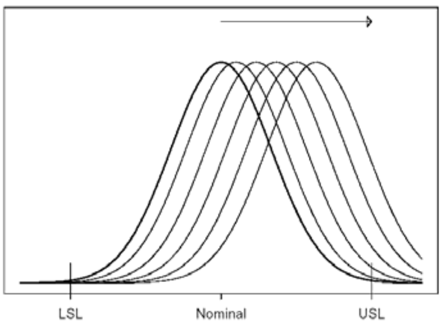

Figure 1.3. Various processes with the same value of C_p.