Chapter 1

Reflection and Refraction

1.1 Main Laws and Formulae

1.1.1 The laws of reflection and refraction

a. The directions of the incident ray R_1, the reflected ray R'_1 and the normal N to the surface AB (Fig. 1.1) are coplanar. The angle of incidence α_1 and the angle of reflection α'_1 are equal.

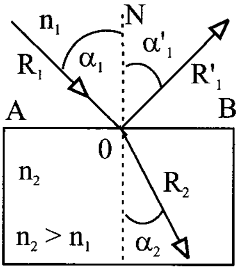


Fig. 1.1

b. The directions of the incident ray R_1, the refracted ray R_2 and the normal N (Fig 1.1) are coplanar. The angle of incidence α_1 and the angle of refraction α_2 are related by the law

$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2 \tag{1.1}$$

where n_1 and n_2 are called refractive indices of the two transparent, uniform and isotropic media separated by the plane surface AB.

If the medium above the surface *AB* is denser than that below it ($n_1 > n_2$), the angle of refraction α_2 will always be greater than the angle of incidence α_1. So there is a limit angle $\alpha_{1\,\max}$ that allows refraction given by

$$\alpha_{1\max} = \arcsin\left(\frac{n_2}{n_1}\right) \tag{1.2}$$

For $\alpha_1 > \alpha_{1\,\max}$ there will be only reflection.

1.1.2 Sign convention

a. Assuming a Cartesian coordinate system a light ray moves from left to right along the *z* axis in the plane *yz* (Fig. 1.2). The height of an object is positive if it is above the plane *xz*.

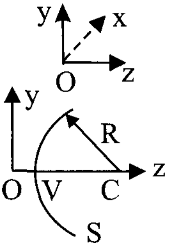


Fig.1.2

b. If *S* (Fig. 1.2) is a section in the plane *yz* of a spherical surface of center *C* and radius *R* the right line through the points *V* and *C* is assumed as *z* axis and called optic axis. More spherical surfaces having the same optic axis form a coaxial system.

c. The radius *R* is positive/negative if *S* presents its convex/concave surface to the incident ray.

d. We consider only rays which make small angles with *z* axis. These are called paraxial. With this assumption we can write

$$\sin\alpha = \alpha \quad \tan\alpha = \alpha \quad \cos\alpha = 1 \tag{1.3}$$

d. The angle between a ray and *z* axis is positive if its trigonometric tangent is positive, lengths being measured from the corner *C* where the right angle is situated (Fig. 1.3).

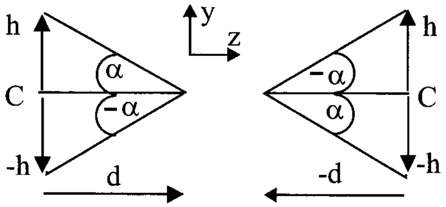


Fig. 1.3 Positive/negative values of h, d and α

1.1.3 Reflection and refraction on a spherical surface

The laws of reflection and refraction relate angles. Their application to spherical surfaces establishes a reciprocal relationship between distances *z* of an object and *z*' of an image from a spherical surface (Fig. 1.4). Two corresponding points of object and image are called conjugate.

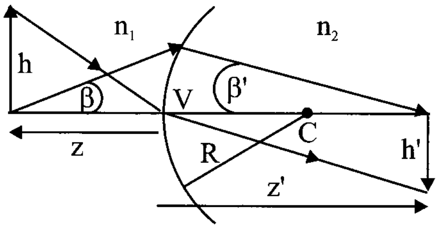


Fig. 1.4

For a refracting spherical surface z, z', h, h', β and β' are related by the following relations

$$\frac{n_2}{z'} = \frac{n_1}{z} + \frac{n_2 - n}{R} \tag{1.4.1}$$

$$M_T = \frac{h'}{h} = \frac{n_1 z'}{n_2 z} \tag{1.4.2}$$

$$M_\beta = \frac{\beta'}{\beta} = \frac{z}{z'} \tag{1.4.3}$$

where M_T and M_β are called transverse and angular magnification.

For a reflecting spherical surface (concave/convex mirrors) the previous relations become

$$\frac{1}{z'}=-\frac{1}{z}+\frac{2}{R} \tag{1.5.1}$$

$$M_T=-\frac{z'}{z} \tag{1.5.2}$$

$$M_\beta=-\frac{1}{M_T}=\frac{z}{z'} \tag{1.5.3}$$

For z approaching infinite from the (1.5.1) we have the focal length of spherical mirrors

$$f=\frac{R}{2} \tag{1.6}$$

where R is negative/positive for concave/convex mirrors.

An example of ray tracing for a concave spherical mirror is given in Fig. 1.5 where $R = OC$ and $f = OF = R/2$ are negative values. The rays *1'*, *2'* and *3'* are the corresponding reflected rays of the incident rays *1*, *2* and *3*.

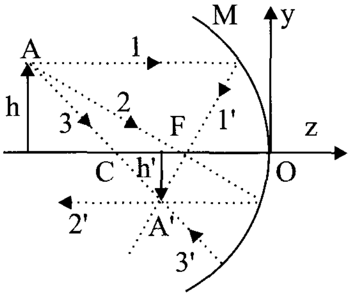


Fig. 1.5 Ray tracing for a spherical concave mirror

1.1.4 Matrix and vector form for light rays

The matrix treatment of light rays is useful when an optical system, assumed as a serial combination of many simple constituent, must be examined. If the optical elements have the individual matrices

$$[M_1], [M_2], [M_3], \ldots, [M_n] \tag{1.7}$$

the final ray emerging from the optical system is given by the matrix multiplication

$$[M] = [M_n] \ldots [M_3][M_2][M_1] \tag{1.8}$$

We consider now the ray matrices for the simplest optical components.

a. *A ray traveling through an homogeneous medium* (Fig. 1.6) can be set

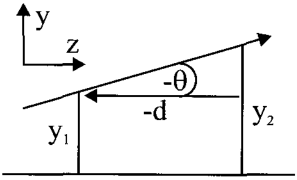


Fig. 1.6 A ray moves in a single homogeneous medium

forth in the two following statements

$$y_2 = y_1 + (-d) \times (-\theta) \tag{1.9.1}$$

$$\theta = \theta_1 = \theta_2 \tag{1.9.2}$$

that can also be easily obtained multiplying the matrix on the right side of (1.10) by the adjacent vector

$$\begin{vmatrix} y_2 \\ \theta \end{vmatrix} = \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ \theta \end{vmatrix} \tag{1.10}$$

b. *A ray reflected by a plane mirror* described by the following statements (Fig. 1.7)

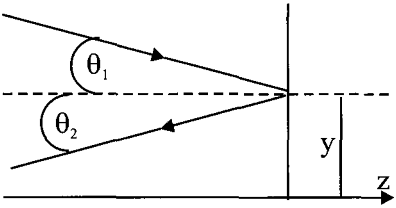


Fig. 1.7 A ray reflected by a plane mirror

$$y = y_1 = y_2 \quad \theta = \theta_2 = -\theta_1 \tag{1.11}$$

can assume the following matrix form

$$\begin{vmatrix} y \\ \theta_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} y \\ -\theta_1 \end{vmatrix} \tag{1.12}$$

c. For *a refracted ray from a plane surface*, the statements (Fig. 1.8)

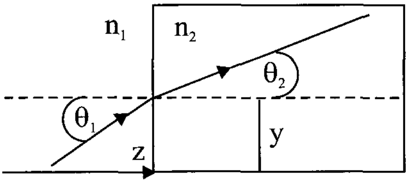


Fig. 1.8 A ray refracted by a plane surface

$$y = y_1 = y_2 \qquad \theta_2 = \frac{n_1}{n_2}\theta_1 \tag{1.13}$$

become

$$\begin{vmatrix} y \\ \theta_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{vmatrix} \begin{vmatrix} y \\ \theta_1 \end{vmatrix} \tag{1.14}$$

d. For *a ray reflected by a spherical surface* the statements (Fig. 1.9)

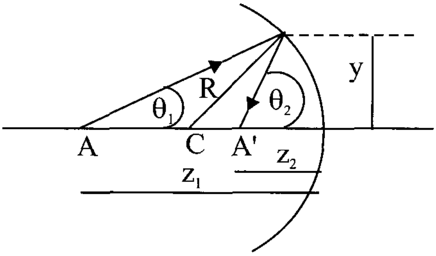


Fig. 1.9 A ray reflected by a spherical surface

$$y = y_1 = y_2 \tag{1.15}$$

$$\frac{1}{z_2} = -\frac{1}{z_1} + \frac{1}{f} \quad \Rightarrow \quad \frac{\theta_2}{y} = -\frac{\theta_1}{y} + \frac{1}{f} \tag{1.16}$$

become

$$\begin{vmatrix} y \\ \theta_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{f} & -1 \end{vmatrix} \begin{vmatrix} y \\ \theta_1 \end{vmatrix} \tag{1.17}$$

e. For *a ray refracted by a spherical surface* (Fig. 1.10) we have

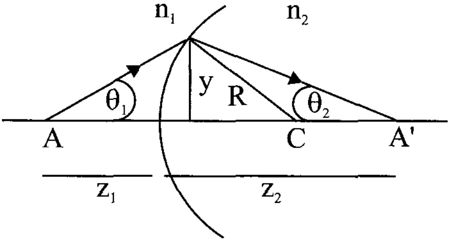


Fig. 1.10 A ray refracted by a spherical surface

$$y = y_1 = y_2 \tag{1.18}$$

$$\frac{n_2}{z_2} = \frac{n_1}{z_1} + \frac{n_2 - n_1}{R} \tag{1.19.1}$$

$$\frac{n_2\theta_2}{y} = \frac{n_1\theta_1}{y} + \frac{n_2 - n_1}{R} \tag{1.19.2}$$

$$\theta_2 = (1 - \frac{n_1}{n_2})\frac{1}{R}y + \frac{n_1}{n_2}\theta_1 \tag{1.19.3}$$

and the corresponding matrix form is

$$\begin{vmatrix} y \\ \theta_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ (1 - \frac{n_1}{n_2})\frac{1}{R} & \frac{n_1}{n_2} \end{vmatrix} \begin{vmatrix} y \\ \theta_1 \end{vmatrix} \tag{1.20}$$

1.1.5 Reflection and refraction as electromagnetic waves propagation

A light ray is really a propagation of oscillating electric and magnetic fields (Fig. 1.11) with the electric field ***E*** perpendicular to the magnetic field ***H*** and both perpendicular to the direction of propagation, defined by the wave vector ***k*** having the magnitude *2π/λ*. A light ray conveys energy that is usually considered per unit time and per unit surface and called intensity or, in the jargon of the Fourier transform, power spectrum or spectral power density. As an example, when a light ray gives rise to reflection and refraction, portion of its intensity is carried by the reflected ray and the remainder by the refracted ray assuming that there is not absorption of energy by the second medium.

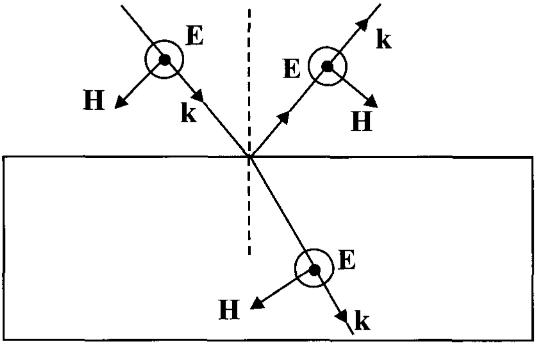


Fig. 1.11 At a fixed time the field H in the plane of the figure and E normal to the plane and directed upwards

The intensity is proportional to the square of the magnitude of the electric field by means of the constant A (see Appendix 1)

$$I = A \cdot E^2 \tag{1.21}$$

Considering a light ray as propagation of electric and magnetic fields, the basic laws of reflection and refraction can be derived from the Fresnel equations under the following assumptions:

a. *the media are homogeneous and isotropic*

b. *the waves are plane and harmonic*

c. *the electric and magnetic fields are continuous* at the boundary surface of the two media.

If the direction of electric field $\boldsymbol{E}_1$ of the incident ray makes an angle different from 90° with the plane of figure (plane of incidence) the magnitude of its components will be E_{1w} in this plane and E_{1y} normal to this plane. The corresponding magnitudes of the components of the electric field for the reflected (E'_{1w}, E'_{ly}) and refracted (E_{2w}, E_{2y}) rays are given by the following Fresnel equations

$$E'_{1w} = E_{1w} \frac{\tan(\alpha_1 - \alpha_2)}{\tan(\alpha_1 + \alpha_2)} \tag{1.22.1}$$

$$E'_{1y} = -E_{1y} \frac{\sin(\alpha_1 - \alpha_2)}{\sin(\alpha_1 + \alpha_2)} \tag{1.22.2}$$

$$E_{2w} = E_{1w} \frac{2 \sin\alpha_2 \cos\alpha_1}{\sin(\alpha_1 + \alpha_2)\cos(\alpha_1 - \alpha_2)} \tag{1.22.3}$$

$$E_{2y} = E_{1y} \frac{2 \sin\alpha_2 \cos\alpha_1}{\sin(\alpha_1 + \alpha_2)} \tag{1.22.4}$$

with α_1 the angle of incidence and α_2 the angle of refraction, related by the Snell's law (1.1).

