I. FUNCTORIALITY AND NORMS

Summary

The symmetric square lifting for admissible and automorphic representations, from the group $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$, to the group $\mathbf{G} = \mathrm{PGL}(3)$, is defined by means of character relations. Its basic properties are derived: the lifting is proven for induced, trivial and special representations, and both spherical functions and orthogonality relations of characters are studied. The definition is compatible with dual group homomorphisms

$$\lambda_0 = \mathrm{Sym}^2 : \widehat{H} = \mathrm{PGL}(2,\mathbb{C}) = \mathrm{SO}(3,\mathbb{C}) \hookrightarrow \widehat{G} = \mathrm{SL}(3,\mathbb{C})$$

and $\lambda_1 : \widehat{H}_1 = \mathrm{SL}(2,\mathbb{C}) \to \widehat{G}$, where $\mathbf{H}_1 = \mathrm{PGL}(2)$. Of course it will be compatible with the computation of orbital integrals (stable and unstable) in chapters II and III.

Introduction

In this chapter we define the symmetric square lifting in terms of character relations, and derive its basic properties. This work is required for the study of the lifting of automorphic forms of $\mathbf{H}(\mathbb{A})$ to $\mathbf{G}(\mathbb{A})$, where $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$ and $\mathbf{G} = \mathrm{PGL}(3)$, by means of the trace formula.

The lifting is suggested by the symmetric square, or adjoint, representation $\lambda_0\colon \widehat{H} \to \widehat{G}$ of the dual group $\widehat{H} = \mathrm{PGL}(2,\mathbb{C})$ of $\mathbf{H}$ in $\widehat{G} = \mathrm{SL}(3,\mathbb{C})$. Put ${}^tg$ =transpose of g, and

$$\sigma(g) = J\,{}^tg^{-1}J, \qquad J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

The group $\mathbf{H}$ is a σ-endoscopic group of $\mathbf{G}$ (see [KS]). Indeed, $\widehat{H} = \mathrm{SO}(3,\mathbb{C})$ is the group $Z_{\widehat{G}}(\sigma) = \{g \in \widehat{G}; \sigma g = g\}$ of points fixed by σ in $\widehat{G}$. It is elliptic ($\widehat{H}$ is not contained in a σ-invariant proper parabolic subgroup of $\widehat{G}$). But $\mathbf{G}$ has another elliptic σ-endoscopic group, which is $\mathbf{H}_1 = \mathrm{PGL}(2)$:

$$\lambda_1 \colon \widehat{H}_1 = \mathrm{SL}(2,\mathbb{C}) = Z_{\widehat{G}}(s\sigma) = \{g \in \widehat{G}; s\sigma(g)s = g\} \hookrightarrow \widehat{G},$$

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto h_1 = \begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}.$$

Via the Satake isomorphism, the maps λ_i formally define the lifting $\pi = \lambda_i(\pi_i)$ of unramified H_i-modules π_i to unramified G-modules π. Moreover, we introduce in section 1 (of this chapter I) the dual maps $\lambda_i^* \colon \mathbb{H} \to \mathbb{H}_i$ from the Hecke algebra $\mathbb{H}$ of G to the Hecke algebra $\mathbb{H}_i$ of H_i. It follows from the definitions that if $f_i = \lambda_i^*(f)$ then the spherical functions f and f_i have matching orbital integrals on the split tori.

In section 2 we define lifting, denoted $\pi_i = \lambda_i(\pi)$, of admissible representations π_i of H_i to such representations π of G, by means of character relations. The definition generalizes the spherical case, and uses packets rather than a single irreducible. Basic examples of the stable lifting λ_0 are given. These concern induced, trivial, and special representations.

Section 3 concerns orthogonality relations for characters, needed in our study of the local lifting. The cases of cuspidal G-modules and Steinberg π are standard but useful. We also record the twisted orthogonality relation for two tempered G-modules which are not relevant. The proof follows closely that of the nontwisted case by Kazhdan [K2]. It depends on the twisted analogue of the crucial appendix of [K2]; this is proven in [F1;II] for a general group, and in chapter V, (1.8), in our case.

I.1 Hecke algebra

1.1 Dual groups. Let F be a global or local field of characteristic zero. Put $\mathbf{G} = \mathrm{PGL}(3)$, $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$, and $\mathbf{H}_1 = \mathrm{PGL}(2) = \mathrm{SO}(3)$, viewed as $\mathbb{Z}$-groups. For any field k denote by $\mathbf{G}(k)$, $\mathbf{H}(k)$ and $\mathbf{H}_1(k)$ the group of k-rational points of $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{H}_1$. We write G' for the group $\mathbf{G}'(F)$ of F-rational points, for any algebraic group $\mathbf{G}'$ over F. Fix an algebraic closure $\overline{F}$ of F.

Let $\widehat{G} = \mathrm{SL}(3,\mathbb{C})$ be the connected dual group of G (for any reductive group G the connected dual group $\widehat{G}$ is defined in [Bo2], where it is denoted by ${}^L G^0$). Consider the semidirect product $\widehat{G}' = \widehat{G} \rtimes \langle\sigma\rangle$; $\langle\sigma\rangle$ denotes the group generated by the automorphism $\sigma(g) = J{}^t g^{-1} J$ of G of order 2.

The dual group $\widehat{H}$ of $\mathbf{H}$ is $\mathrm{PGL}(2,\mathbb{C}) \xrightarrow{\sim} \mathrm{SO}(3,\mathbb{C})$. It is isomorphic to the centralizer of $1 \times \sigma$ in the connected component of 1 in $\widehat{G}'$, and to the σ-centralizer $\widehat{G}_1^\sigma = \{g \text{ in } \widehat{G}; g^{-1}\sigma(g) = 1\}$ of 1 in $\widehat{G}$. The isomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{x}\begin{pmatrix} a^2 & ab\sqrt{2} & b^2 \\ ac\sqrt{2} & ad+bc & bd\sqrt{2} \\ c^2 & cd\sqrt{2} & d^2 \end{pmatrix} \qquad (x = ad - bc).$$

This map will be denoted by λ and by $\lambda_0\colon \widehat{H} \to \widehat{G}$.

The dual group $\widehat{H}_1$ of $\mathbf{H}_1 = \mathrm{PGL}(2)$ is $\mathrm{SL}(2,\mathbb{C})$, and the map

$$\lambda_1\colon h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto h_1 = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}$$

embeds $\widehat{H}_1$ in $\widehat{G}$. The image is the centralizer of $s \times \sigma$ in $\widehat{G}$, where s is the diagonal matrix $\mathrm{diag}(-1,-1,1)$. Equivalently, it is the σ-centralizer $\widehat{G}_s^\sigma = \{g \in \widehat{G}; s\sigma(g)s^{-1} = g\}$ of s in $\widehat{G}$.

1.2 Hecke algebra. Let F be a p-adic field, $R = \{x \text{ in } F;\ |x| \le 1\}$ its ring of integers, and $K = \mathbf{G}(R)$ the standard maximal compact subgroup of G. Fix a Haar measure dg on G. The Hecke algebra $\mathbb{H} = \mathbb{H}_G$ is the convolution algebra $C_c(K\backslash G/K)$ of complex valued compactly supported K-biinvariant measures $f dg$ on G. Such $f dg$ are called *spherical*.

Let π be an admissible irreducible representation of G on a complex vector space V. A representation π is called *smooth* if each vector is fixed by an open subgroup of G. It is called *admissible* ([BZ1]) if it is smooth and if the subspace of V of vectors fixed by any open subgroup is finite dimensional. A smooth irreducible representation is admissible by a well-known theorem of Bernstein.

Put ${}^\sigma\pi(g) = \pi(\sigma g)$ (g in G). Then ${}^\sigma\pi$ is an admissible irreducible representation of G on V. We say that π is *σ-invariant* if π is equivalent to ${}^\sigma\pi$. In this case there is an invertible operator $A\colon V \to V$ with $\pi(\sigma g) = A\pi(g)A^{-1}$ (g in G). Since π is irreducible and A^2 intertwines π with itself, Schur's lemma ([BZ1]) implies that A^2 is a scalar. Multiplying A by $1/\sqrt{A^2}$, we assume that $A^2 = 1$. Then A is unique up to a sign. We put $\pi(\sigma) = A$, and define the operator $\pi(f dg \times \sigma) = \pi(f dg\sigma) = \pi(f dg)\pi(\sigma)$ to be the map $v \mapsto \int f(g)\pi(g)Av\, dg$.

If fdg is spherical (in $\mathbb{H}_G$) then $\pi(fdg)$ factorizes through the projection to the space π^K of K-fixed vectors in (π, V). If π is irreducible, $\dim_{\mathbb{C}} \pi^K \leq 1$. The representation π is called *unramified* if $\pi^K \neq 0$. Then $(k \in K)$ acts as the identity on π^K. If π is irreducible, $\pi(fdg) \neq 0$ implies that the image π^K of $\pi(fdg)$ is one dimensional.

If π is unramified, it lies in a representation $I = I(\eta)$ of G induced from an unramified character η of the upper triangular Borel subgroup $B = TN$ (e.g., [Bo3]). Here N denotes the unipotent upper triangular subgroup, and T denotes the diagonal subgroup. In fact π is the unique unramified constituent in the composition series of I.

Fix v in V so that $w = \pi(fdg \times \sigma)v$ is nonzero. Since $\sigma(K) = K$, Aw is also a K-fixed vector, and $Aw \neq 0$, since $A(Aw) = w \neq 0$. Hence there is a constant c with $Aw = cw$. As $A^2 = 1$, c is 1 or -1. We replace A by cA to have $Aw = w$. This normalization is compatible with the normalization for generic representations, see chapter V, (1.1.1).

The character η is given by

$$\eta(\delta) = \mu_1(a)\mu_2(b)\mu_3(c)$$

at an element $\delta = \operatorname{diag}(a, b, c)$ in the diagonal torus T of G. Here μ_i are characters of $F^\times$ with $\mu_1\mu_2\mu_3 = 1$. The induced representation $I = I(\eta)$ consists of all (right) smooth $\phi : G \to \mathbb{C}$ with

$$\phi(n\delta g) = \boldsymbol{\delta}^{1/2}(\delta)\eta(\delta)\phi(g), \qquad g \in G, \qquad n \in N, \qquad \delta \in T.$$

The action is by right translation: $(I(g)\phi)(h) = \phi(hg)$. The value of the factor

$$\boldsymbol{\delta}(\delta) = |\det(\operatorname{Ad}(\delta)|\operatorname{Lie} N)| \qquad \text{is} \qquad |a/c|^2.$$

Here $\operatorname{Lie} N$ denotes the Lie algebra of N.

Let $\boldsymbol{\pi}$ be a generator of the maximal ideal in the ring R of integers of F. Consider the element

$$t = \operatorname{diag}(\mu_1(\boldsymbol{\pi}), \mu_2(\boldsymbol{\pi}), \mu_3(\boldsymbol{\pi}))$$

in the diagonal torus $\widehat{T}$ of $\widehat{G}$. Then the equivalence class of the unramified representation π is uniquely determined by the conjugacy class in $\widehat{G}$ of t.

1.3 Orbital integrals. Fix a Haar measure da on the diagonal torus T. The normalized orbital integral

$$F(\delta, fdg) = \Delta(\delta)\int f(g\delta g^{-1})\frac{dg}{da} \qquad (g \in G/T),$$

where

$$\Delta(\delta) = \boldsymbol{\delta}^{-1/2}(\delta)|\det(1-\mathrm{Ad}(\delta))|\mathrm{Lie}\, N| = \left|\frac{a-b}{a}\frac{b-c}{b}\frac{a-c}{c}\right|,$$

depends only on the image of

$$\delta = \mathrm{diag}(a,b,c) \qquad \text{in} \qquad T/\mathbf{T}(R) \simeq X_*(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$$

when fdg is spherical. Indeed, writing $g = an_1k$ (and $dg/da = dn\,dk$) and introducing n by $n_1^{-1}\delta n_1 = \delta n$, changing variables on n in the orbital integral gives the factor

$$|1-\mathrm{Ad}(\delta^{-1})|^{-1} = |\mathrm{Ad}(\delta)||1-\mathrm{Ad}(\delta)|^{-1}.$$

Hence

$$F(\delta, fdg) = \boldsymbol{\delta}^{1/2}(\delta)\int_N f^K(\delta n)dn, \qquad \text{where} \qquad f^K(g) = \int_K f(k^{-1}gk)dk.$$

We denote this value of the orbital integral by $F(\mathbf{n}, fdg)$, $\mathbf{n}$ being the image of δ in

$$X_*(\mathbf{T}) \simeq \{(n_1,n_2,n_3);\ n_i \in \mathbb{Z}\}/\{(n,n,n);\ n \in \mathbb{Z}\}.$$

For $t = \mathrm{diag}(t_1,t_2,t_3)$ in $\widehat{T}$ and $\mathbf{n} = (n_1,n_2,n_3)$ in $X_*(\mathbf{T})$, we put $\mathbf{n}(t) = t_1^{n_1}t_2^{n_2}t_3^{n_3}$.

The Satake transform $(fdg)^\vee$ of fdg is abbreviated to $\check{f}$ and is defined by

$$\check{f}(t) = |\mathbf{T}(R)|\sum_{\mathbf{n}} F(\mathbf{n}, fdg)\mathbf{n}(t) \qquad (\mathbf{n} \in X^*(\widehat{T}) \simeq X_*(\mathbf{T})),$$

where $|\mathbf{T}(R)|$ denotes the volume of $\mathbf{T}(R) = T \cap K$ with respect to dt. The map $fdg \mapsto \check{f}$ is an isomorphism from the algebra $\mathbb{H}_G$ to the algebra $\mathbb{C}[\widehat{T}]^W$ of finite Laurent series in $t \in \widehat{T}$ which are invariant under the action of the Weyl group W of $\widehat{T}$ in $\widehat{G}$.

Let $C_c^\infty(G)$ denote the space of all smooth compactly supported complex valued functions on G. If π is an admissible representation, for any f in $C_c^\infty(G)$ the operator $\pi(fdg) = \int_G f(g)\pi(g)dg$ has finite rank. We write $\operatorname{tr}\pi(fdg)$ for its trace. If π is irreducible but not equivalent to ${}^\sigma\pi$, then $\operatorname{tr}\pi(fdg\times\sigma)$ is zero. If π is irreducible and unramified, and fdg is spherical, then $\pi(fdg)$ is a scalar multiple of the projection on the K-fixed vector w. If, moreover, $\pi \simeq {}^\sigma\pi$, then $\pi(\sigma)$ acts as 1 on w, and $\operatorname{tr}\pi(fdg\times\sigma) = \operatorname{tr}\pi(fdg)$ is this scalar. Let us compute it.

1.4 LEMMA. *Suppose that π is unramified and $t = t(\eta) = t(\pi)$ is a corresponding element in $\widehat{T}$. If ${}^\sigma\eta = \eta$, then for any fdg in $\mathbb{H}_G$ we have*

$$\operatorname{tr}\pi(fdg\times\sigma) = \operatorname{tr}\pi(fdg) = \check{f}(t).$$

PROOF. Corresponding to $g = nak$ there is a measure decomposition $dg = \boldsymbol{\delta}^{-1}(a)dndadk$. For a test function $f \in C_c^\infty(G)$ the convolution operator $\pi(fdg) = \int_G \pi(g)f(g)dg$ maps $\phi \in \pi$ to

$$\begin{aligned}(\pi(fdg)\phi)(h) &= \int_G f(g)\phi(hg)dg = \int_G f(h^{-1}g)\phi(g)dg \\ &= \int_N\int_T\int_K f(h^{-1}n_1ak)(\boldsymbol{\delta}^{1/2}\eta)(a)\phi(k)\boldsymbol{\delta}^{-1}(a)dn_1dadk.\end{aligned}$$

The change of variables $n_1 \mapsto n$, where n is defined by $n^{-1}ana^{-1} = n_1$, has the Jacobian $|\det(1-\operatorname{Ad}a)|\operatorname{Lie}N|$. The trace of $\pi(fdg)$ is obtained on integrating the kernel of the convolution operator — viewed as a trivial vector bundle over K — on the diagonal $h = k \in K$. Hence

$$\begin{aligned}\operatorname{tr}\pi(fdg) &= \int_K\int_N\int_T(\Delta\eta)(a)f(k^{-1}n^{-1}ank)dndadk \\ &= \int_T \eta(a)\left[\Delta(a)\int_{G/T} f(gag^{-1})\frac{dg}{da}\right]da.\end{aligned}$$ □

1.5 DEFINITION. For δ in T put

$$\Phi(\delta\sigma, fdg) = \int_{T^\sigma\backslash G} f(g\delta\sigma(g)^{-1})\,\frac{dg}{da}.$$

Here $T^\sigma = \{a \in T; \sigma(a) = a\}$ is the group of σ-fixed points in T. Also put $\tilde{\delta} = J\delta J$ ($= \operatorname{diag}(c,b,a)$ if $\delta = \operatorname{diag}(a,b,c)$), and $T^{1-\sigma} = \{t\sigma(t)^{-1}; t \in T\}$.

The involution σ defines (via differentiation) an involution, which we denote again by σ, on the Lie algebra $\operatorname{Lie} G$ of G. It stabilizes $\operatorname{Lie} N$. Define

$$F(\delta\sigma, fdg) = \Delta(\delta\sigma)\Phi(\delta\sigma, fdg)$$

where

$$\Delta(\delta\sigma) = \boldsymbol{\delta}^{-1/2}(\delta)|\det(1 - \operatorname{Ad}(\delta)\sigma)|\operatorname{Lie} N|.$$

Note that

$$|\det(1 - \operatorname{Ad}(\delta)\sigma)|\operatorname{Lie} N| = \left|\left(1 - \frac{a}{c}\right)\left(1 + \frac{a}{c}\right)\right|, \qquad \boldsymbol{\delta}^{1/2}(\delta) = |a/c|,$$

hence $\Delta(\delta\sigma) = \Delta_0(N\delta)$, where $N\delta = \operatorname{diag}(a/c, c/a)$. Here

$$\Delta_0(\operatorname{diag}(x,y)) = |(x-y)^2/xy|^{1/2}$$

is the usual Δ-factor on $\mathrm{GL}(2)$. We usually use indices 0, 1 or 2 for objects related to $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$, $\mathbf{H}_1 = \mathrm{PGL}(2)$, and $\mathrm{GL}(2)$, respectively.