1.1.6 Harmonic waves. Principle of superposition. Complex notation

The simplest wave model having a sinusoidal shape is called harmonic (Fig. 1.12). For a progressive harmonic wave the oscillations along the y

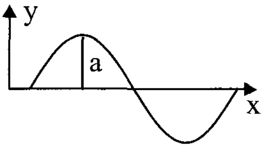


Fig. 1.12

axis, function of x and t, are given by

$$y = a\sin(kx-\omega t) \tag{1.23}$$

where

$$k=\frac{2\pi}{\lambda} \quad \omega=\frac{2\pi}{T}=2\pi\nu \quad v=\frac{\omega}{k} \tag{1.24}$$

with k the magnitude of the wave vector, λ the wavelength, ω the angular frequency, T the period, ν the frequency and v the velocity of propagation.

The varying values of y in (1.23) represent an electric or a magnetic field. Because the Maxwell's equations for vacuum are linear differential equations, if at a point P in empty space different sources produce the (electric, for example) fields

$$\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots, \vec{F}_n \tag{1.25}$$

the resulting field in P is equal to the vector sum

$$\vec{F}=\vec{F}_1+\vec{F}_2+\vec{F}_3, \ldots, +\vec{F}_n \tag{1.26}$$

The optical interference, for exsample, is based on this principle of linear superposition of electromagnetic fields.

Therefore, for example, if at a point P and time t there is a superposition of a progressive wave

$$\mathrm{y}_1=a\sin(kx-\omega t) \tag{1.27}$$

and a regressive one

$$y_2 = a\sin(kx+\omega t) \tag{1.28}$$

the resultant wave in *P* at a time *t* will be

$$\begin{aligned} y_P &= y_1 + y_2 = \\ &= a\sin(kx-\omega t) + a\sin(kx+\omega t) = \\ &= 2\mathrm{a}\cos\omega t \sin kx \end{aligned} \tag{1.29}$$

and the corresponding intensity at time *t* will be (*A* is a constant)

$$I_P = A \cdot 4a^2 \cos^2 \omega t \tag{1.30}$$

The results (1.29) and (1.30) are usually obtained representing the harmonic wave with complex numbers. Remembering the Euler formula

$$e^{i\varphi} = \cos\varphi + i\,sin\varphi$$

we can write

$$\begin{aligned} y_P = y_1 + y_2 &= a(e^{i(kx-\omega t)} + e^{i(kx+\omega t)}) = \\ &= e^{ikx}(e^{-i\omega t} + e^{i\omega t}) = \end{aligned} \tag{1.31.1}$$

$$\begin{aligned} &= a(\cos kx + i\sin kx) 2\cos\omega t = \\ &= 2a\cos kx\cos\omega t + i\,2a\sin kx\cos\omega t \end{aligned} \tag{1.31.2}$$

The result (1.29) appears here as the coefficient of the imaginary part of the complex number (1.31.2).

In a straightforward way the intensity is determined using complex notation for the waves. In fact the main purpose in optics in not to ascertain the resulting waveform (1.29) from superposition of the component waves in a point *P* but the whole energy or intensity (1.30) in this point. Multiplying the complex value (1.31.1)

$$y_P = e^{ikx}(e^{-i\omega t} + e^{i\omega t})$$

by its complex conjugate

$$y^*_P = e^{-ikx}(e^{+i\omega t} + e^{-i\omega t})$$

we have directly the intensity (1.30), omitting the constant *A*)

$$I = y_P\, y^*_P = a e^{ikx}(e^{-i\omega t} + e^{i\omega t})\, a e^{-ikx}(e^{+i\omega t} + e^{-i\omega t})$$

$$a^2(1 + e^{-2i\omega t} + e^{+2i\omega t} + 1) = a^2(1 + 2\cos 2\omega t + 1) =$$

$$= 2a^2(1 + \cos 2\omega t) = 4a^2 \cos^2 \omega t$$

The resulting waveform (1.29) represents (Fig. 1.13) a stationary wave.

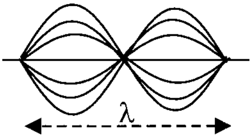


Fig. 1.13

Its profile expands and shrinks vertically but does not move forwards or backwards. In a stationary wave the energy is not transferred in either direction. There is a power density localized in the space but not uniformly and constantly distributed.

1.2 Problems

1.2.1 A simple concave mirror

An object y has a distance z from the vertex O of a concave spherical mirror having focus F and center C (Fig. 1.14). Use ray tracing to find images corresponding to the six situations of Fig. 1.14. With the focal distance $f = -15$ mm determine the image distance z' from O and the magnification m_T varying z, with a step of 5mm, in the interval (-35, 10) mm. Give the plots of z' and m_T varying z.

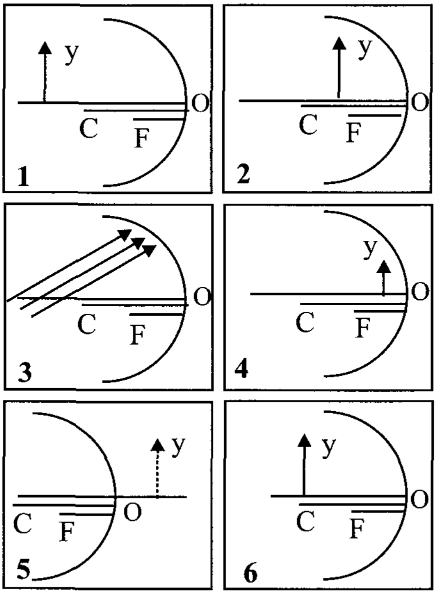
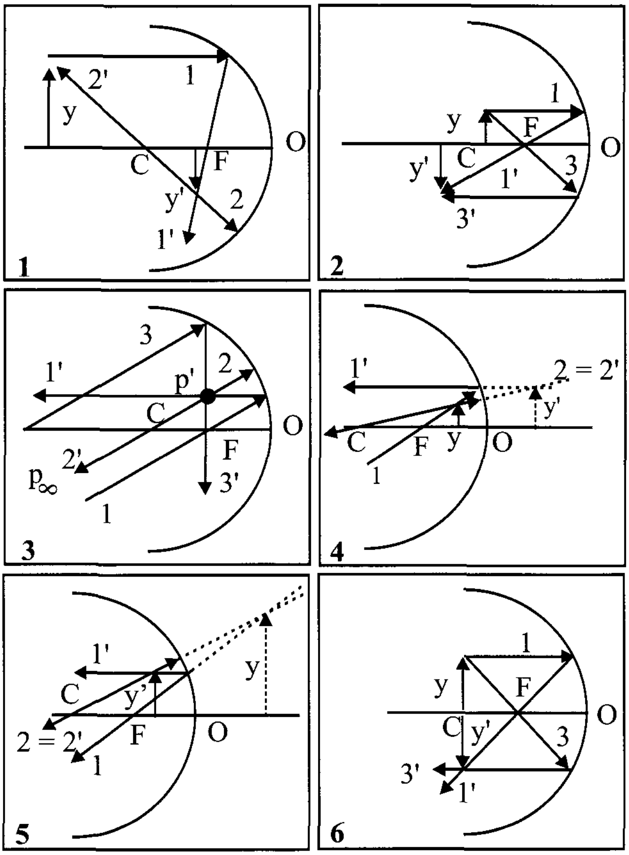


Fig. 1.14

Solution:

The ray tracing is given in Fig. 1.15. The table 1.1 and Fig. 1.16 give the values of z' and m_T and the corresponding plots.


2'
1
y
O
C
F
y'
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1'
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1
y
F
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y'
C
3
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Fig. 1.15

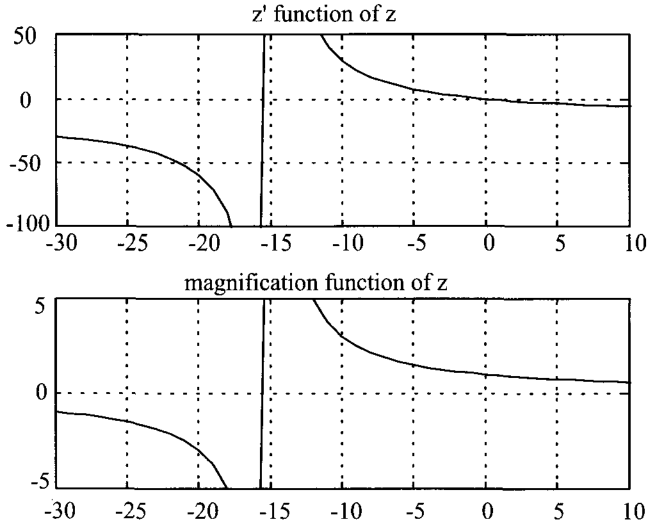


Fig. 1.16

Table 1.1

z (mm)	z' (mm)	m_T
-35	-26.25	-0.75
-30	-30	-1
-25	-37.51	-1.5
-20	-60	-3
-15	∞	∞
-10	30	3
-5	7.5	1.5
0	0	0/0
5	-3.75	0.75
10	-6	0.6

1.2.2 A simple convex mirror

An object y has a distance z from the vertex O of a convex spherical mirror having focus F and center C (Fig. 1.17). Find with ray tracing the images corresponding to the four situations of Fig. 1.17. With the focal length $f = 15$ mm determine the image distance z' from O and the magnification m_T varying z, with a step of 5 mm, in the interval (-30, 30) mm. Give the corresponding plots.

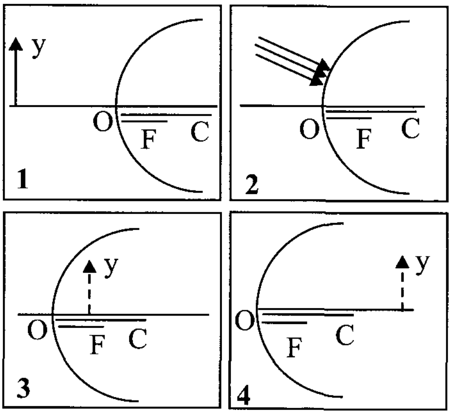


Fig. 1.17.

Solution:

The ray tracing is given in Fig. 1.18. The table 1.2 and Fig. 1.19 contain the values z' and m_T and the corresponding plots.

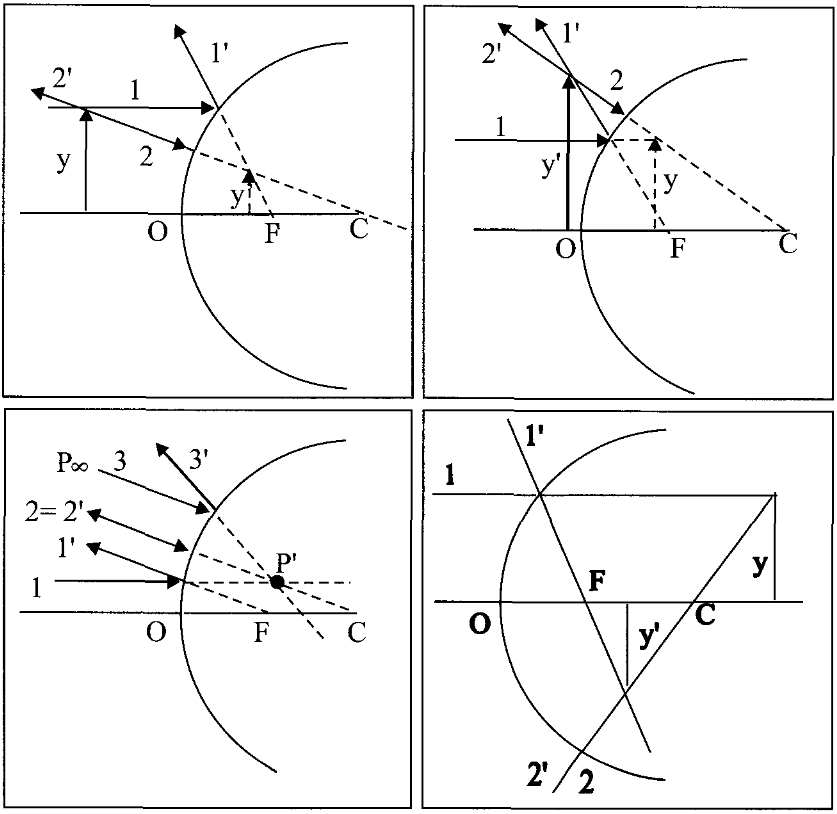


Fig.1.18

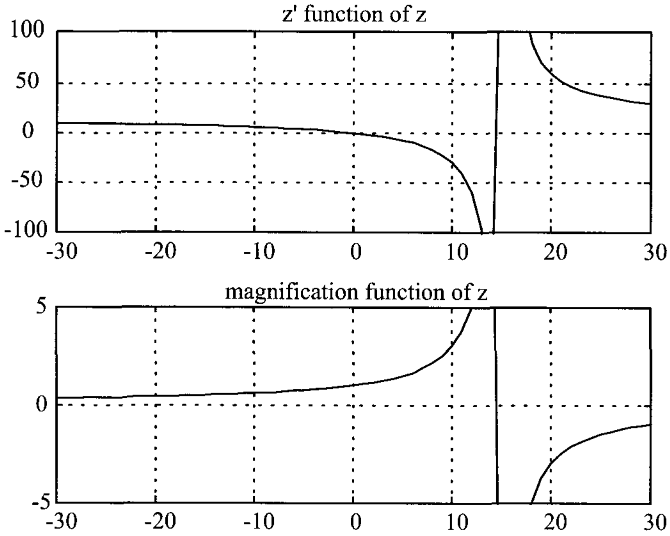


Fig.1.19

Table 1.2

z (mm)	z' (mm)	m_T
-30	10	0.33
-25	9.38	0.38
-20	8.57	0.43
-15	7.50	0.50
-10	6.00	0.60
-5	3.75	0.75
0	0	0/0
5	-7.50	1.50
10	-30	3.00
15	$-\infty$	$-\infty$
20	60.00	-3.00
25	37.50	-1.50
30	30	-1.00

1.2.3 A refracting surface

A box (Fig. 1.20) is filled with water ($n = 4/3$). An object whose height is $y = 6$ m is placed at a distance $z = -10$ m from the point O of the plane

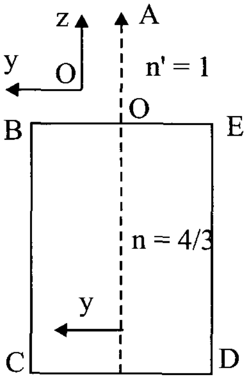


Fig.1.20

boundary between water and air ($n' = 1$).

Determine the position z' and the height y' of the image seen by an observer located in A. Give the corresponding ray tracing.

Solution:

For a refracting spherical surface we have

$$\frac{n'}{z'}=\frac{n}{z}+\frac{n'-n}{R} \qquad M_T=\frac{nz'}{n'z}$$

If the refracting surface is plane (R is infinite) the previous become

$$\frac{n'}{z'}=\frac{n}{z} \qquad z'=\frac{n'z}{n}=\frac{1\times(-10)}{4/3}=-7.5\text{m}$$

$$M_T=\frac{nz'}{n'z} \quad \longrightarrow \quad \frac{y'}{y}=\frac{nz'}{n'z} \qquad y'=\frac{nz'}{n'z}y=\frac{n}{n'z}z'y=\frac{1}{z'}z'y=y=6\text{m}$$

The observer perceives the object in the water at a distance $z' = -7.5$ m but having the real height.

Rotating of 90° clockwise the box the ray tracing is simpler (Fig. 1.21).

1.2.4 Prism 1

A ray of monochromatic light is incident at A (Fig. 1.22) with an angle β on a prism whose refractive index is n. Let α be the angle at the edge B.

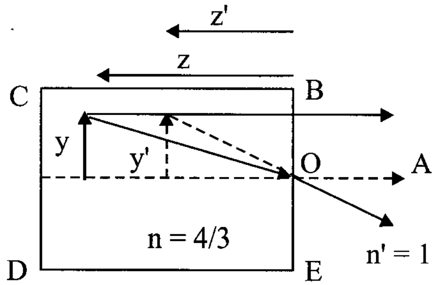


Fig. 1.21

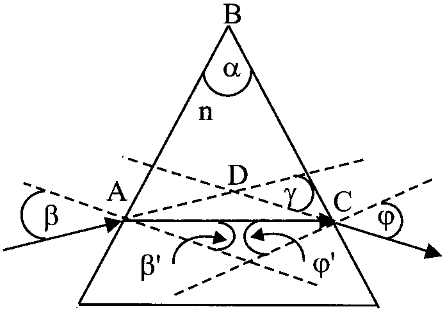


Fig. 1.22

Prove that the angle of deviation between incident beam on A and the emerging beam from C is $\gamma = (n - 1)\alpha$ if α is small.

Solution:

Using the law of refraction in A and in C

$$\sin\beta = n\sin\beta' \tag{1.32}$$

$$\sin\varphi = n\sin\varphi' \tag{1.33}$$

From triangle ABC

$$\beta'+\varphi'=\alpha \tag{1.34}$$

If α is small, it follows from (1.34) that also β' and φ' are small. Hence from (1.32) and (1.33) also β and φ are small. Then

$$\beta = n\beta' \qquad \varphi = n\varphi'$$

From triangle ADC

$$\gamma = (\beta - \beta') + (\varphi - \varphi') = (\beta + \varphi) - (\beta' + \varphi') \tag{1.35}$$

Using the previous formulae, (1.35) becomes

$$\gamma = n\beta' + n\varphi' - (\beta' + \varphi') = (n-1)(\beta' + \varphi') = (n-1)\alpha$$

1.2.5 A reflecting plate

A parallel beam of monochromatic light is incident with an angle θ_1 on the surface AC (Fig. 1.23) of a box of a transparent and homogeneous material. Find the minimum value of refractive index n which allows total internal reflection on the surfaces AB and CD for every value of θ_1.

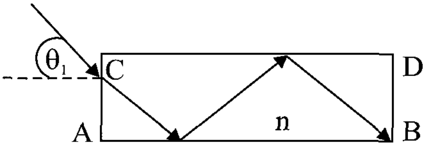


Fig 1.23

Solution:

On the separation surface AC we have

$$\sin\theta_1 = n\sin\theta_2 \tag{1.36}$$

On lateral surfaces must be (Fig. 1.24)

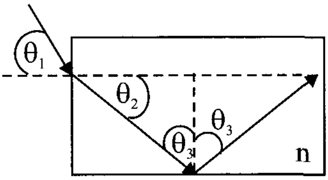


Fig 1.24

$$n\sin\theta_3 = 1 \qquad \theta_3 = \arcsin(\frac{1}{n}) \tag{1.37}$$

which define the minimum θ_3 that allows total internal reflection. From geometry, (1.36) becomes

$$\sin\theta_1 = n\sin(90° - \theta_3) \qquad \sin\theta_1 = n\cos\theta_3$$

Hence (Fig. 1.25)

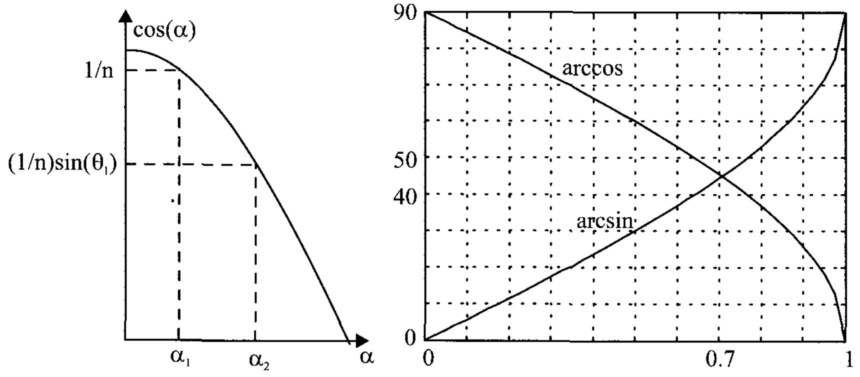


Fig.1.25

$$\theta_3 = \arccos(\frac{1}{n}\sin\theta_1) \geq \arccos(\frac{1}{n}) \tag{1.38}$$

Using (1.37) and (1.38)

$$\theta_3 = arcsin(\frac{1}{n}) = \arccos(\frac{1}{n} sin\,\theta_1) \geq \arccos(\frac{1}{n})$$

Because (Fig. 1.25)

$$\arcsin(\frac{1}{n}) \geq \arccos(\frac{1}{n})$$

only if sin(α) is greater than 0.7071, it follows

$$\frac{1}{n} \geq 0.7071 \qquad\qquad n \leq 1.4142 \tag{1.39}$$

1.2.6 Prism 2

A ray of monochromatic light is incident at right angle on the face AB (Fig. 1.26) of a prism having a vertex angle α and a refractive index n.

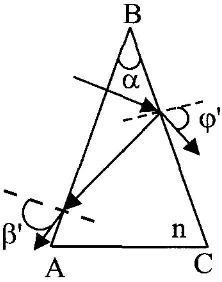


Fig. 1.26

The beam is partially reflected on the face BC and refracted with an angle $\varphi' = 7.26°$. The reflected ray is refracted again on the face AB with an angle $\beta' = 14.58°$.

Find values of α and n.

Solution:

The angles φ and α are equal, as both complementary of γ (Fig. 1.27)

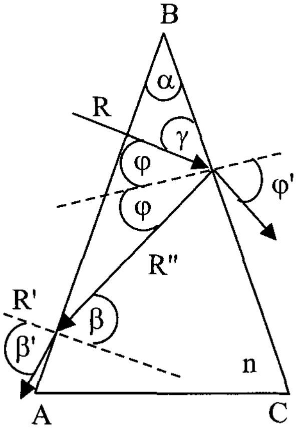


Fig. 1.27

$$\varphi = \alpha \tag{1.40}$$

From geometry

$$\beta = 2\varphi \tag{1.41}$$

From the law of refraction on the faces CB and AB

$$n \sin\varphi = \sin\varphi' \tag{1.42}$$

$$n \sin\beta = \sin\beta' \tag{1.43}$$

Using (1.40) and (1.41), (1.42) and (1.43) become

$$n \sin\alpha = \sin\varphi' \qquad n \sin 2\alpha = \sin\beta'$$

and dividing

$$\frac{n \sin 2\alpha}{n \sin\alpha} = \frac{\sin\beta'}{\sin\varphi'} \qquad 2\cos\alpha = \frac{\sin\beta'}{\sin\varphi'}$$

$$\alpha = \arccos\left(\frac{\sin\beta'}{2\sin\varphi'}\right) = \arccos(0.996) = 5.1^\circ$$

Hence from (1.42) using (1.40) we have

$$n = \frac{\sin\varphi'}{\sin\varphi} = \frac{\sin\varphi'}{\sin\alpha} = 1.42$$

1.2.7 A refracting plate

A ray of monochromatic light is incident, with an angle α, on a plate (Fig. 1.28) having a constant thickness s = 9 mm and a refractive index

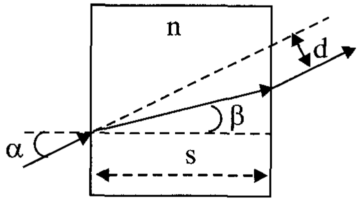


Fig.1.28

$n = 1.56$. Compute for the maximum value of α allowing the transmission of the light through the plate, the corresponding β_{max} and d_{max}. Are β_{max} and/or d_{max} functions of n?

Solution:

On the first face is (Fig. 1.29)

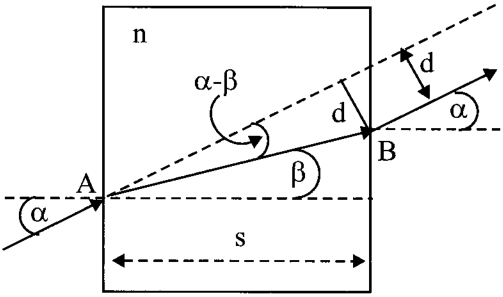


Fig. 1.29

$$\sin\alpha = n\sin\beta \tag{1.44}$$

By geometry

$$s = AB\cos\beta \qquad AB = \frac{s}{\cos\beta} \tag{1.45}$$

and

$$d = AB\sin(\alpha-\beta) \tag{1.46}$$

Substituting from (1.45) AB into (1.46)

$$d = \frac{s}{\cos\beta}\sin(\alpha-\beta) \tag{1.47}$$

For $\alpha = \alpha_{max} = 90°$ (1.47) becomes

$$d_{max} = \frac{s}{\cos\beta}\cos\beta = s = 9\,\text{mm}$$

The value of $d_{\max}$ is not dependent from n.
The corresponding maximum value of β (Fig. 1.30) is

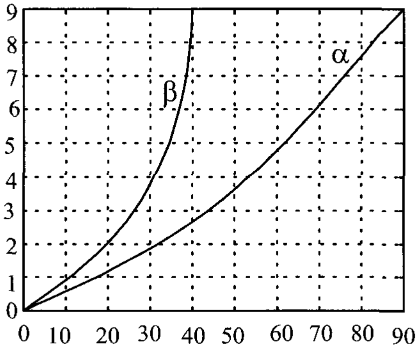


Fig.1.30

$$\beta_{\max} = \arcsin\left(\frac{1}{n}\right) = 39.9°$$

which is dependent from n.

1.2.8 Prism 3

A ray of monochromatic light is incident (Fig. 1.31) horizontally on a prism, whose section is an equilateral triangle, having a refractive index n = 1.5. The ray emerging from the prism must be reflected by a plane vertical mirror that must be rotated by an angle δ so that the beam has to return back horizontally. Calculate δ.