1.6 LEMMA. *For any character η of T we have*

$$\operatorname{tr} I(\eta; fdg \times \sigma) = \int_{T^{1-\sigma}\backslash T} \frac{1}{2}[\eta(a) + \eta(\tilde{a})]F(a\sigma, fdg)\,da.$$

PROOF. For $\pi = I(\eta)$, we have

$$(\pi(\sigma fdg)\phi)(h) = \int_G f(g)\phi(\sigma(h)g)dg = \int_G f(\sigma(h)^{-1}g)\phi(g)dg$$
$$= \int_N \int_T \int_K f(\sigma(h^{-1})nak)(\boldsymbol{\delta}^{1/2}\eta)(a)\phi(k)\boldsymbol{\delta}^{-1}(a)dndadk.$$

Hence

$$\operatorname{tr}\pi(\sigma fdg) = \int_K \int_N \int_T f(\sigma(k)^{-1}n_1ak)(\boldsymbol{\delta}^{-1/2}\eta)(a)dn_1dadk.$$

We change variables $n_1 \mapsto n$, where $\sigma(n)^{-1}ana^{-1} = n_1$, which has the same Jacobian as if $na\sigma(n)^{-1}a^{-1} = n_1$, which is $|\det(1 - \operatorname{Ad}(a)\sigma)|\operatorname{Lie} N|$, to get

$$\operatorname{tr}\pi(fdg \times \sigma) = \operatorname{tr}\pi(\sigma fdg) = \int_{T/T^{1-\sigma}} \eta(a)\Delta(a\sigma)da \int_{T^\sigma\backslash G} f(\sigma(g)^{-1}ag)\frac{dg}{da}.$$

□

1.7 Cases of H and $\mathbf{H}_1$. Considerations analogous to (1.3), (1.4) apply in the cases of the groups $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$ and $\mathbf{H}_1 = \mathrm{PGL}(2) \simeq \mathrm{SO}(3)$, with respect to the maximal compact subgroups $K_i = \mathbf{H}_i(R)$. Unramified representations π_0, π_1 are associated with $I_0(\mu_1, \mu_2)$, $I_1(\mu, \mu^{-1})$ and their classes are represented by

$$t_0 = \mathrm{diag}(z_1, z_2), \qquad t_1 = \mathrm{diag}(z, z^{-1})$$

in $\widehat{H}_0$, $\widehat{H}_1$. Here $z_i = \mu_i(\boldsymbol{\pi})$, $z = \mu(\boldsymbol{\pi})$. For $f_i dh_i$ in the Hecke algebras $\mathbb{H}_i$ of compactly supported K_i-biinvariant measures on H_i, the Satake transform is

$$\check{f}_0(\mathrm{diag}(z_1, z_2)) = |\mathbf{T}_0(R)| \sum_n F(n, f_0 dh_0)(z_1/z_2)^n,$$
$$\check{f}_1(\mathrm{diag}(z, z^{-1})) = |\mathbf{T}_1(R)| \sum_n F(n, f_1 dh_1) z^n.$$

The symbol $|\mathbf{T}_i(R)|$ denotes the volume of $\mathbf{T}_i(R) = T_i \cap K_i$ with respect to da_i. The expression $F(n, f_i dh_i)$ denotes the normalized orbital integral of $f_i dh_i$ at regular elements $\mathrm{diag}(a, b)$ in T_i (diagonal subgroup of H_i) with valuations $(n, -n)$ $(i = 0)$ and (m_1, m_2), $m_1 - m_2 = n$ $(i = 1)$. It depends on the choice of Haar measures dh_i, da_i on H_i, T_i; but $\check{f}_i$ depends only on dh_i.

The standard computation of (1.3) shows that for spherical $f_i dh_i$, π_i, we have

$$\mathrm{tr}\, \pi_i(f_i dh_i) = \check{f}_i(t_i) \qquad (t_i = t_i(\pi_i)).$$

Recall (1.1) that we have maps $\lambda_i \colon \widehat{H}_i \to \widehat{G}$ and ((1.2), (1.5)) classes t_i, t in $\widehat{H}_i$, $\widehat{G}$ for unramified representations π_i, π of H_i, H $(i = 0, 1)$.

1.8 Definition. The unramified representation π_i *lifts* to π through λ_i if $t = \lambda_i(t_i)$. In this case we write $\pi = \lambda_i(\pi_i)$.

The maps $\tilde{\lambda}_i^* \colon \mathbb{H} \to \mathbb{H}_i$ dual to λ_i are defined by $f_i dh_i = \tilde{\lambda}_i^*(f dg)$ if $\check{f}_i(t_i) = \check{f}(\lambda_i(t_i))$ for all t_i in $\widehat{T}_i$. Equivalently, $f_i dh_i = \tilde{\lambda}_i^*(f dg)$ if $\mathrm{tr}\, \pi_i(f_i dh_i) = \mathrm{tr}\, \pi(f dg \times \sigma)$ for all π_i and $\pi = \lambda_i(\pi_i)$. Note that $\pi = \lambda_i(\pi_i)$ if and only if $\check{f}_i(t_i) = \check{f}(t)$, where $t_i = t_i(\pi_i)$, $t = t(\pi)$, for all $f dg$ and $f_i dh_i = \tilde{\lambda}_i^*(f dg)$.

Note that $I_0(\mu) \stackrel{\mathrm{dfn}}{=} I_0(\mu, 1)$, $I_1(\mu) \stackrel{\mathrm{dfn}}{=} I_1(\mu, \mu^{-1})$ both lift (through λ_0, λ_1) to $I(\mu, 1, \mu^{-1})$.

There are several formal consequences concerning orbital integrals of measures fdg, $f_i dh_i$ related by $f_i dh_i = \tilde{\lambda}_i^*(fdg)$, as these integrals are the coefficients of $\check{f}$ and $\check{f}_i$.

1.9 LEMMA. *If*

$$\delta = \operatorname{diag}(a,b,c), \qquad \gamma = \operatorname{diag}(a/c, c/a), \qquad \textit{and} \qquad \gamma_1 = \operatorname{diag}(a,c),$$

then

$$F(\delta\sigma, fdg) = F(\gamma, f_0 dh_0) \qquad \textit{and} \qquad F(\delta\sigma, fdg) = F(\gamma_1, f_1 dh_1).$$

PROOF. If $t_1 = \operatorname{diag}(t, t^{-1})$ lies in $\widehat{T}_1$ then

$$|\mathbf{T}(R)| \sum_{\mathbf{m}=(m_1,m_2,m_3)} F(\mathbf{m}, fdg) t^{m_1-m_3} = \check{f}(\lambda_1(t_1))$$

$$= \check{f}_1(t_1) = |\mathbf{T}_1(R)| \sum_n F(n, f_1 dh_1) t^n.$$

Comparing coefficients of t^n we obtain

$$|\mathbf{T}_1(R)| F(n, f_1 dh_1) = \sum_{\{\mathbf{m}; m_1-m_3=n\}} |\mathbf{T}(R)| F(\mathbf{m}, fdg).$$

A simple change of variables shows that this is the product of $|\mathbf{T}^\sigma(R)|$, where

$$\mathbf{T}^\sigma(R) = \{t \in \mathbf{T}(R);\ t = \sigma(t)\},$$

and

$$F(n\sigma, fdg) = \Delta(\delta\sigma) \int f(g^{-1}\delta\sigma(g))\, dg,$$

where

$$\delta = \operatorname{diag}(a,b,c), \qquad \gamma = \operatorname{diag}(a/c, c/a), \qquad |a/c| = |\boldsymbol{\pi}|^n.$$

It is clear that the integral depends only on n, but not on the choice of δ.

In the case of $\mathbf{H}_0 = \mathrm{SL}(2)$, taking a representative $t_0 = (t,1)$ in $\widehat{T}_0$ we have

$$|\mathbf{T}(R)| \sum_{\mathbf{m}} F(\mathbf{m}, fdg) t^{m_1-m_3} = \check{f}(\lambda_0(t_0))$$

$$= \check{f}_0(t_0) = |\mathbf{T}_0(R)| \sum_n F(n, f_0 dh_0) t^n.$$

Hence $F(n\sigma, f dg) = F(n, f_0 dh_0)$. □

REMARK. (1) We normalize the measures so that $|\mathbf{T}_i(R)| = |\mathbf{T}^\sigma(R)|$; the groups T_i and T^σ are isomorphic to the multiplicative group $\mathbb{G}_m$.
(2) Every $\check{f}_1$ is so obtained from some $\check{f}$, hence the $\check{f}_1$ separate the π_1. Every $\check{f}_0$ is so obtained from some $\check{f}$, hence the $\check{f}_0$ separate the π_0.

I.2 Norms

2.1 Stability. To extend the study of lifting from the unramified case to any admissible σ-invariant representation, we need to define norm maps N and N_1 to extend the definition suggested by the formal Lemma 1.9 on diagonal matrices. Thus for $\delta = \mathrm{diag}(a, b, c)$ we put:

$$N(\delta) = \mathrm{diag}(a/c, c/a) \qquad \text{and} \qquad N_1(\delta) = \mathrm{diag}(a, c).$$

These norm maps will be used to relate orbital integrals and characters, so they should be defined in terms of (twisted) conjugacy classes. More precisely, the norm will be defined to be a map from the set of regular stable σ-conjugacy classes in G to the sets of regular stable conjugacy classes in H and H_1. We begin with a description of these classes.

Let F be a local or global field of characteristic 0. Fix an algebraic closure $\overline{F}$ of F. Let $\mathbf{G}$ be a reductive group defined over F and $G = \mathbf{G}(F)$ the group of F-rational points of $\mathbf{G}$. Denote by σ an automorphism of $\mathbf{G}$ defined over F. The elements δ, δ' of G are called *σ-conjugate* if there is h in G with $\delta' = h\delta\sigma(h^{-1})$. They are called *stably σ-conjugate* if there is h in $\mathbf{G}(\overline{F})$ with $\delta' = h\delta\sigma(h^{-1})$. The term (stable) conjugacy (no mention of σ) is employed if σ is the trivial automorphism.

The stable σ-conjugates of δ in G are described by the set $A(\delta)$ of g in $\mathbf{G}(\overline{F})$ with $g\delta\sigma(g^{-1})$ in G. The map

$$A(\delta) \xrightarrow{\alpha'} H^1(F, Z_{\mathbf{G}}(\delta\sigma)), \qquad g \mapsto \{\tau \mapsto g_\tau = g^{-1}\tau(g)\},$$

where

$$Z_{\mathbf{G}}(\delta\sigma) = \{g \in \mathbf{G}; g\delta\sigma(g^{-1}) = \delta\},$$

factors through

$$1 \longrightarrow D(\delta) \xrightarrow{\alpha} H^1(F, Z_{\mathbf{G}}(\delta\sigma)) \longrightarrow H^1(F, \mathbf{G}),$$

where the double coset space $D(\delta) = G\backslash A(\delta)/Z_{\mathbf{G}}(\delta\sigma)(\overline{F})$ parametrizes the σ-conjugacy classes within the stable σ-conjugacy class of δ.