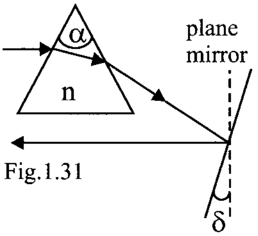


Fig.1.31

Solution:

The plane section of the prism is an equilateral triangle, hence α = 60°. By law of refraction (Fig. 1.32)

$$\sin\beta = n\sin\beta' \tag{1.48}$$

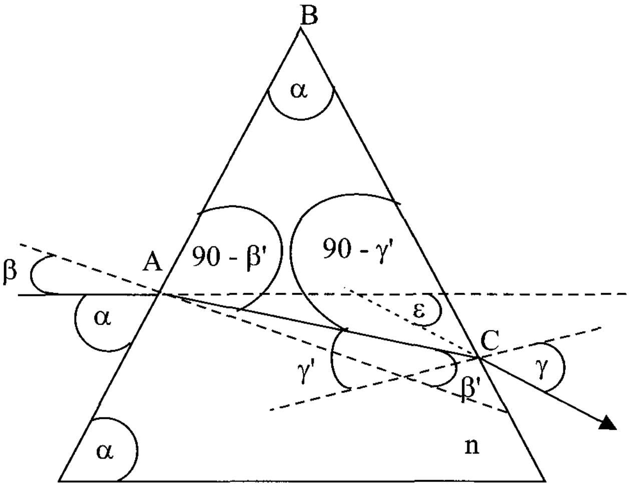


Fig. 1.32

By geometry

$$\beta = 90° - \alpha = 30° \tag{1.49}$$

Using (1.48) and (1.49 we have)

$$\beta' = \arcsin(\frac{\sin\beta}{n}) = 19.5°$$

In addition by geometry

$$\alpha = \beta' + \gamma' \qquad \gamma' = \alpha - \beta' = 40.5°$$

and from the law of refraction

$$n\sin\gamma' = \sin\gamma$$

Hence

$$\gamma = \arcsin(n\sin\gamma') = 77.0°$$

Once more by geometry for the angle of deviation ε

$$\varepsilon=(\beta-\beta')+(\gamma-\gamma')=\beta+\gamma-\alpha=47.0°$$

From Fig. 1.33 follows

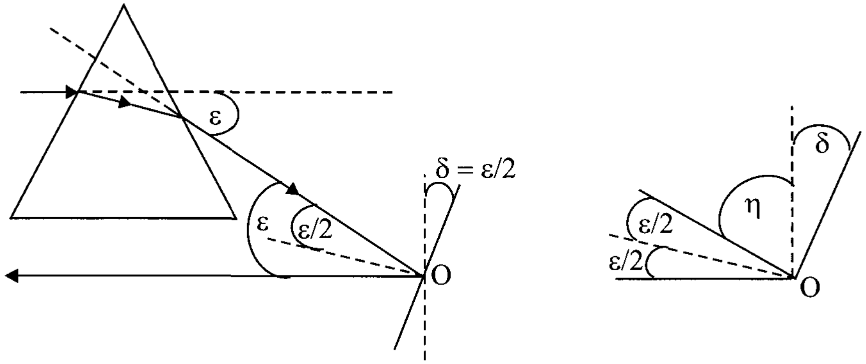


Fig. 1.33

$$\delta=\varepsilon/2=23.5°$$

1.2.9 A hemisphere

A spherical concave mirror, whose section is a semicircle of radius R = OC = 30 cm, is filled with water (n = 1.33).A ray of monochromatic light is incident, with an angle α = 4°, on the superior surface of water. Fixing an xyz system of coordinates with origin in O (Fig. 1.34) determine the position (x, y, z) of the image.

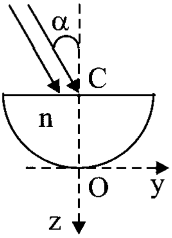


Fig.1.34

Solution:

From the law of refraction (Fig. 1.35) we obtain

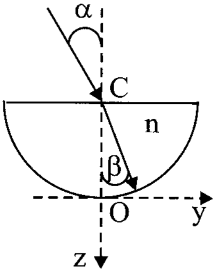


Fig. 1.35

$$\beta = \arcsin(\frac{\sin\alpha}{n}) = 0.0698 = 3.01°$$

Fig. 1.36 shows a ray (r_1) crossing the focus F and a ray (r_2) the center C

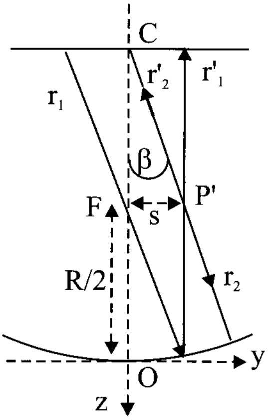


Fig.1.36

of the mirror with corresponding reflected rays (r'_1 and $r'_2 = r_2$). From geometry follows

$$s = \frac{R}{2}\tan\beta = 0.8\,\text{cm}$$

The coordinates of the point image P' are $x = 0$, $y = s = 0.8$cm and $z = -R/2 = -15$ cm

1.2.10 Plane and spherical surfaces

A spherical concave mirror, having radius R = -30 cm, is filled with a solid, homogeneous and transparent material M of refractive index $n' = 1.55$ (Fig. 1.37). Assuming $d = -5$ cm, find the position z' of the image of an object having distance, from external and vertical face of M, $z_1 = -100$ cm, the object lateral magnification and the focus of the spherical concave mirror filled with M.

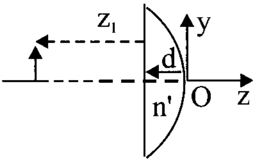


Fig. 1.37

Solution:

The optical system is compounded of a plane refracting surface and a mirror.

At the plane refracting surface we have (Fig. 1.38) for the position z'_1

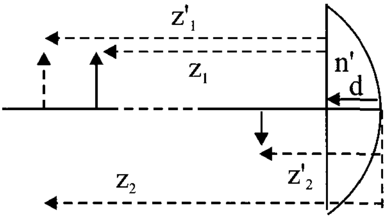


Fig.1.38

$$\frac{n'}{z'_1} = \frac{n}{z_1} \qquad z'_1 = \frac{n'}{n} z_1 = -155\,\text{cm}$$

Fig. 1.39 gives the related ray tracing. The virtual image y' becomes an object at distance

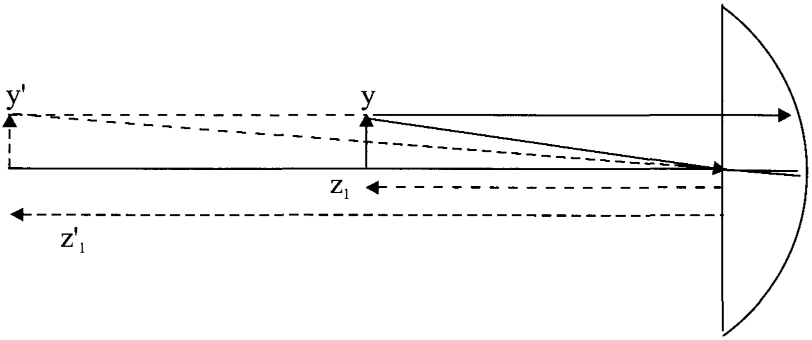


Fig. 1.39

$$z_2 = z'_1 + d = -160\,\text{cm}$$

from the mirror.

The corresponding distance of the image is

$$z'_2 = \frac{f z_2}{-f + z_2} = -16.6\,\text{cm}$$

with $f = R/2$.

The magnification is

$$m_T = -\frac{z'_2}{z_2} = -0.1$$

The final real image is reversed (Fig. 1.38) and its height is one tenth of that of the real object.

The focus of the spherical concave mirror filled with M is

$$f_{\text{new}} = \frac{z z'}{z + z'} = -14.2\,\text{cm}$$

assuming $z = z_1 = -100\text{cm}$ and $z' = z'_2 = -16.6$ cm.

The focus f_{new} is dependent on d (Fig. 1.40); on the contrary the focus of a simple spherical mirror

$$f = \frac{R}{2}$$

varies only with R.

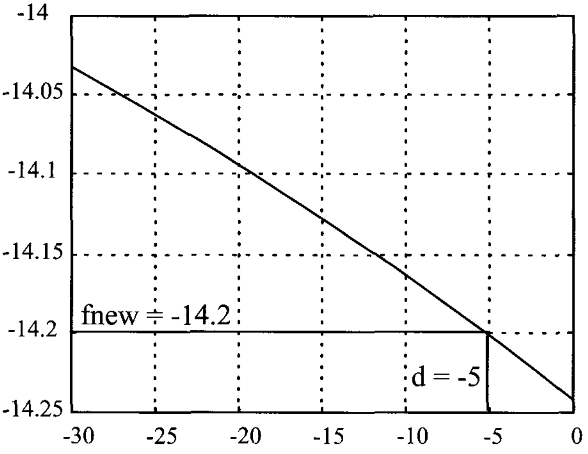


Fig. 1.40 f_{new} varying with d (cm)

1.2.11 Prism 4

A beam of light $R_{1\text{-}2}$, consisting of two wavelengths λ_1 and λ_2, is incident on a prism with apex angle $\alpha = 60°$. The refractive indexes for λ_1 and λ_2 are respectively $n_1 = 1.618$ and $n_2 = 1.652$. Assume that the prism is in position of minimum deviation for the ray R_1 corresponding to the light of wavelength λ_1 (Fig. 1.41). Calculate the angular deviations γ_1 e γ_2 of the rays R_1 and R_2 from the original direction of ray $R_{1\text{-}2}$. Assume that a

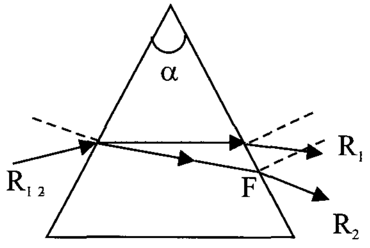


Fig. 1.41

convergent thin lens (f = 15 cm) has its optical axis coincident with R_2.and the front focus on point F where the ray R_2 emerges from prism. Find the distance z' from the lens of the point image of R_1 and its height y' from the optical axis.

Solution:

The position of minimum deviation for R_1 (Fig. 1.42) needs

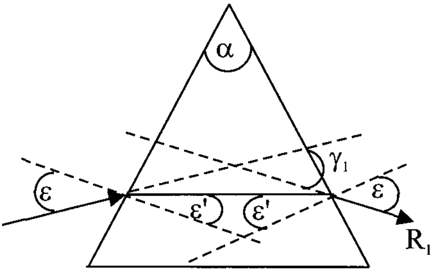


Fig.. 1.42

$$\varepsilon' = \frac{\alpha}{2} = 30°$$

$$\sin\varepsilon = n_1 \sin 30° \qquad \varepsilon = \arcsin(\frac{1.618}{2}) = 54.0°$$

$$\gamma_1 = 2\varepsilon - \alpha = 48.0°$$

For R_2 we have (Fig. 1.43),

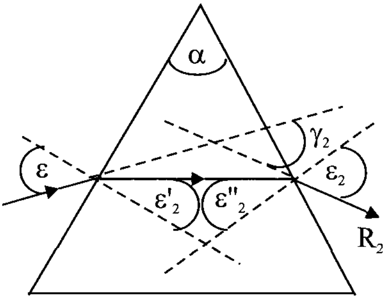


Fig. 1.43

$$\sin\varepsilon = n_2 \sin\varepsilon'_2 \qquad \varepsilon'_2 = \arcsin(\frac{\sin\varepsilon}{n_2}) = 29.3°$$

From the geometry (Fig. 1.43)

$$\varepsilon'_2 + \varepsilon''_2 = \alpha \qquad \varepsilon''_2 = 60 - 29.3 = 30.7°$$

$$n_2 \sin \varepsilon''_2 = \sin \varepsilon_2 \qquad \varepsilon_2 = 57.5°$$

and

$$\gamma_2 = \varepsilon + \varepsilon_2 - \alpha = 54.0 + 57.5 - 60 = 51.5°$$

From the geometry (Fig. 1.44)

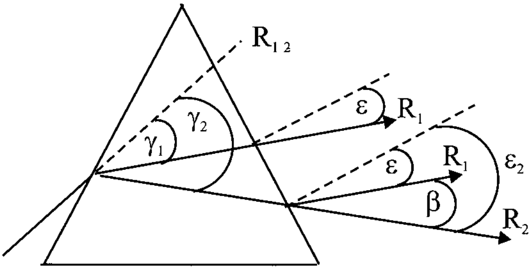


Fig. 1.44

$$\beta = \varepsilon_2 - \varepsilon = 57.5 - 54.0 = 3.5°$$

or

$$\beta = \gamma_2 - \gamma_1 = 51.5 - 48.0 = 3.5°$$

Fig. 1.45 gives $z' = f = 15$ cm and

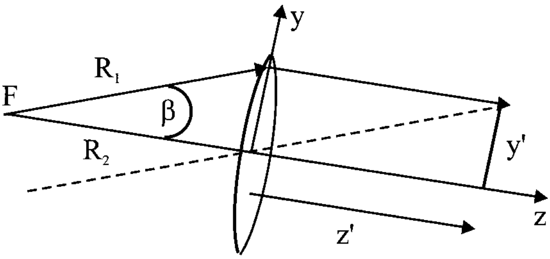


Fig. 1.45

$$y' = 15 \cdot \tan \beta = 0.92\,\text{cm}$$

1.2.12 A sphere

A ray of monochromatic light is incident with angle α (Fig. 1.46) on a transparent sphere, having a refractive index $n = 1.66$. There is on B a partial or total reflection? Find the angular deviation δ, between the ray incident on *A* and the ray emerging from D, as a function of α. Define the value $\alpha_{\min}$ for which the function $\delta(\alpha)$ has a minimum. Determine the values of β and δ when $\alpha = \alpha_{\min}$.

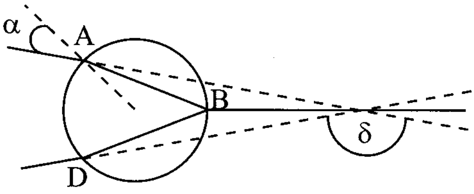


Fig.1.46

Solution:

The angle of incidence on B is equal to the angle of refraction on A as the triangle ACB is isosceles (Fig. 1.47).

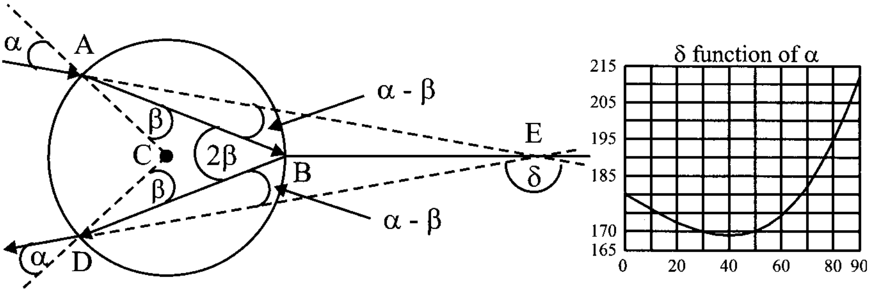


Fig.1.47

From the law of refraction on A we have

$$\beta = \arcsin\left(\frac{1}{n}\sin\alpha\right) \tag{1.50}$$

The angle of total reflection would be

$$\beta^* = \arcsin(\frac{1}{n}) \tag{1.51}$$

Because from (1.50) and (1.51)

$$\frac{1}{n}\sin\alpha < \frac{1}{n}$$

and

$$\beta < \beta^*$$

it follows that there is only partial reflection on B.

The angular deviations on A, B and C are respectively α -β, π -2β and α -β and their sum is

$$\delta = \pi + 2\alpha - 4\beta \tag{1.52}$$

We find the critical value of δ, using (1.52), as a function of α

$$\frac{d\delta}{d\alpha} = \frac{d}{d\alpha}(\pi + 2\alpha - 4\beta) =$$

$$= 2 - 4\frac{d\beta}{d\alpha} = 2 - 4\frac{d}{d\alpha}(\arcsin(\frac{\sin\alpha}{n})) =$$

$$= 2 - 4(\frac{1}{\sqrt{1 - \frac{\sin^2\alpha}{n^2}}}\frac{\cos\alpha}{n}) = 2 - 4(\frac{\cos\alpha}{\sqrt{n^2 - \sin^2\alpha}}) = 0 \tag{1.53}$$

From (1.53) follows

$$\frac{\cos\alpha}{\sqrt{n^2 - \sin^2\alpha}} = \frac{1}{2} \qquad 4\cos^2\alpha = n^2 - \sin^2\alpha$$

$$4 - 4\sin^2\alpha = n^2 - \sin^2\alpha \qquad \sin\alpha = \sqrt{\frac{4 - n^2}{3}}$$

Hence

$$\alpha^{*}=\arcsin\sqrt{\frac{4-n^{2}}{3}}=40.1^{\circ}$$

The value of the second derivative of δ is

$$\frac{d^{2}\delta}{d\alpha^{2}}(\alpha^{*})=1.357$$

Then $\alpha_{\min} = \alpha^*$ and from (1.50) follows $\beta = 22.8^\circ$ and from (1.52) $\delta = 168.9^\circ$.

1.2.13 A moving mirror

A plane mirror M (Fig. 1.48) rotates clockwise around an axis normal in O the plane yz of the figure. A source (having a fixed position in S of coordinates $y = 30$ cm, $z = -5$ cm) emits a beam of light that is reflected on O and impinges on the point S' of a screen AB distant $h = 100$ cm from M. Calculate the coordinates of the real image S' and of the corresponding virtual image S_0 below the mirror.

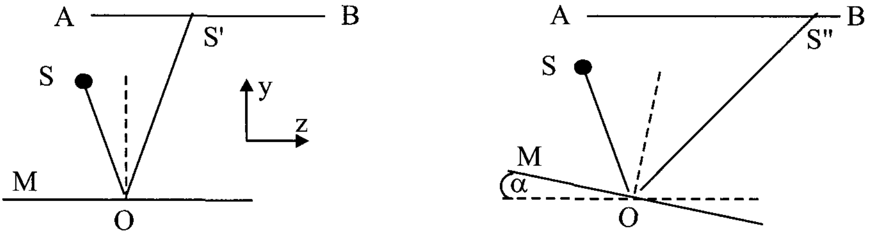


Fig. 1.48

After the rotation of the mirror M of an angle $\alpha = 2^\circ$, without moving the source S and the screen AB calculate the new coordinates of the real image S'' and of the corresponding virtual image S_2 below the mirror. Determine the equation of the plane figure passing through the three points S, S_0 and S_2.

Solution:

From coordinates of S (Fig. 1.49) we have for β, without regard to its sign

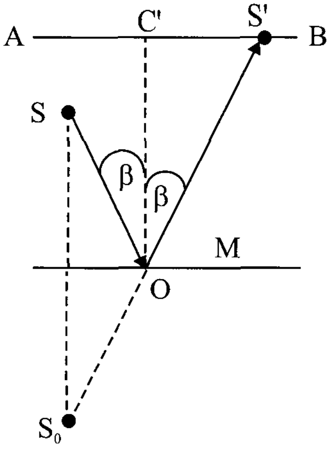


Fig.1.49

$$\beta = \arctan\frac{5}{30} = 9.46°$$

The coordinates of S' are (0, 100, 16.7) cm, because

$$C'S' = h\tan\beta = 100\cdot\tan(0.1651) = 16.7\text{cm}$$

From geometry (Fig. 1.50)

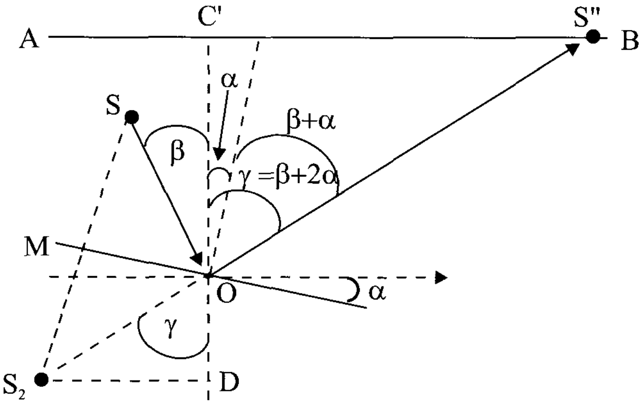


Fig. 1.50

$$\gamma = \beta + 2\alpha = 13.46°$$

hence S" has coordinates (0, 100, 23.9) cm because

$$C'S''=h\tan\gamma=100\cdot\tan(0.2350)=23.9\,\mathrm{cm}$$

The coordinates of virtual image S_0 are (0,-30,-5)
Hence R = SO = S_0O

$$R=\sqrt{30^2+5^2}=30.4\mathrm{cm}$$

After rotation there is R = OS_2 = OS. The positions S, S_0 and S_2 are points of a circle having radius R, center in O and equation

$$z^2+y^2=R^2=924.16\,\mathrm{cm}^2$$

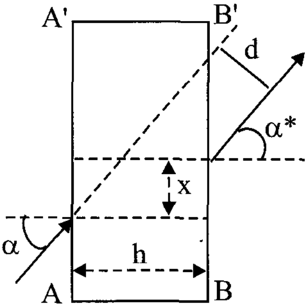
The coordinates of S_2, are (0, -29.6, -7.1) cm because

$$DS_2=R\sin\gamma=30.4\cdot\sin(0.2350)=7.1\mathrm{cm}$$

$$OD=R\cos\gamma=30.4\cdot\cos(0.2350)=29.6\mathrm{cm}$$

1.2.14 A transparent plate

A beam of monochromatic light is incident with an angle α on a transparent plate (Fig. 1.51) of thickness h = 1 cm. If the refractive index is n



Fig, 1.51

= 1.45 determine α^*, x and d when α = 2°, 4°, 6°, …, 60°. If the refractive index is a linear function of the distance between AA' and BB' with values $n' = 1.45$ on AA' and $n'' = 2.5$ on BB' determine the corresponding values of x. Plot as a function of α the values x and d if n is fixed and if n is varying linearly. Give in the two conditions the values of x and d for $\alpha = 40°$.

Solution:

When the refractive index is fixed we have

$$\sin\alpha = n\sin\beta \qquad \beta = \arcsin(\frac{1}{n}\sin\alpha) \tag{1.54}$$

and from geometry (Fig. 1.52)

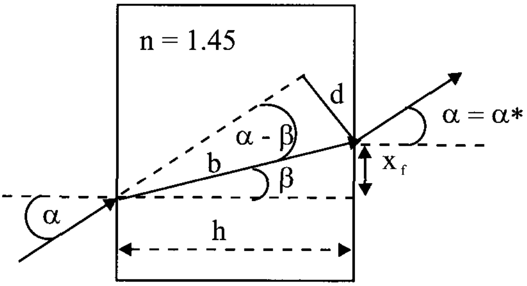


Fig. 1.52

$$b = \frac{h}{\cos\beta} \qquad x = h\cdot\tan\beta \qquad d = b\cdot\sin(\alpha-\beta)$$

The refracted ray from the first face is the reflected ray on the second face, hence

$$n\sin\beta = \sin\alpha^* \tag{1.55}$$

From (1.55) and (1.54) follows $\alpha = \alpha^*$.

The continuous linear function between AA' and BB' of the refractive index is

$$n(z) = n' - (n'' - n')\frac{z}{h}$$

with z varying in the interval $(0, h)$.Assuming $N = 100$ (for example) and

$$\Delta n = \frac{n'' - n'}{N}$$

we have the following discrete values of n

$$n_1, n_2, \ldots, n_i, n_{i+1}, \ldots, n_N$$

with

$$n_{i+1} = n_i + \Delta n_1 \qquad i = 1, 2, \ldots, N-1 \qquad n_1 = n'$$

Assuming also $\Delta h = h/N$ we have (Fig. 1.53)

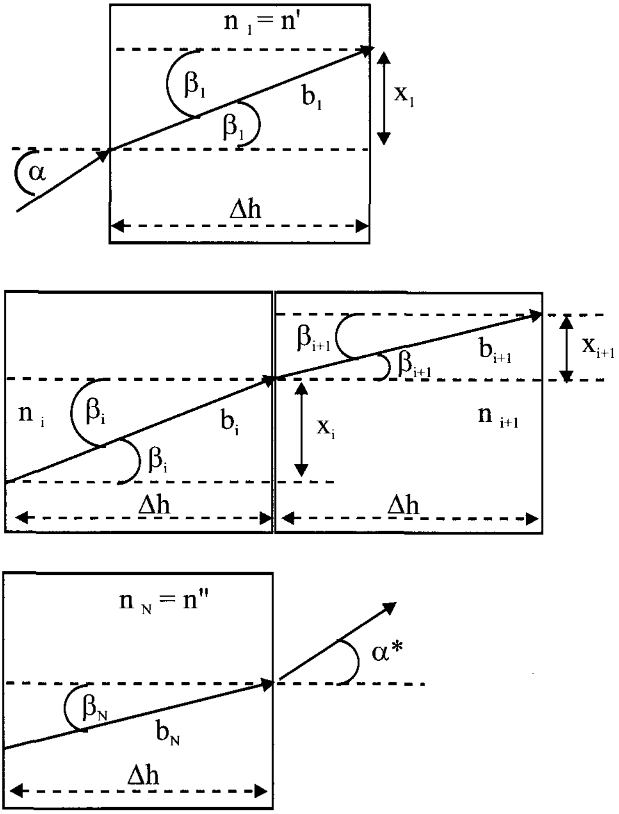


Fig. 1.53

$$\sin\alpha = n_1 \sin\beta_1 = \ldots = n_i \sin\beta_i = \ldots = n_N \sin\beta_N = \sin\alpha$$

and for every β_i

$$x_i = \Delta h \times \tan\beta_i \qquad b_i = \frac{\Delta h}{\cos\beta_i} \qquad d_i = b_i \times \sin(\beta_{i-1} - \beta_i)$$

The final values will be (Fig. 1.54)

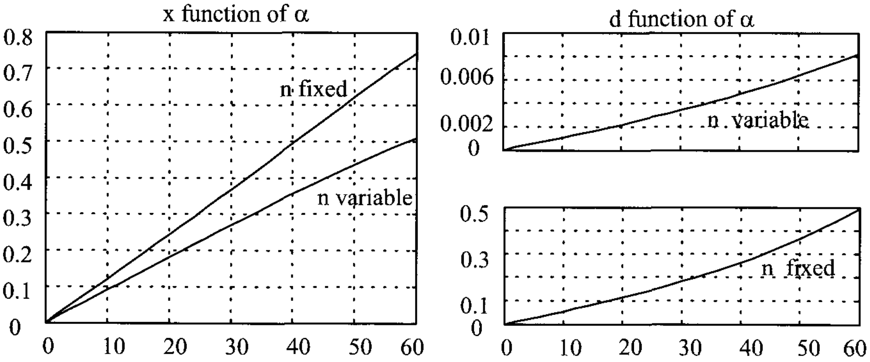


Fig.1.54

$$x = \sum_{1}^{N} x_i \qquad d = \sum_{1}^{N} d_i$$

For $\alpha = 40°$ we have

Table 1.3

$\alpha = 40°$	n fixed	n variable
x (mm)	4.9	3.6
d (mm)	2.6	0.1

1.2.15 A wave guide

A beam of monochromatic light ($\lambda = 0.65\ \mu$) is subjected in free space ($n = 1$) to multiple reflections between the two plane parallel mirrors S and S' distant h (Fig. 1.55). It is required that the wave incident in A and the

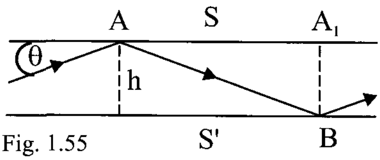


Fig. 1.55

wave reflected in B must have the same phase.