The definitions introduced above will be used with $\mathbf{G} = \mathrm{PGL}(3)$ and the (involution) outer automorphism $\sigma(g) = J{}^t g^{-1} J$, and also with $\mathbf{H} = \mathbf{H}_0 = SL(2)$, $\mathbf{H}_1 = \mathrm{PGL}(2) = \mathrm{SO}(3)$ and the trivial σ. If $\gamma \in H$, $Z_{\mathbf{H}}(\gamma)$ denotes the centralizer of γ in $\mathbf{H}$. Similarly, $Z_{\mathbf{H}_1}(\gamma_1)$ is the centralizer of $\gamma_1 \in H_1$ in $\mathbf{H}_1$.

Note that every conjugacy class of H_1 (and of $\mathrm{GL}(n,F)$ or $\mathrm{PGL}(n,F)$) is stable. Indeed, the centralizer of a semisimple element γ in $\mathrm{GL}(n,F)$ is a product $\prod_j E_j^\times$, where E_j are field extensions of F with $\sum_j [E_j : F] = n$. We have $H^1(F, \mathbb{G}_m) = \{0\}$, hence $D(\gamma)$ is trivial for $\mathrm{GL}(n,F)$ or $\mathrm{PGL}(n,F)$.

However, for $\mathbf{H} = \mathrm{SL}(2)$, the centralizer in H of a nonsplit γ is $E^1 = \ker N_{E/F}$, where $E = F(\gamma)$ is the extension generated by γ. Hence the set of conjugacy classes within the stable conjugacy class of a regular γ in H is parametrized by $F^\times/NE^\times$, which is $\mathbb{Z}/2\mathbb{Z}$ when F is local and γ is elliptic, and $\{0\}$ when the eigenvalues of γ are in $F^\times$. For this we need to compute $H^1(F, \mathbf{T}) = H^1(\mathrm{Gal}(E/F), E^\times)$ where $\mathbf{T}$ is $\mathbb{G}_m$ over E and $\sigma \neq 1$ in $\mathrm{Gal}(E/F)$ acts on $\mathbf{T}(E) = E^\times$ by $\sigma(x) = \overline{x}^{-1}$ ($\overline{x}$ is the conjugate of x in E over F). Then a cocycle is $b = b_\sigma \in E^\times$ with $1 = b_{\sigma^2} = b_\sigma \sigma(b_\sigma) = b/\overline{b}$, thus $b \in F^\times$. The coboundaries are $b/\sigma(b) = b\overline{b}$, thus $N_{E/F}E^\times$.

There is of course an easy way in the case of $\mathrm{SL}(2,F)$ (and more generally $\mathrm{SL}(n,F)$) to realize the stable conjugacy in $\mathrm{GL}(n,F)$. If $E = F(\sqrt{A})$, a γ in H splitting over E, thus with eigenvalues $a \pm b\sqrt{A}$, is equal to $\begin{pmatrix} a & bA \\ b & a \end{pmatrix}$ up to stable conjugacy. A γ' in H stably conjugate but not conjugate to γ has the same eigenvalues as γ, hence it is conjugate to γ in $\mathrm{GL}(2,F)$, thus it is conjugate in $\mathrm{SL}(2,F)$ to $\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & bA \\ b & a \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}$, where $D \in F^\times - N_{E/F}E^\times$. Indeed, if $D \in NE^\times$ then $\mathrm{diag}(D,1) \in T(F)\,\mathrm{SL}(2,F)$ where $T(F)$ is the centralizer of γ in $\mathrm{GL}(2,F)$.

To realize γ' as $g^{-1}\gamma g$, $g \in \mathrm{SL}(2,\overline{F})$, we solve $\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} x & yA \\ y & x \end{pmatrix}$, $x = x_1' + x_2'\sqrt{A} \in E$, $y = y_1' + y_2'\sqrt{A} \in E$, thus $x^2 - y^2 A = D$. The solutions are $x = x_1(x_2 + 1) + x_2\sqrt{A}$, $y = x_2 + 1 + \frac{x_1 x_2}{A}\sqrt{A}$, provided

$2x_2+1=\frac{D}{x_1^2-A}$. We take $x_1=0$. Then $x_2=-\frac{1}{2}(\frac{D}{A}+1)$, $x=x_2\sqrt{A}$, $y=x_2+1=\frac{1}{2}(1-\frac{D}{A})$. Then

$$g=\frac{1}{D}\begin{pmatrix} -\frac{1}{2}(\frac{D}{A}+1)\sqrt{A} & -\frac{A}{2}(1-\frac{D}{A}) \\ \frac{-1}{2}(1-\frac{D}{A}) & -\frac{1}{2}(\frac{D}{A}+1)\sqrt{A}\end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 1\end{pmatrix}$$

satisfies

$$g_\sigma=g\sigma(g)^{-1}=h_0^{-1}\begin{pmatrix} -A/D & 0 \\ 0 & -D/A\end{pmatrix}h_0^{-1}$$

where

$$h_0=\begin{pmatrix} 1 & \sqrt{A} \\ \frac{-1}{2\sqrt{A}} & \frac{1}{2}\end{pmatrix},\qquad \begin{pmatrix} a & bA \\ b & a\end{pmatrix}=h_0^{-1}\begin{pmatrix} a+bA & 0 \\ 0 & a-bA\end{pmatrix}h_0,$$

$$h_0\sigma(h_0)^{-1}=\begin{pmatrix} 0 & 2\sqrt{A} \\ \frac{-1}{2\sqrt{A}} & 0\end{pmatrix},\qquad h_0g\sigma(h_0g)^{-1}=\begin{pmatrix} 0 & \frac{-2A\sqrt{A}}{D} \\ \frac{D}{2A\sqrt{A}} & 0\end{pmatrix},$$

and

$$h_0g=\begin{pmatrix} \sqrt{A}/D & 0 \\ 0 & 1/\sqrt{A}\end{pmatrix}h_0\begin{pmatrix} D & 0 \\ 0 & 1\end{pmatrix}$$

so that g satisfies $\gamma'=g^{-1}\gamma g$.

2.2 The Norm. Let δ be an element of G. The set of eigenvalues of $\delta\sigma(\delta)$ is of the form $\{\lambda, 1, \lambda^{-1}\}$. Indeed, if λ is an eigenvalue of $\delta\sigma(\delta)$ then there is an eigenvector v with ${}^t(\delta\sigma(\delta))v=\lambda v$. Hence

$$\lambda^{-1}v={}^t(\delta\sigma(\delta))^{-1}v,\qquad\text{and}\qquad \lambda^{-1}(\delta Jv)=\delta J{}^t\delta^{-1}J(\delta Jv),$$

that is, λ^{-1} is also an eigenvalue. It is clear that $\lambda\in F^\times$ or that $[F(\lambda):F]=2$.

The element δ of G is called *σ-regular* if the eigenvalues λ, 1, λ^{-1} of $\delta\sigma(\delta)$ are distinct. In this case *let $N\delta$ be the class in H determined by the eigenvalues λ, λ^{-1}* and *$N_1\delta$ the class in H_1 with eigenvalues of ratio λ* if $\mathbf{H}_1$ is viewed as PGL(2), or *with eigenvalues λ, 1, λ^{-1}* if $\mathbf{H}_1$ is viewed as SO(3).

For any $h=\begin{pmatrix} x & y \\ z & t\end{pmatrix}$ in $\mathrm{GL}(2,F)$ we put

$$h_1=\begin{pmatrix} x & 0 & y \\ 0 & 1 & 0 \\ z & 0 & t\end{pmatrix},\qquad e=\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix},\qquad w=\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

Assume that δ is σ-regular. Replacing $\delta\sigma(\delta)$ by a conjugate $g^{-1}\delta\sigma(\delta)g$, hence δ by a σ-conjugate $g^{-1}\delta\sigma(g)$, we may assume that $\delta\sigma(\delta)$ is of the form

h_1. Since δJ takes λ-eigenvectors of ${}^t(\delta\sigma(\delta))$ to λ^{-1}-eigenvectors of $\delta\sigma(\delta)$, the assumption $\delta\sigma(\delta) = h_1$ implies that δJ fixes the subspaces $\begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix}, \begin{pmatrix} * \\ 0 \\ * \end{pmatrix}$. So does δ. Hence multiplying by a scalar we have $\delta = a_1$ for some a in $\mathrm{GL}(2, F)$.

Note that if $\delta = (ae)_1$, then $N\delta = h_1$; here

$$h = aew{}^ta^{-1}ew = \frac{-1}{\det a}a^2.$$

If $\delta' = (a'e)_1$ and $\delta' = \beta^{-1}\delta\sigma(\beta)$ [hence $\delta'\sigma(\delta') = \beta^{-1}\delta\sigma(\delta)\beta$ and $\beta = b_1$ for some b in $\mathrm{GL}(2, F)$], then $a'e = b^{-1}aew{}^tb^{-1}w$ and

$$a' = b^{-1}a(ew){}^tb^{-1}(ew)^{-1} = \frac{1}{\det b}b^{-1}ab.$$

Hence δ, δ' are (stably) σ-conjugate if and only if a, a' are projectively (stably) conjugate.