Define the N angles θ_m which satisfy this condition. Plot θ_m as a function of m with $h = 5\ \mu$.

Find how N varies with h for a fixed value of λ.

With $h = 5\ \mu$ and the minimum angle θ_m, determine the light velocity along the direction A-A$_1$, the distance AB and the time necessary to light to move from A to B.

Solution:

The passage from A to B, and not to C, in the interval Δt, is due to reflection on A. From geometry (Fig. 1.56)

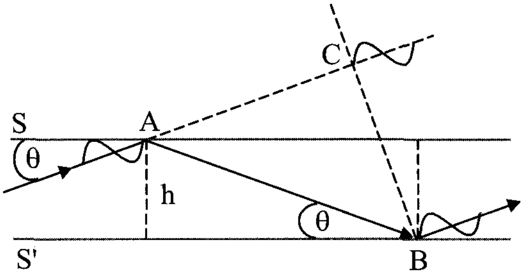


Fig. 1.56

$$AB = \frac{h}{\sin\theta} \qquad AC = AB\cos 2\theta$$

The path difference is

$$s = AB - AC = AB(1-\cos 2\theta) = 2h\sin\theta$$

and the corresponding phase difference, assuming two additional phase shifts of π due to the reflection on A and B, is

$$\varphi = 2\pi + k(AB - AC) = 2\pi + \frac{2\pi}{\lambda}2h\sin\theta$$

The wave in B replicates the phase of wave in A if

$$2\pi + \frac{2\pi}{\lambda}2h\sin\theta = p2\pi$$

where p is an integer. Hence

$$\frac{2h\sin\theta}{\lambda}=p-1 \tag{1.56}$$

with $p = 2, 3, \ldots$. For $p = 1$ there isn't reflection because $\theta = 0$. The (1.56) can be written

$$2h\sin\theta=m\lambda \qquad m=1,2,3,\ldots \qquad m=p-1 \tag{1.57}$$

Rounding we have $N = 2h/\lambda = 15$, the maximum value of m, for $\sin\theta = 1$. For a fixed λ, N is a linear function of h.

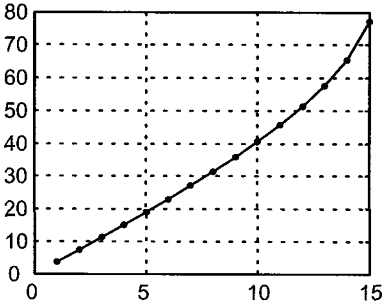


Fig. 1.57 θ_m function of m

From (1.57) the angles θ_m are (Fig. 1.57 and Table 1.4)

$$\theta_m=\arcsin\left(\frac{m\lambda}{2h}\right) \qquad m=1,2,\ldots,15$$

Table 1.4

m	1	2	3	4	5	...	11	12	13	14	15
θ_m(°)	3.7	7.5	11.2	15.1	19.0	...	45.6	51.3	57.7	65.5	77.2

With $h = 10\mu$ and $\theta_m = 3.7°$, the light speed from A to A_1 is

$$v=\frac{s}{t}=\frac{AA_1}{AB/c}=c\frac{AA_1}{AB}=c\frac{AB\cos\theta_m}{AB}=c\times\cos\theta_m=2.99\times10^8\text{m/sec}$$

The paths AB and AA_1 are (in micron)

$$AB = \frac{h}{\sin\theta} = 76.9 \qquad AA_1 = AB\cos\theta = 76.8$$

and the light travels from A to B in the time $t_{AB} = AB/c = 0.3$ picosec

1.2.16 Two mirrors

Between two flat parallel mirrors, S and S' (Fig. 1.58), separated by a distance $d = 15$ cm and with a coefficient ρ (ratio of reflected intensity to the incident intensity) equal to 0.95, there is free space ($n = 1$). A thin beam of light, perpendicular to S and S', is reflected back and forth. The light has a wavelength varying from 0.4 to 0.7 μ. Dividing this interval with a step of 3 nm, find the values of λ which satisfy the condition of standing waves calculating and plotting their relative intensities

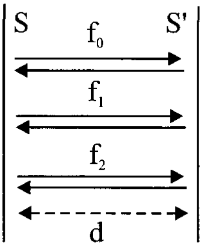


Fig. 1.58

Solution:

If the wavelengths λ have to be standing waves with d fixed, must be (Fig. 1.59) $d = m(\lambda/2)$ with m integer, or $m = 2d/\lambda$.

For example, if $d = 15$cm and $\lambda = 0.5\mu$ would be $m = 600000$. Scanning, with a step of 3nm, values of λ the corresponding values of m can be calculated. Defining with ε the ratio of the amplitude of the reflected to incident wave,

$$\varepsilon = \sqrt{\rho}$$

(The reflectivity ρ in Sec. 3.1.3 is called R^*)

$$\varphi = 2\frac{2\pi}{\lambda}d = \frac{4\pi d}{\lambda} \tag{1.58}$$

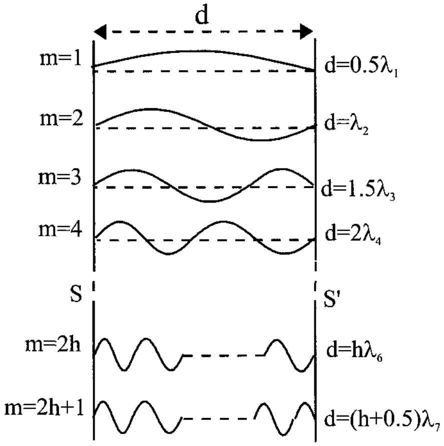


Fig. 1.59

the phase change after a round trip of the ray between S and S', and α the initial phase of f_0 we have for the first wave, considered harmonic

$$f_0 = a\sin\alpha = a e^{i\alpha}$$

and for the following ones

$$f_1 = \varepsilon a e^{i(\alpha+\varphi)} = \varepsilon e^{i\varphi} f_0 \quad f_2 = \varepsilon^2 e^{i2\varphi} f_0 \quad \dots \quad f_n = \varepsilon^n e^{in\varphi} f_0$$

The resulting wave will be

$$f = f_0 (1 + \varepsilon e^{i\varphi} + \varepsilon^2 e^{i2\varphi} + \dots + \varepsilon^n e^{in\varphi})$$

and the corresponding relative intensity (see Appendix 2)

$$I_r = \frac{1}{1+\varepsilon^2 - 2\varepsilon\cos\varphi} \tag{1.59}$$

The maximum of (1.59) is

$$I_{r\max} = \frac{I_0}{(1-\varepsilon)^2}$$

for

$$\varphi = 2m\pi \tag{1.60}$$

Using (1.58) and (1.60) we have again the initial condition

$$\frac{4\pi d}{\lambda} = 2m\pi \qquad d = m\frac{\lambda}{2}$$

In the range (0.4, 0.7) μ , using a step of 3 nm, we have N values of λ and of the corresponding value of m.

From the N values of m are picked out the N_1 values m_1 that are integer or near to an integer and the corresponding N_1 values of λ_1.

The corresponding N_1 values of φ_1, given by the (1.58), and of I_{r1}, given by (11.59), are calculated.

The N_1 values of λ_1, m_1, φ_1 and I_{r1} are given in the Table 1.5. The values of I_{r1} as a function of λ_1 are plotted in Fig. 1.60.

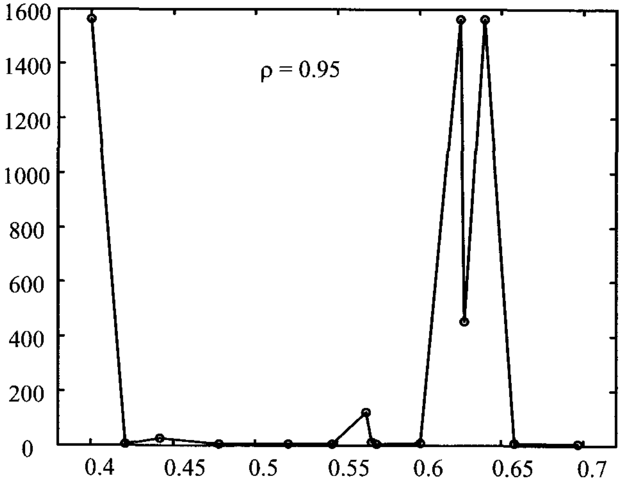


Fig.1.60 Relative intensity for λ in the range (0.4-0.7) μ

Table 1.5

λ (μ)	m	φ (°)	I_r
0.400	**750000**	**0**	**1560**
0.421	712589	26.5	1.3
0.442	678733	11.4	6.5
0.478	627615	22.6	1.7
0.520	576923	27.7	1.2
0.547	548446	25.0	1.4
0.568	528169	5.07	32.2
0.571	525394	16.4	3.2
0.574	522648	30.1	1.0
0.601	499168	19.2	2.4
0.625	**480000**	**0**	**1560**
0.628	477707	2.29	145.3
0.640	**468750**	**0**	**1560**
0.658	455927	18.6	2.5
0.697	430416	24.8	1.5

1.2.17 Three mirrors

Three plane mirrors are placed in free space (n = 1) as in Fig. 1.61.

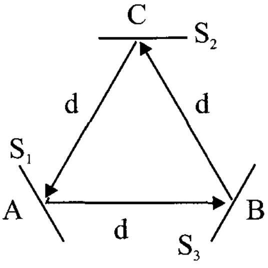


Fig. 1.61

Assume that each mirror has a coefficient $\rho = 0.95$ (see Sec. 1.2.16) and introduces a phase shift of π. The point A is distant d = 15cm from B, and so B from C and C from A. Also assume that a beam of light, represented as a harmonic wave, is reflected back and forth on the mirrors without escaping. Find the relation between d and λ if the waves have to be considered standing. If λ is specified in the interval (0.4 - 0.7) μ with a step of 0.5 nm calculate the values of λ that determine a maximum of relative intensity I_r of the reflected light. Plot the highest 20 values of I_r versus the related λ.

Solution:

The distance d must be (Fig. 1.62, Fig. 1.63 and Fig. 1.64) equal to an odd integer number of $\lambda/2$

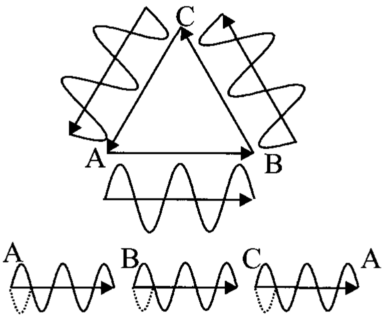


Fig. 1.62

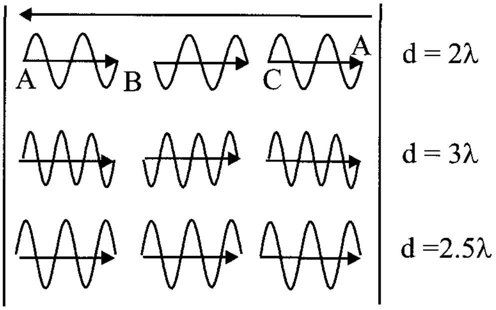


Fig. 1.63 Only the third case is valid

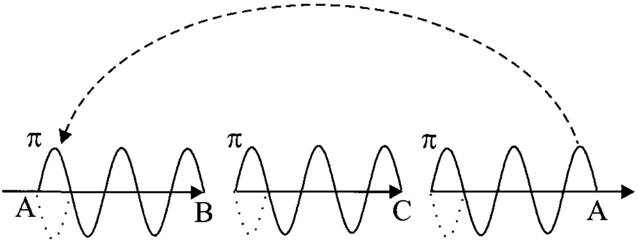


Fig.1.64 The additional shift of π due to the reflection is displayed

$$d = (2m_1 + 1)\frac{\lambda}{2} \qquad m_1 = 0, 1, 2, \ldots$$

Through a roundtrip must be

$$g = 3d = 3(2m_1 + 1)\frac{\lambda}{2} = (2m + 1)\frac{\lambda}{2} \tag{1.61}$$

with $m = (3m_1 + 1)$ or, (Table 1.6), $m = 1, 4, 7, \ldots$

Table 1.6

m_1	0	1	2	3
$d\,(\lambda/2)$	1	3	5	7
m	1	4	7	10
$g\,(\lambda/2)$	3	9	15	21

Starting from A the beam is represented by a wave function f_0 with initial phase α

$$f_0 = a \sin\alpha = a e^{i\alpha}$$

After the first running along the path A-B-C the wave function is

$$f_1 = a\varepsilon e^{i(\alpha+\varphi)} = f_0 \varepsilon e^{i\varphi}$$

with ε defined in Sec. 1.2.16, and a change in phase (Fig. 1.65)

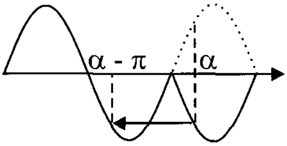


Fig. 1.65

$$\varphi = kg - 3\pi \qquad \longrightarrow \qquad \varphi = kg - \pi \tag{1.62}$$

and for the following passages along A-B-C

$$f_2 = f_0 \varepsilon^2 e^{i2\varphi} \qquad \ldots\ldots \qquad f_n = f_0 \varepsilon^n e^{in\varphi}$$

Their sum is

$$f = f_0(1+\varepsilon e^{i\varphi} + \varepsilon^2 e^{i2\varphi} ... + \varepsilon^n e^{in\varphi})$$

and the corresponding relative intensity (see Appendix 2)

$$I_r = \frac{I}{I_0} = \frac{1}{1+\varepsilon^2 - 2\varepsilon\cos\varphi} \tag{1.63}$$

The (1.63) has the maximum

$$I_{r\max} = \frac{1}{(1-\varepsilon^2)}$$

for

$$\varphi = 2m\pi \tag{1.64}$$

Using (1.62) and (1.64) we get back again the (1.61)

$$kg - \pi = 2m\pi \qquad \frac{2\pi}{\lambda}g - \pi = 2m\pi \qquad g = (2m+1)\frac{\lambda}{2}$$

In Fig. 1.66 the requested relative intensity is plotted as a function of λ.