2.3 LEMMA. *For any given regular γ in H there is a unique stable σ-conjugacy class of δ with $N\delta = \gamma$. The σ-conjugacy classes within such a stable class are parametrized by u in $F^\times/NE^\times$, $E = F(\delta\sigma(\delta))$. A set of representatives is given by $\delta = (uae)_1$.*

PROOF. If the eigenvalues $\lambda, 1, \lambda^{-1}$ of $\delta\sigma(\delta)$ are distinct then they lie in a quadratic extension of F (or in F) and define a stable conjugacy class $N\delta$ in H with eigenvalues λ, λ^{-1}, and a conjugacy class $N_1\delta$ in H_1 with eigenvalues λ, 1, λ^{-1} in $\mathrm{SO}(3, F)$ or λ, 1 in $\mathrm{PGL}(2, F)$. Given λ there exist α, β in $F(\lambda)^\times$ with $\alpha/\beta = -\lambda$. Here $\beta = \overline{\alpha}$ and we use Hilbert Theorem 90 if $\lambda \notin F$. The pair α, β is determined up to a multiple by a scalar u in $F^\times$. The matrix $\delta\sigma(\delta)$ (where $\delta = (ae)_1$) has eigenvalues λ, 1, λ^{-1} iff a has eigenvalues α, β so that $\frac{-1}{\det a}a^2$ has eigenvalues $-\alpha/\beta$, $-\beta/\alpha$. Hence the norm map is onto the set of regular elements of H, and the δ in G with a fixed regular $N\delta$ make a single stable σ-conjugacy class, as a and ua (u in $F^\times$) are projectively stably conjugate.

But a and $a' = u^{-1}a$ are projectively conjugate only if $u^{-1}a = \frac{1}{\det b}b^{-1}ab$ for some b in $\mathrm{GL}(2, F)$. Then $u^2 = \det b^2$, and $u = \pm\det b$. If $u = -\det b$ then $-a = b^{-1}ab$, a has eigenvalues γ, $-\gamma$ and $h = I$ does not have eigenvalues different than 1. Hence $u = \det b$, $a = b^{-1}ab$ and $u = \det b$ lies in $N_{E/F}E^\times$, where $E = F(a)$. □

Thus the norm map has a particularly simple description in the case where $\delta\sigma(\delta)$ has distinct eigenvalues. Up to a σ-conjugacy such δ can be assumed to be of the form $\delta = (ae)_1$. Then $\gamma = N\delta = (-1/\det a)a^2$.

2.3.1 COROLLARY. *Let F be a global field, u a place of F, and δ, δ' stably σ-conjugate but non-σ-conjugate elements of $\mathbf{G}(F)$. Then there is a place $v \neq u$ of F such that δ, δ' are not σ-conjugate in $\mathbf{G}(F_v)$.*

2.4 DEFINITION. If $N\delta$ is regular put $\tilde{\delta} = \frac{1}{2}[\delta J + {}^t(\delta J)]J$. Note that $\tilde{\delta}\sigma(\tilde{\delta}) = 1$. Hence $\tilde{\delta}J$ is symmetric $(= {}^t(\tilde{\delta}J))$. Define $\boldsymbol{\kappa}(\delta)$ to be 1 if $\mathrm{SO}(3, \tilde{\delta}J)$ is split and -1 if not.

The function $\boldsymbol{\kappa}$ depends only on the σ-conjugacy class of δ. Indeed if δ is replaced by $\beta\delta J {}^t\beta J$ then $\delta J + {}^t(\delta J)$ is replaced by

$$\beta\delta J {}^t\beta + \beta J {}^t\delta {}^t\beta = \beta[\delta J + {}^t(\delta J)]{}^t\beta,$$

and the form $\delta J + {}^t(\delta J)$ splits if and only if $\beta[\delta J + {}^t(\delta J)]{}^t\beta$ does.

If δ, δ' are stably σ-conjugate with regular norm, but they are not conjugate, then the forms $\tilde{\delta}J$ and $\tilde{\delta}'J$ are not equivalent, and $\boldsymbol{\kappa}(\delta') = -\boldsymbol{\kappa}(\delta)$. Thus if $\delta = (ae)_1$ and $\delta' = (uae)_1$, then $\boldsymbol{\kappa}(\delta') = \chi(u)\boldsymbol{\kappa}(\delta)$, χ being the quadratic character of $F^\times$ trivial on $NE^\times$, $E = F(\delta\sigma(\delta))$.

If $N\delta = \gamma$ is regular in H then $Z_{\mathbf{G}}(\delta\sigma) \simeq Z_{\mathbf{H}}(\gamma)$. Indeed, if

$$g^{-1}\delta\sigma(g) = \delta \qquad \text{then} \qquad g^{-1}\delta\sigma(\delta)g = \delta\sigma(\delta);$$

if $\delta = (ae)_1$ then $g = b_1$ and $b^{-1}ab = a$, since $\delta\sigma(\delta) = h_1$, $h = \frac{-1}{\det a}a^2$. Hence

$$b^{-1}aew{}^tb^{-1}we = a, \qquad \text{namely} \qquad \frac{1}{\det b}b^{-1}ab = a,$$

so that $\det b = 1$. It is clear that $Z_{\mathbf{H}}(\gamma) = Z_{\mathbf{H}}(a)$.

The norm map can be extended to classes of δ in G which are not σ-regular. This is done next.

2.5 Identity. We now deal with the (two) cases where all eigenvalues of $\delta\sigma(\delta)$ are 1.

If $\delta\sigma(\delta) = 1$ we write $N\delta = 1$ and $N_1\delta = 1$. Then $\delta J = {}^t(\delta J)$ is symmetric, any two symmetric matrices are equivalent over F, hence for each δ' with $\delta'\sigma(\delta') = 1$ there is S in G with $\delta J = S\delta' J {}^tS$, so that $\delta = S\delta'\sigma(S^{-1})$, and the δ with $\delta\sigma(\delta) = 1$ form a single stable σ-conjugacy class.

For such δ the σ-centralizer

$$Z_{\mathbf{G}}(\delta\sigma) \qquad \text{is} \qquad (\mathrm{PO}(3,\delta J) =) \qquad \mathrm{SO}(3,\delta J),$$

the (projective =) special orthogonal group with respect to the form δJ. Replacing δ by a σ-conjugate $u\delta\sigma(u^{-1})$ or δJ by $u\delta J{}^t u$, implies replacing $Z_{\mathbf{G}}(\delta\sigma)$ by its conjugate $uZ_{\mathbf{G}}(\delta\sigma)u^{-1}$. Hence if F is $\mathbb{R}$ or p-adic then there are two σ-conjugacy classes in the stable σ-conjugacy class of the δ with $N\delta = 1$, corresponding to the split and nonsplit forms δJ. Put $\boldsymbol{\kappa}(\delta) = 1$ if $Z_{\mathbf{G}}(\delta\sigma) = \mathrm{SO}(3,\delta J)$ splits and $\boldsymbol{\kappa}(\delta) = -1$ if it is anisotropic. If we put $\gamma = N\delta$ $(= 1)$ then there is a natural surjection

$$\varphi : Z_{\mathbf{H}}(\gamma) = \mathrm{SL}(2) \to Z_{\mathbf{G}}(\delta\sigma) = \mathrm{SO}(3,\delta J)$$

with kernel $\{\pm 1\}$. The morphism φ is defined over F only if $\mathrm{SO}(3,\delta J)$ is split.

2.6 Unipotent. If $\delta\sigma(\delta)$ is unipotent but not 1 we check by matrix multiplication that it is a regular unipotent (not conjugate to $\left(\begin{smallmatrix}1&0&1\\0&1&0\\0&0&1\end{smallmatrix}\right)$). Alternatively, $\delta\sigma(\delta)v = v$ if and only if $(\delta J - {}^t(\delta J))w = 0$, where $w = {}^t(\delta J)^{-1}v$. Thus the 1-eigenspace of $\delta\sigma(\delta)$ has the same dimension as the zero-eigenspace of the skew-symmetric matrix $\delta J - {}^t(\delta J)$, namely 1 or 3, and $\delta\sigma(\delta) \neq 1$ is regular unipotent. Up to stable σ-conjugacy we may assume that $\delta\sigma(\delta) = \left(\begin{smallmatrix}1&1&1/2\\0&1&1\\0&0&1\end{smallmatrix}\right)$, a σ-invariant matrix. Hence δ commutes with $\sigma(\delta)$ and $\delta\sigma(\delta)$, and it is unipotent of the form $\left(\begin{smallmatrix}1&x&y\\0&1&x\\0&0&1\end{smallmatrix}\right)$. These make a single σ-conjugacy class. The σ-centralizer $Z_{\mathbf{G}}(\delta\sigma)$ is the additive group $\mathbb{G}_a$, $H^1(F,\mathbb{G}_a)$ is trivial, hence there is a unique σ-conjugacy class of δ with $\delta\sigma(\delta) =$ unipotent $\neq 1$, and we put $N\delta =$ unipotent in H.

If $\gamma = N\delta$ is unipotent then $Z_{\mathbf{H}}(\gamma) = \{\pm 1\} \times \mathbb{G}_a$ and there is a natural surjection $\varphi : Z_{\mathbf{H}}(\gamma) \to Z_{\mathbf{G}}(\delta\sigma)$ with kernel $\{\pm 1\}$.

2.7 Negative identity. It remains to deal with the case where two eigenvalues of $\delta\sigma(\delta)$ are -1. Here $Z_{\mathbf{G}}(\delta\sigma) \simeq Z_{\mathbf{H}}(\gamma)$, as we see next.

If $\delta\sigma(\delta) = h_1$ and $h = -I$ in $\mathrm{GL}(2,F)$ then $a^2 = \det a$ $(\delta = (ae)_1)$ and a is a scalar $\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)$. We put $N\delta = -I$, and note that all δ with $N\delta = -I$ form a single σ-conjugacy class, since

$$\begin{pmatrix}\alpha&0\\0&\alpha\end{pmatrix} = \frac{\alpha}{\beta}\begin{pmatrix}\alpha/\beta&0\\0&1\end{pmatrix}\begin{pmatrix}\beta&0\\0&\beta\end{pmatrix}\begin{pmatrix}\beta/\alpha&0\\0&1\end{pmatrix}.$$

2.8 Negative unipotent. If $\delta\sigma(\delta) = h_1$ and $h = -$unipotent $\neq -I$ in $\mathrm{GL}(2, F)$, then up to conjugacy $h = -\begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix}$, hence $a = u^{-1}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ with $\alpha \in F^\times$, $u \in F^\times$. But a is equal to

$$\frac{1}{u}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & \alpha u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

hence it is projectively conjugate to

$\begin{pmatrix} 1 & \alpha u \\ 0 & 1 \end{pmatrix}$. Now $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ $(\alpha, \beta \in F^\times)$

are (projectively) conjugate only if α/β is a square in $F^\times$; they are clearly stably conjugate. Hence the σ-conjugacy classes within the single stable σ-conjugacy class of our δ are parametrized by $F^\times/F^{\times 2}$. If

$$\delta = (ae)_1, \qquad a = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \qquad \alpha \neq 0,$$

we let $N\delta$ be the stable conjugacy class of h in H, and define $N_1\delta$ to be the conjugacy class in H_1 of elements which generate $F(\sqrt{\alpha})$ over F, and the quotient of whose eigenvalues is -1. Such an element of $\mathrm{GL}(2, F)$ is $\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$.

I.3 Local lifting

3.1 Orbital integrals. Let F be a local field. Fix a Haar measure dg on G. For any σ-regular δ, the σ-centralizer $Z_G(\delta\sigma)$ of δ in G is a torus. Fix a Haar measure dt on it. If δ' in G is stably σ-conjugate to δ, $Z_G(\delta\sigma)$ is isomorphic to $Z_G(\delta'\sigma)$. We choose dt and dt' on these groups to assign their maximal compact subgroups the same volumes. The measures dg, dt determine a measure on the quotient $G/Z_G(\delta\sigma)$. Let $f \in C_c^\infty(G)$ be a smooth compactly supported function on G. Put

$$\Phi(\delta\sigma, f dg) = \int_{G/Z_G(\delta\sigma)} f(g\delta\sigma(g)^{-1})\frac{dg}{dt}.$$

If δ is σ-regular, put

$$\Phi^{st}(\delta\sigma, f dg) = \sum_{\delta'} \Phi(\delta'\sigma, f dg).$$

The sum is over a set of representatives for the σ-conjugacy classes in the stable σ-conjugacy class of δ.

If f_0 is a smooth compactly supported function on H define

$$\Phi(\gamma, f_0 dh) = \int_{H/Z_H(\gamma)} f_0(h\gamma h^{-1})\frac{dh}{dt}.$$

Here dh is a Haar measure on H and dt on the centralizer $Z_H(\gamma)$. Also put

$$\Phi^{\mathrm{st}}(\gamma, f_0 dh) = \sum_{\gamma'} \Phi(\gamma', f_0 dh).$$

If $\gamma = N\delta$ is regular then $Z_H(\gamma) \simeq Z_G(\delta\sigma)$. The measures on the two groups are related by assigning the maximal compact subgroup the same volume.

The measures fdg and $f_0 dh$ are said to have *matching orbital integrals* and we write $f_0 dh = \lambda^*(fdg)$ if for all γ, δ with regular $\gamma = N\delta$ they satisfy the relation

$$\Phi^{\mathrm{st}}(\gamma, f_0 dh) = \Phi^{\mathrm{st}}(\delta, fdg).$$

3.2 Weyl integration formula. Let $\{T_0\}$ denote a set of representatives for the conjugacy classes of tori of H over F. The regular set H^{reg} of H (distinct eigenvalues) is the union over $\{T_0\}$ of $\mathrm{Int}(H/T_0)(T_0^{\mathrm{reg}})$. The Jacobian of the morphism

$$T_0 \times H/T_0 \to H, \qquad (t,h) \mapsto \mathrm{Int}(h)t = hth^{-1},$$

is

$$D_0(t) = |\det(1 - \mathrm{Ad}\ t)|\,\mathrm{Lie}(H/T_0)|.$$

We have the Weyl integration formula

$$\int_H f_0(h)\,dh = \sum_{\{T_0\}} |W(T_0)|^{-1} \int_{T_0} \Delta_0(t)^2\,dt \int_{H/T_0} f_0(hth^{-1})\,\frac{dh}{dt}.$$

Here $W(T_0)$ is the Weyl group of T_0 (normalizer/centralizer), and $\Delta_0(t)^2 = D_0(t)$. It is $\mathbb{Z}/2$ if T_0 splits over F or -1 lies in $N_{E/F}E^\times$, and $\{0\}$ otherwise, as the normalizer of $T_0 \simeq E^1$ is $x \mapsto \overline{x}$, realized by $\mathrm{Int}(\mathrm{diag}(-1,1))$ with the choices of section 2.1.

Let $\{T_0\}_s$ denote a set of representatives for the stable conjugacy classes of tori of H over F. It consists of a representative, say the diagonal torus, for the tori which split over F, and elliptic tori, which are parametrized by the quadratic field extensions E of F, where $T_0 = E^1$. The Weyl group of T_0 in $A(T_0)$ (see section 2.1) is $\mathbb{Z}/2$. Hence

$$\int_H f_0(h)\,dh = \frac{1}{2}\sum_{\{T_0\}_s}\int_{T_0}\Delta_0(t)^2\,dt\sum_{t'}\int_{H/Z_H(t')} f_0(ht'h^{-1})\,\frac{dh}{dt}.$$

The sum over t' ranges for a set of representatives for the conjugacy classes within the stable conjugacy class of t in T_0.

Next we write an analogue of the Weyl integration formula in the twisted case. We use the observation of (1.9) that each σ-regular element in G is σ-conjugate to an element $\delta = (ae)_1$ with a in $\mathrm{GL}(2,F)$. Recall that $\delta = (ae)_1$ and $\delta' = (a'e)_1$ are σ-conjugate if and only if $a' = (1/\det b)b^{-1}ab$. Hence we may take the a in $N\mathbf{Z}(E)\backslash T_E$, where T_E ranges over a set of representatives for the conjugacy classes of tori T_2 of GL(2) over F. If T_2 splits over E (= F or a quadratic extension of F), we denote it by T_E. We denote by $\mathbf{Z}$ the center of GL(2) and by N the norm map form E to F.

Every σ-regular element of G has the form

$$g\delta\sigma(g)^{-1}, \qquad \delta \in T = T(T_E/N\mathbf{Z}(E)), \qquad g \in G/Z_G(T\sigma),$$

for some E. Here

$$T(T_E/N\mathbf{Z}(E)) = \{\delta_a = (ae)_1; a \in T_E/N\mathbf{Z}(E)\}.$$

The σ-centralizer of T in G,

$$Z_G(T\sigma) = \{g \in G; g\delta\sigma(g)^{-1} = \delta, \forall\delta \in T\},$$

is isomorphic to $Z_H(NT)$ where $NT = N(T) = T_E^1 (= T_E \cap \mathrm{SL}(2,F))$.

The expression is unique up to the action of the σ-normalizer, which consists of the w with $w^{-1}\delta\sigma(w) = \delta' = \delta'(\delta) \in T$ for all δ in T. Then $w^{-1}\delta\sigma(\delta)w = \delta'\sigma(\delta')$. Modulo the centralizer there are two w's, $w = (e)_1$ represents the nontrivial one with the choices made in section 2.1.

The Jacobian of the morphism

$$T \times G/Z_G(T\sigma) \to G, \qquad (\delta, g) \mapsto g\delta\sigma(g)^{-1}$$

is

$$\Delta(\delta\sigma)^2 = |\det[1 - \mathrm{Ad}(\delta)\sigma] | \operatorname{Lie}(G/T^\sigma)|.$$

The twisted Weyl integration formula is then (put δ_a for $(ae)_1$)

$$\int_G f(g)\, dg = \frac{1}{2}\sum_E \int_{T_E/N\mathbf{Z}(E)} \Delta(\delta_a\sigma)^2 da \int_{G/Z_G(T\sigma)} f(g\delta_a\sigma(g)^{-1})\frac{dg}{da}.$$

Let us compute $\Delta(\delta\sigma)^2$ explicitly. We may assume δ is $\mathrm{diag}(a,b,c)$. $\operatorname{Lie}\mathbf{G}$ consists of $X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$ modulo center. Thus we assume that $x_5 = 0$ to fix representatives. Note that $\operatorname{Lie} Z_{\mathbf{G}}(\delta\sigma) = \{\mathrm{diag}(x,0,-x)\}$, since

$$-\sigma X = J^t X J = \begin{pmatrix} x_9 & -x_6 & x_3 \\ -x_8 & x_5 & -x_2 \\ x_7 & -x_4 & x_1 \end{pmatrix},$$

$$X - \mathrm{Ad}(\delta)\sigma X = \begin{pmatrix} x_1+x_9 & x_2-\frac{a}{b}x_6 & (1+\frac{a}{c})x_3 \\ x_4-\frac{b}{a}x_8 & 2x_5 & x_6-\frac{b}{c}x_2 \\ (1+\frac{c}{a})x_7 & x_8-\frac{c}{b}x_4 & x_1+x_9 \end{pmatrix}.$$

Recalling that $x_5 = 0$, and noting that in $\operatorname{Lie}\mathbf{G}/Z_{\mathbf{G}}(\delta\sigma)$ the $x_1 + x_9$ is a single variable (alternatively, in X we could replace x_9 by zero and x_1 by $x_1 + x_9$), we conclude that

$$\Delta(\delta\sigma)^2 = \left|\left(1-\frac{a}{c}\right)\left(1+\frac{a}{c}\right)\left(1-\frac{c}{a}\right)\left(1+\frac{c}{a}\right)\right|.$$

The 4 factors correspond to change of variables on: (x_2, x_6), x_3, (x_4, x_8), x_7. This $\Delta(\delta\sigma)^2$ is then equal to $\Delta_0(\gamma)^2$, $\gamma = N\delta$. Indeed we may assume that $\gamma = \mathrm{diag}(a/c, c/a)$, and then

$$\Delta_0(\gamma)^2 = \left|\left(\frac{a}{c} - \frac{c}{a}\right)^2\right| = \left|\frac{a^2-c^2}{ac}\right|^2 = \Delta(\delta\sigma)^2.$$

3.3 Characters. Let F be a local (archimedean or not) field, f_i a compactly supported smooth function on H_i, π_i an admissible irreducible representation of H_i, and $\pi_i(f_i dh_i)$ the convolution operator $\int f_i(g)\pi_i(g)\, dg$. This operator has finite rank, see (1.3).