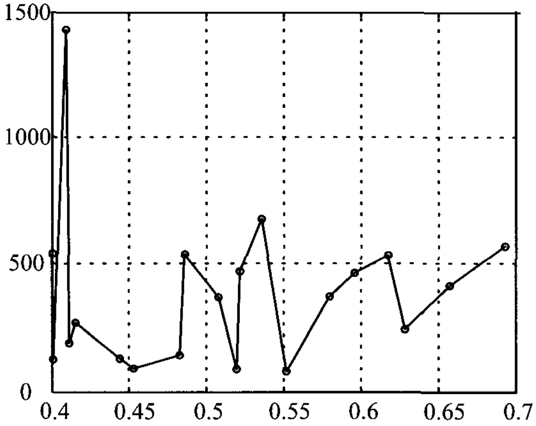


Fig. 1.66 Relative intensity function of λ (μ)

For λ = 0.409 μ we find the highest value of relative intensity (I_r = 1431) and the corresponding values of m (1100244) and φ (360°).

1.2.18 Cavity 1

Between two plane mirrors S and S', separated by a distance $d = 0.5$ m a beam of light, normal to the mirrors, is bouncing back and forth in free space (fig. 1.67). The light has a wavelength in the interval (6328 ± 0.016)

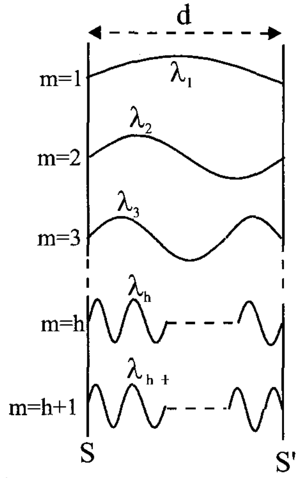


Fig.1.67

Å. S and S' have a coefficient of reflexivity $\rho = 0.95$ (su Sec. 1.2.16). Define the λ_i and the corresponding m_i for which the condition of standing waves $d = m_i(\lambda_i/2)$ holds giving the plot of the relative intensity $I_{r\,i}$. Determine the positions between S and S' where $I_{r\,i}$ has a maximum value and the elapsing time between two maxima of $I_{r\,i}$ in the same position

Solution:

1. Table 1.7 gives the first pair of λ_i and m_i when the condition of standing waves holds for a fixed d.

Table 1.7

$m = 1$	$\lambda_1 = 2d$
$m = 2$	$\lambda_2 = 2d/2$
$m = 3$	$\lambda_3 = 2d/3$
....	....
$m = h$	$\lambda_h\ 2d/h$

The relative intensities are given by (see Sec. 1.2.16)

$$I_{ri} = \frac{1}{1+\varepsilon^2 - 2\varepsilon \cos\varphi_i}$$

where ε is the square root of ρ (see Sec. 1.2.16) and

$$\varphi_i = 2k_i d = \frac{4\pi}{\lambda_i} d \tag{1.65}$$

The intensity has a maximum when

$$\varphi_i = 2m_i\pi \tag{1.66}$$

Using (1.65) and (1.66) the condition of standing waves is retrieved. Numerically are searched the N pair of λ_i and m_i and the maxima of $I_{r\,i}$ are calculated. Table 1.8 and Fig. 1.68 give the results.

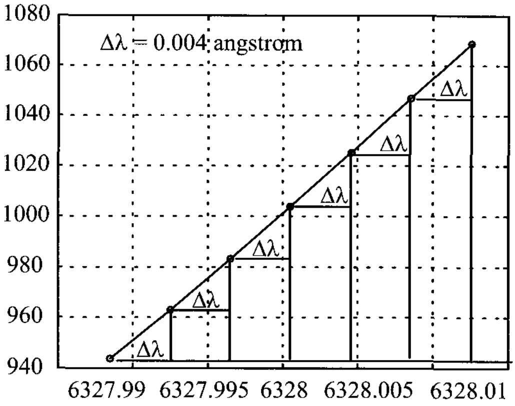


Fig. 1.68 The highest values of I_r function of λ

Table 1.8

λ_i (Å)	m_i	φ_i (°)	I_{ri}
6327.988	1580281	358.8	944
6327.992	1580280	358.8	963
6327.996	1580279	358.9	983
6328.000	1580278	358.9	1004
6328.004	1580277	358.9	1025
6328.008	1580276	359.0	1047
6328.012	1580275	359.0	1069

From Table 1.8 (first column) and Fig. 1.68 we see that the difference between λ_{i+1} and λ_i is always 0.004 Å.
If a standing wave λ_i can be represented as harmonic plane wave it has the form

$$y = a\cos\omega t \sin k_i x \qquad k_i = \frac{2\pi}{\lambda_i}$$

with x a point of an axis normal to S and S'. The standing wave y has its amplitude $A = a\cos\omega t$, varying with t, on the positions

$$k_i x = (2p+1)\frac{\pi}{2} \qquad x_p = (2p+1)\frac{\lambda_i}{4} \qquad p = 0,1,2,\ldots,p_{\max}$$

For $p = 0$ and $\lambda_1 = 6327.988$ Å, the first position of A (in micron) is

$$x_0 = \frac{\lambda_1}{4} = \frac{6327.988}{4} = \frac{0.63}{4} = 0.16$$

The amplitude A has a maximum when

$$\omega t = n2\pi \qquad t = nT = n\frac{\lambda}{c} \qquad n = 1,2,3,\ldots$$

For n = 1 and $\lambda_1 = 6327.988$ Å, the first time for the maximum of amplitude A is

$$t = T = \frac{\lambda}{c} = 2.1 \cdot 10^{-15}\,\text{sec}$$

On the position $x_0 = 0.16$ μ, and the next x_p points, a peak, having a maximum of A and a relative intensity 944 (Table 1.8) and (Fig. 1.69), will occur about every two femtosec.

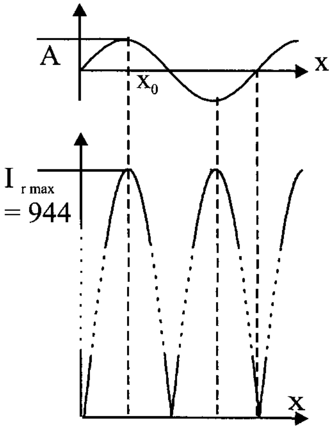


Fig.1.69

1.2.19 Cavity 2

Between two plane mirrors S and S', separated by a distance $d = 0.5$m a beam of light, normal to the mirrors, is bouncing back and forth in free space ($n = 1$). The light has a wavelength in the interval (6328 ± 0.016) Å. S and S' have a coefficient of reflexivity $\rho = 0.95$ (see Sec. 1.2.16). Define the bandwidth $\Delta\nu$ of the frequency of light in the assigned interval and the number N of frequencies ν_i corresponding to the λ_i for which holds the condition of standing waves. Give the plot of the relative intensity $I_{r\,i}$ as a function of the ν_i.

Solution:

1. The maximum and minimum values and their difference are for λ (in Å)

$$\lambda_{\min} = 6327.984 \qquad \lambda_{\max} = 6328.016 \qquad \Delta\lambda = 0.032$$

and for ν (in Hz)

$$\nu_{\min} = 4.740822 \cdot 10^{14} \quad \nu_{\max} = 4.740846 \cdot 10^{14} \quad \Delta\nu = 2.4 \cdot 10^{9} \quad (1.67)$$

The standing wave condition as a function of the frequency becomes

$$\lambda_i = \frac{2d}{m_i} \qquad \frac{c}{\nu_i} = \frac{2d}{m_i} \qquad \nu_i = m_i \frac{c}{2d} = m_i \Delta F$$

with *ΔF* (in Hz)

$$\Delta F = \frac{c}{2d} = 3 \cdot 10^8 \tag{1.68}$$

The frequencies, proportional to m_i, are separated by ΔF (Fig. 1.70) that is the inverse of the time required for a round trip of the light between S and S'.

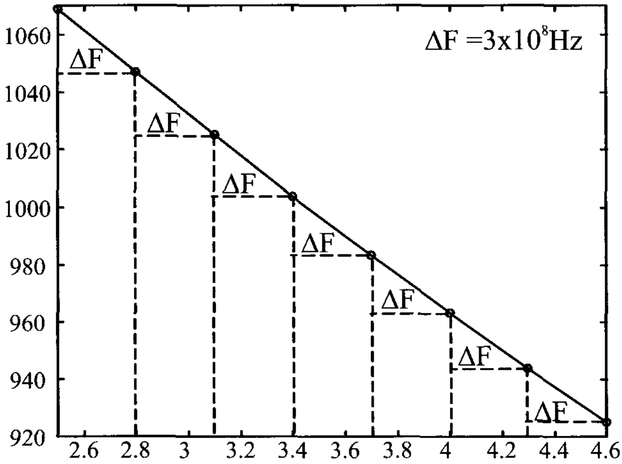


Fig. 1.70 In abscissa are shown only the fifth and sixth decimal digits (see Table 1.9) of the frequency

Using (1.67) and (1.68) we have

$$N = \frac{\Delta\nu}{\Delta F} = \frac{2.4 \cdot 10^9 \text{ Hz}}{3 \cdot 10^8 \text{ Hz}} = 8$$

The relative intensity is (see Sec. 1.2.18 also for definition of ε and φ_i)

$$I_{ri} = \frac{1}{1 + \varepsilon^2 - 2\varepsilon \cos\varphi_i}$$

The eight best values of the frequencies and related values are calculated using a numerical procedure (Table 1.9 and Fig. 1.70).

Table 1.9

m_i	$\nu_i\,(10^{14}\text{Hz})$	$\varphi_i\,(°)$	$I_{r\,i}$
1580275	4.740**825**	359.0	1069
1580276	4.740**828**	359.0	1047
1580277	4.740**831**	358.9	1025
1580278	4.740**834**	358.9	1004
1580279	4.740**837**	358.9	983
1580280	4.740**840**	358.8	963
1580281	4.740**843**	358.8	944
1580282	4.740**846**	358.8	925

1.2.20 Cavity 3

Two concave spherical mirrors A and B (Fig. 1.71) are set in free space with a distance d from O_A to O_B. Their radii are R_A = -10 mm and R_B = 10 mm. A beam of parallel light, considered a paraxial ray, is reflected back and forth on A and B. Assume that the first ray starts from P (distant y_A from the optical axis O_A - O_B) and makes an angle θ_A = 5° with the optical axis. Determine the range of values of d that inhibits the escaping of the ray outside the mirrors. This is called condition of resonance. Known a value of d, that fulfills the resonance condition, define the N allowed values of the pair y_A and θ_A that permits the condition of resonance.

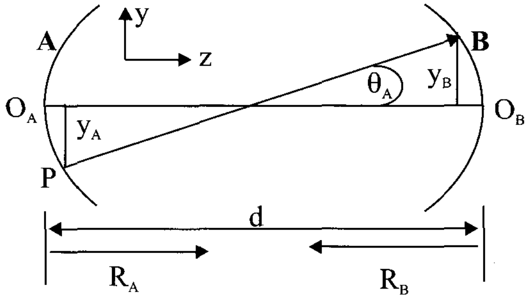


Fig. 1.71

Solution:

The passage of the ray from A to B (Fig. 1.72**C2**) takes on the matrix form

(see Sec. 1.1.4)

$$T=\begin{vmatrix} y_B \\ \\ \theta_A \end{vmatrix}=\begin{vmatrix} 1 & -d \\ \\ 0 & 1 \end{vmatrix}\begin{vmatrix} y_A \\ \\ \theta_A \end{vmatrix} \tag{1.69}$$

that allows to ascertain the point where the ray hit the mirror B, $y_B = T(1)$. The ray is reflected from the mirror B with an angle θ_B (Fig. 1.72**C3**)

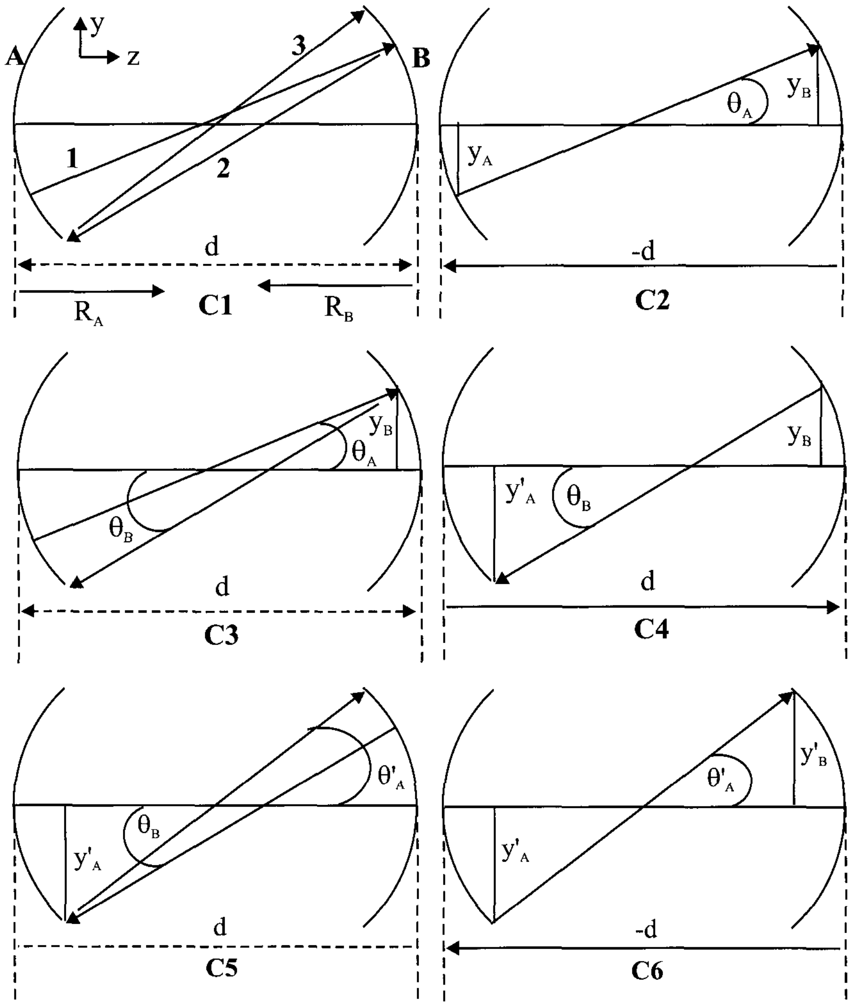


Fig. 1.72

$$S=\begin{vmatrix} y_B \\ \theta_B \end{vmatrix}=\begin{vmatrix} 1 & 0 \\ \dfrac{1}{f_B} & -1 \end{vmatrix}\begin{vmatrix} y_B \\ \theta_A \end{vmatrix}$$

whose value, $\theta_B = S(2)$, is given by the previous matrix equation.
For the ray traveling from B to A we write (Fig. 1.72**C4**)

$$X=\begin{vmatrix} y'_A \\ \theta_B \end{vmatrix}=\begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix}\begin{vmatrix} y_B \\ \theta_B \end{vmatrix}$$

and get $y'_A = X(1)$.
Then the ray is reflected on A with an angle θ'_A (Fig. 1.72**C5**). The following equation

$$W=\begin{vmatrix} y'_A \\ \theta'_A \end{vmatrix}=\begin{vmatrix} 1 & 0 \\ \dfrac{1}{f_A} & -1 \end{vmatrix}\begin{vmatrix} y'_A \\ \theta_B \end{vmatrix}$$

We have $\theta'_A = W(2)$.
For the ray traveling from A to B we can write (Fig. 1.72**C6**)

$$Z=\begin{vmatrix} y'_B \\ \theta'_A \end{vmatrix}=\begin{vmatrix} 1 & -d \\ 0 & 1 \end{vmatrix}\begin{vmatrix} y'_A \\ \theta'_A \end{vmatrix} \tag{1.70}$$

and have another value, $y'_B = Z(1)$, of the point where the ray hits the mirror B. The final equation (1.70) has the same form as the first one (1.69). If the initial values θ_A , R_A and R_B are known, a first value of y_A is $R_A \tan\theta_A$. With a value assigned to d is an easy task the recursive multiplication of the previous matrices (Sec. 1.1.4) to obtain subsequent pair (y_B θ_B), (y_A θ_A), etc. Using a MATLAB program the pair can be calculated and plotted (Fig. 1.73) in a straightforward way varying d in a reasonable range between 5 mm (R_A /2) and 30mm (3 R_A).
From (Fig. 1.73) it follows that if d is between 14 mm and 19 mm the values of θ_A and θ_B are stable below 5°. If we assume d = 15 mm, for example, we have repeatedly (Fig. 1.74) the following three values for

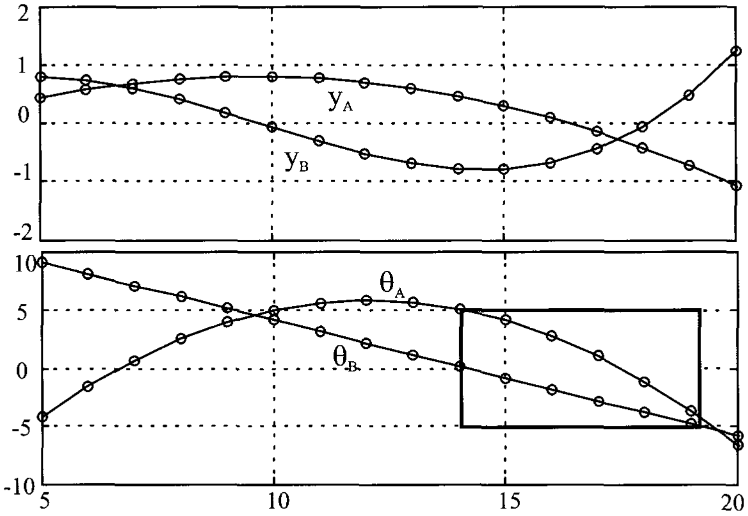


Fig. 1.73 The distance d is in abscissa

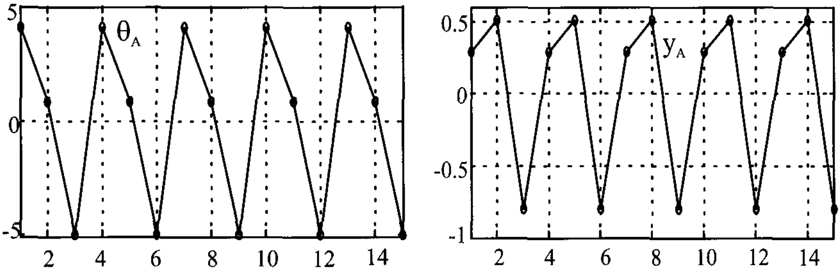


Fig.1.74 The first 15 consecutive θ_A an dy_A for d = 15mm

y_A = 0.2910 mm, 0.5090 mm, -0.8000 mm
and for
θ_A = 4.1673°, 0.8327°, -5.0000°

1.2.21 Fresnel formulae

A beam of monochromatic and linearly polarized light (Fig. 1.75) is incident with an angle θ_1 on the boundary surface between free space and a perfectly transparent medium (n = 1.6). The electric vector oscillates in the plane defined by coordinates wy for the incident ray and by w'y for the reflected ray; the y axis is normal to the plane of figure where lie the axes x, z, w,u, w' and u'. Assume that for the incident ray the amplitude

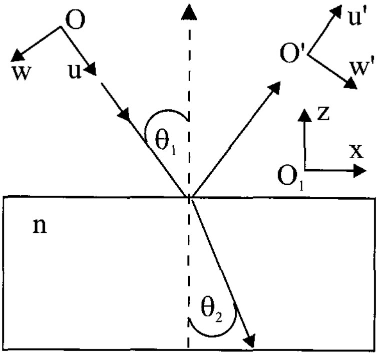


Fig.1.75 The plane of incidence O_1xz and the y axis are shared equally by the three coordinate systems xyz, wuy and w'u'y

of the electric vector is $E = 1.4142$ and the angle between the electric vector and the plane of incidence is $\alpha = 45°$. Hence the components of amplitude of the electric vector $\boldsymbol{E}$ parallel to the w and y axis are

$$E_w = E_y = 1$$

The Fresnel formulae give for the reflected ray the magnitude of the components of the electric vector $\boldsymbol{E}'$ parallel to the w' and y axis

$$E'_{w'} = \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)} \tag{1.71}$$

$$E'_y = -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)} \tag{1.72}$$

Define and plot the magnitude of the electric vector $\boldsymbol{E}'$ and the angle α' between the direction of $\boldsymbol{E}'$ and the w' axis when θ_1 varies in the interval (0, 90°). Discuss the condition and value of phase difference between incident and reflected wave.

Solution:

The angle of refraction θ_2 is a function of θ_1

$$\theta_2 = \arcsin(\frac{\sin\theta_1}{n})$$

Because the incident ray propagates from free space to a medium of refractive index $n>1$, is always $\theta_1 > \theta_2$. Hence is always $E'_y < 0$. On the contrary $E'_w > 0$ if $\theta_1 + \theta_2 < 90°$ and $E'_w < 0$ ° (Fig. 1.76) if $\theta_1 + \theta_2 > 90°$.

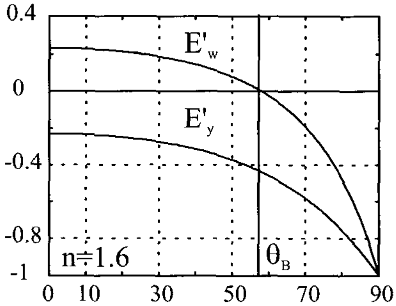


Fig. 1.76

For $\theta_1 + \theta_2 = 90°$, $E'_w = 0$ and the angle of incidence becomes θ_B = arctan(n) which is known as Brewster angle.

Varying θ_1 in the interval (0, 90°) and using (1.71) and (1.72) the magnitude E' of the electric vector

$$E' = \sqrt{(E'_{w'})^2 + (E'_y)^2}$$

and the angle α' (between E' and w')

$$\alpha' = \arctan\frac{E'_y}{E'_{w'}}$$

can be calculated and plotted (Fig. 1.77 and Fig. 1.78).

The amplitude of the electric vector $\boldsymbol{E}'$ changes from $E' = 0.3264$ for $\theta_1 = 0°$ to $E' = 1.4142$ when $\theta_1 = 90°$ (Fig. 1.77 and Fig. 1.79).

The angle α' between the direction of the electric vector $\boldsymbol{E}'$ and the w' axis changes from $\alpha' = -45°$ to $\alpha' = -135°$ (Fig. 1.78 and Fig. 1.79).

When θ_1 varies in the interval (0, 90°) the values of E (= 1.4142) and α (= 45°) of the incident ray remain constant but E' and α' of the reflected ray change.

E and E' are the amplitude of waves oscillating with incident and reflected ray. If the second medium is optically denser than the first, because, whichever the values of θ_1, E' and α', E and E' have different signs the phase of the reflected wave differ by 180° from the phase of the incident wave. The phase remains equal if the second medium is optically less dense than the first.

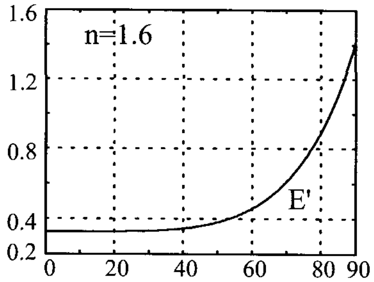


Fig. 1.77

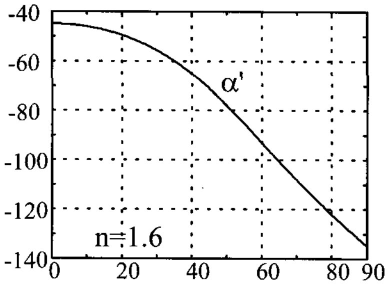


Fig. 1.78 α' is the angle between E' and the w axis on the plane wy varying the angle of incidence

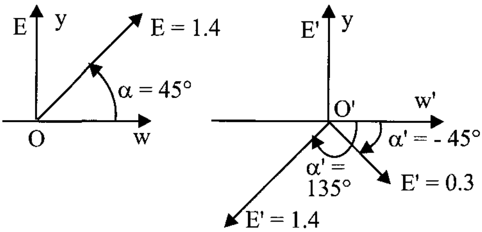


Fig.1.79