A well-known result of Harish-Chandra ([HC2]) asserts that there exists a complex-valued conjugacy-class function $\chi_i = \chi_{\pi_i}$ on H_i which is smooth on the regular set such that for all measures $f_i dh_i$ on the regular set

$$\operatorname{tr} \pi_i(f_i dh_i) = \int f_i(g)\chi_i(g)\, dg.$$

It is called the *character* of π_i. It is locally integrable on H_i.

The twisted analogue of [HC2] (see [Cl2]) asserts that given a σ-invariant admissible irreducible representation π of G, there exists a complex-valued σ-conjugacy class function $\chi_\pi^\sigma : g \mapsto \chi_\pi(g\sigma)$ on G which is smooth on the σ-regular set, such that

$$\operatorname{tr} \pi(f dg \times \sigma) = \int f(g)\chi_\pi^\sigma(g)\, dg$$

for all measures $f dg$ on the σ-regular set. It is called the *twisted character* of π. It is locally integrable on G, hence the identity extends to all measures $f dg$.

Note that χ_π^σ is the *twisted* character of π. It is not the character in the usual sense. We also write $\chi_\pi(g\sigma)$ for $\chi_\pi^\sigma(g)$. Note that the (twisted) character is defined only on the (σ-) regular set. We need the character and its properties for the orthogonality relations, as well as for the study of the approximation in section V.1, and lifting in section V.2.

A function χ on H is called a conjugacy *class function* if $\chi(h) = \chi(h')$ whenever h, h' are regular and conjugate in H. For example, characters of representations are class functions. We shall later show that characters are dense in the space of class functions. A class function is called a *stable* class function if $\chi(h) = \chi(h')$ whenever h, h' are regular and stably conjugate in H ($h' = ghg^{-1}$ for some $g \in \mathbf{H}(\overline{F})$).

Let $\{\pi_0\}$ be a set of irreducible admissible representations of H such that $\chi_{\{\pi_0\}}$, the sum of $\chi_{\pi_0'}$ where π_0' ranges over $\{\pi_0\}$, is a stable class function. We say that $\{\pi_0\}$ is a *stable* set. Similar definition can be made for a set with multiplicities. But in our case it turns out that the stable class functions that we need are all of the form $\chi_{\{\pi_0\}}$. Note that $\chi_{\{\pi_0\}}$ is the character of the reducible admissible representation $\oplus\pi_0'$, sum over the π_0' in $\{\pi_0\}$.

DEFINITION. The representation π_0 of H_0 *lifts* to the representation π of G if π is σ-invariant and there is a stable set $\{\pi_0\}$ including π_0 such that whenever $\gamma = N\delta$ is a regular element of H we have

$$\chi_\pi(\delta\sigma) = \chi_{\{\pi_0\}}(\gamma).$$

In this case we write $\pi = \lambda_0(\pi_0)$ or $\pi = \lambda_0(\{\pi_0\})$.

REMARK. This definition is based on the definition of the norm N in (2.2). The norm relates stable σ-conjugacy classes in G and stable conjugacy classes in H. To lift, $\gamma \mapsto \chi_{\{\pi_0\}}(\gamma)$ has to be a stable class function. To be a lift of π_0 the twisted character χ_π^σ of π has to be a stable σ-class function, namely $\chi_\pi^\sigma(\delta) = \chi_\pi^\sigma(\delta')$ if δ and δ' are stably σ-conjugate.

3.4 LEMMA. *We have $\pi = \lambda_0(\pi_0)$ if and only if for all fdg, f_0dh with $f_0dh = \lambda_0^*(fdg)$ we have $\operatorname{tr}\pi(fdg \times \sigma) = \operatorname{tr}\{\pi_0\}(f_0dh)$.*

PROOF. Suppose that $\operatorname{tr}\pi(fdg \times \sigma) = \operatorname{tr}\{\pi_0\}(f_0dh)$. We use the Weyl integration formula of (3.2) to write $\operatorname{tr}\pi(fdg \times \sigma) = \int f(g)\chi_\pi^\sigma(g)\,dg$ as

$$\sum_{\{E\}} \frac{1}{2} \int_{N\mathbf{Z}(E)\backslash T_E} \Delta_0(\gamma)^2 \chi_\pi^\sigma((ae)_1)\Phi((ae)_1\sigma, fdg)da.$$

Fix a quadratic extension E of F. Denote by T_E the element of $\{T_2\}$ (i.e., a torus in $\mathrm{GL}(2,F)$) which splits over E. Take fdg so that its twisted orbital integral $\Phi(\delta\sigma, fdg)$ is supported on T_E, namely on the σ-orbits of the $\delta_a = (ae)_1$ with a in T_E. We claim that

$$\operatorname{tr}\pi(fdg \times \sigma) = \frac{1}{2}\int_{Z\backslash T_E} \Delta_0(\gamma)^2\chi_\pi^\sigma(\delta)\Phi^{\mathrm{st}}(\delta\sigma, fdg)da \qquad (\delta = (ae)_1),$$

where $\Phi^{\mathrm{st}}(\delta\sigma, fdg)$ denotes the stable twisted orbital integral of fdg at δ, as in (3.1). To show this, note that the trace $\operatorname{tr}\pi(fdg \times \sigma)$ depends only on the stable twisted orbital integral of fdg, since it is equal to $\operatorname{tr}\{\pi_0\}(f_0dh)$. If we take $f_0 = 0$, then for each a in T_E we have

$$\Phi((uae)_1\sigma, fdg) = -\Phi((ae)_1\sigma, fdg) \qquad (u \in F - N_{E/F}E).$$

Since $\operatorname{tr}\pi(fdg \times \sigma)$ vanishes for such fdg, we have

$$\int_{Z\backslash T_E} \Delta_0(\gamma)^2[\chi_\pi^\sigma((ae)_1) - \chi_\pi^\sigma((uae)_1)]\Phi((ae)_1\sigma, fdg)da = 0.$$

Choosing fdg so that the support of $\Phi((ae)_1\sigma, fdg)$ is small, we deduce that

$$\chi_\pi^\sigma((ae)_1) = \chi_\pi^\sigma((uae)_1)$$

depends only on the stable σ-conjugacy class of $(ae)_1$. Hence the claim follows.

On the other hand,

$$\begin{aligned}\operatorname{tr}\{\pi_0\}(f_0dh) &= \int f_0(g)\chi_{\{\pi_0\}}(g)\,dg \\ &= \sum_{\{T_0\}} [W(T_0)]^{-1} \int_{T_0} \Delta_0(\gamma)^2 \chi_{\{\pi_0\}}(\gamma)\Phi(\gamma, f_0dh)d\gamma \\ &= \frac{1}{2}\int_{T_{0E}} \Delta_0(\gamma)^2\chi_{\{\pi_0\}}(\gamma)\Phi^{\mathrm{st}}(\gamma, f_0dh)d\gamma.\end{aligned}$$

The last equality follows from our assumption on f_0: the stable orbital integral $\Phi^{\mathrm{st}}(\gamma, f_0dh)$ of f_0dh at γ is supported on (the stable conjugacy class of) the torus T_{0E} in $\{T_0\}$ which splits over E. Since the map $F^\times\backslash E^\times \to E^1$ by $z \mapsto z/\overline{z}$ is a bijection and serves to relate measures from $Z\backslash T_E$ to the torus T_{0E} of $\mathrm{SL}(2,F)$, and $f_0dh = \lambda_0^*(fdg)$ means $\Phi^{\mathrm{st}}(\delta\sigma, fdg) = \Phi^{\mathrm{st}}(\gamma, f_0dh)$ for all δ, γ with $N\delta = \gamma$, it follows that $\pi = \lambda_0(\pi_0)$.

The opposite direction is proven by reversing the above steps. □

3.5 Unstable characters. Recall that the norm map N_1 of (2.2) bijects the stable σ-regular σ-conjugacy classes in G with the regular conjugacy classes in $H_1 = \mathrm{SO}(3,F)$. In each stable σ-conjugacy class of elements δ such that $\delta\sigma(\delta)$ has distinct eigenvalues there are two σ-conjugacy classes (unless the eigenvalues of $\delta\sigma(\delta)$ lie in $F^\times$, in which case there is a single σ-conjugacy class). They differ by whether $Z_G(\delta'\sigma)$ is split or not for a representative δ, and we write $\boldsymbol{\kappa}(\delta) = 1$ or -1 accordingly. Here we put $\delta' = \frac{1}{2}(\delta + J{}^t\delta J)$ as in (2.4), and note that the σ-centralizer $Z_G(\delta'\sigma)$ of δ' depends only on the σ-conjugacy class of δ, up to conjugacy in G.

The twisted character χ_π is a *σ-class function* on the σ-regular set, namely,

$$\chi_\pi^\sigma(g\delta\sigma(g)^{-1}) = \chi_\pi^\sigma(\delta)$$

for all g in G. By an *unstable σ-class function* we mean a σ-class function which satisfies $\chi_\pi^\sigma(\delta) = -\chi_\pi^\sigma(\tilde{\delta})$ whenever δ, $\tilde{\delta}$ are stably σ-conjugate but not σ-conjugate.

Note that if $\tilde{\delta}$, δ are stably σ-conjugate, but not conjugate, then up to σ-conjugacy $\delta = (ae)_1$ and $\tilde{\delta} = (uae)_1$ with u in $F^\times$ but not in $N_{E/F}E^\times$, where E/F is a quadratic extension determined by δ.

DEFINITION. The representation π_1 of $H_1 = \mathrm{SO}(3, F)$ *lifts* to the representation π of G if χ_π^σ is an unstable σ-class function and

$$|(1+\gamma')(1+\gamma'')|^{1/2}\chi_\pi^\sigma(\delta) = \chi_{\pi_1}(\gamma_1) \tag{3.5.1}$$

for all γ_1 in H_1 and δ in G such that $Z_G(\delta'\sigma)$ is split and $N_1\delta = \gamma_1$ has distinct eigenvalues as an element of $H_1 = \mathrm{SO}(3, F)$. Here γ', γ'' denote the eigenvalues of γ_1 which are not equal to 1. Note that $\chi_\pi^\sigma(\delta) = -\chi_\pi^\sigma(\delta')$ whenever δ, δ' are stably σ-conjugate but not σ-conjugate. We then write $\pi = \lambda_1(\pi_1)$.

We shall relate orbital integrals on G and on $H_1 = \mathrm{SO}(3, F)$.

3.6 DEFINITION. If $\gamma_1 = N_1\delta$ has eigenvalues 1, γ', γ'' with $\gamma' \neq \gamma''$, put

$$\Phi^{\mathrm{us}}(\delta\sigma, fdg) = \sum_{\delta'} \kappa(\delta')\Phi(\delta'\sigma, fdg).$$

If f_1 is a smooth compactly supported function on H_1 then for all regular semisimple γ_1 we put

$$\Phi(\gamma_1, f_1dh_1) = \int_{H_1/Z_{H_1}(\gamma_1)} f_1(h\gamma_1h^{-1})\frac{dh}{dt}.$$

We say that $f_1dh_1 = \lambda_1^*(fdg)$ if, when the measures $d\gamma_1$, $d\delta$ used in the definition of the orbital integrals assign the same volume to the maximal compact subgroups of $Z_{H_1}(\gamma_1)$ and $Z_G(\delta\sigma)$, we have

$$\Phi(\gamma_1, f_1dh_1) = |(1+\gamma')(1+\gamma'')|^{1/2}\Phi^{\mathrm{us}}(\delta\sigma, fdg)$$

for all $\gamma_1 = N_1\delta$ with distinct eigenvalues.

3.7 LEMMA. *We have* $\mathrm{tr}\,\pi(fdg \times \sigma) = \mathrm{tr}\,\pi_1(f_1dh_1)$ *for all* fdg, f_1dh_1 *with* $f_1dh_1 = \lambda_1^*(fdg)$ *if and only if* $\pi = \lambda_1(\pi_1)$.

PROOF. If $\mathrm{tr}\,\pi(fdg \times \sigma) = \mathrm{tr}\,\pi_1(f_1dh_1)$ for fdg, f_1dh_1 with $f_1dh_1 = \lambda_1^*(fdg)$, then $\mathrm{tr}\,\pi(fdg \times \sigma)$ is equal to $\int f_1(g)\chi_{\pi_1}(g)\,dg$, which by the Weyl integration formula of (3.2), is

$$\sum_{\{T_1\}} \frac{1}{2} \int_{T_1} \Delta_1(\gamma_1)^2 \chi_{\pi_1}(\gamma_1) \Phi(\gamma_1, f_1 dh_1)\, d\gamma_1$$
$$= \sum_{\{T_1\}} \frac{1}{2} \int_{T_1} \Delta_1(\gamma_1)^2 \chi_{\pi_1}(\gamma_1) |(1+\gamma')(1+\gamma'')|^{1/2} \Phi^{\mathrm{us}}(\delta\sigma, f dg)\, d\gamma_1.$$

We write Δ_1 to emphasize that the Δ-factor is on the group H_1. The sum is taken over a set of representatives for the conjugacy classes of tori T_1 of H_1 over F. Recall that $H_1 = \mathrm{SO}(3) \simeq \mathrm{PGL}(2)$, and in H_1 a stable conjugacy class is a conjugacy class.

The element δ, or rather its σ-conjugacy class, is uniquely determined by γ_1 and the requirement that $Z_G(\delta\sigma)$ be split over F. Moreover, $\Phi^{\mathrm{us}}(\tilde{\delta}\sigma, f dg)$ is $-\Phi^{\mathrm{us}}(\delta\sigma, f dg)$ if δ, $\tilde{\delta}$ are stably σ-conjugate but not σ-conjugate.

Define χ_π^σ by the equation (3.5.1) to be an unstable σ-conjugacy class function. Then our sum becomes

$$\sum_{\{T_2\}} \frac{1}{2} \int_{Z\backslash T_2} \Delta_0(\gamma)^2 \chi_\pi^\sigma(\delta) \Phi^{\mathrm{us}}(\delta\sigma, f dg) da.$$

The sum is over conjugacy classes of F-tori T_2 in $\mathrm{GL}(2, F)$, $\delta = (ae)_1$, $\gamma = (-1/\det a)a^2$, and $a \mapsto \gamma_1$ defines an isomorphism of $Z\backslash T_2$ and T_1 for tori T_2, T_1 which share their splitting field. Note that when the eigenvalues of a are u, v, then those of γ are $-u/v$, $-v/u$, we have

$$\Delta_0(\gamma) = \left|\left(\frac{u}{v} - \frac{v}{u}\right)^2\right|^{1/2} = \left|\left(1 + \frac{u}{v}\right)\left(1 - \frac{v}{u}\right)\right| = \left|\left(1 - \frac{u}{v}\right)\left(1 + \frac{u}{v}\right)\frac{v}{u}\right|$$

and

$$\Delta_1(\gamma_1) = \left|\frac{(u-v)^2}{uv}\right|^{1/2} = \left|\left(1 - \frac{u}{v}\right)\left(1 - \frac{v}{u}\right)\right|^{1/2} = \left|\left(1 - \frac{u}{v}\right)\left(1 - \frac{u}{v}\right)\frac{v}{u}\right|^{1/2}.$$

Hence

$$\Delta_0(\gamma)^2 = \Delta_1(\gamma_1)^2 |(1+\gamma')(1+\gamma'')|, \qquad \gamma' = \gamma''^{-1} = \frac{u}{v}.$$

The sum is equal to

$$\sum_{\{T_E\}} \frac{1}{2} \int_{NZ(E)\backslash T_E} \Delta_0(\gamma)^2 \chi_\pi^\sigma(\delta) \Phi(\delta\sigma, f dg) d\delta.$$

This is

$$\int f(g)\chi_\pi^\sigma(g)\,dg$$

by the twisted Weyl formula (3.2). Hence $\pi = \lambda_1(\pi_1)$ by the definition of χ_π^σ and λ_1. □

3.8 Induced. Let $\pi = I(\eta)$ denote the representation of G normalizedly induced from the character $\eta(\mathrm{diag}(a,b,c)) = \mu(a/c)$ of the Borel subgroup B, where μ is a character of $F^\times$. Denote by $\pi_0 = I_0(\mu)$ and $\pi_1 = I_1(\mu)$ the representations of H_0, H_1 normalizedly induced from the characters

$$\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \mapsto \mu(a), \qquad \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto \mu(a/b)$$

of the upper triangular Borel subgroups. Then the computation of (1.6) and the Weyl integration formulae of (3.2) show that the σ-character χ_π^σ of $\pi = I(\eta)$ vanishes at δ unless δ is diagonal (up to σ-conjugacy), where

$$\chi_\pi^\sigma(\delta) = \Delta_0(\gamma)^{-1}(\eta(\delta) + \eta(\tilde{\delta})) \qquad (\tilde{\delta} = J\delta J, \quad \gamma = N\delta).$$

Similar standard computations show that the χ_{π_i} are also supported on the (conjugacy classes of) diagonal elements of H_i. They are given there by

$$\chi_{\pi_0}(\gamma) = \Delta_0(\gamma)^{-1}(\mu(a) + \mu(a^{-1})), \qquad \gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

and

$$\chi_{\pi_1}(\gamma_1) = \Delta_1(\gamma_1)^{-1}(\mu(a) + \mu(a^{-1})), \qquad \gamma_1 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that if $\pi = I(\eta)$, $\pi_0 = I_0(\mu)$, $\pi_1 = I_1(\mu)$, then

LEMMA. $\pi = \lambda_0(\pi_0) = \lambda_1(\pi_1)$, *namely* $I_0(\mu)$ *and* $I_1(\mu)$ *both lift to* $I(\eta)$.

PROOF. The characters of π, π_i are supported on the split tori, and the stable σ-conjugacy class of an element where χ_π^σ does not vanish consists of a single σ-conjugacy class. □

REMARK. Here the field F is any (archimedean or not) local field.

3.9 Special representation. Let F be nonarchimedean. Let ν denote the valuation character of $F^\times$, thus $\nu(x) = |x|$. The composition series of the induced representation $I_0 = I_0(\nu)$ of H consists of the one-dimensional representation $\mathbf{1}_0$ and of the special, or Steinberg, representation sp, of H.

Note that sp is irreducible. But by Lemma 3.8 I_0 lifts to the representation $\pi = I(\eta)$ of G, induced from the character $\eta = (\nu, 1, \nu^{-1})$ of the upper triangular Borel subgroup of G. The composition series of π consists of the trivial representation $\mathbf{1}_3$, the irreducible representation $\pi_{P_1}(\mathrm{sp}(\nu, 1), \nu^{-1})$ normalizedly induced from the representation $\mathrm{sp}(\nu, 1) \times \nu^{-1}$ of the maximal parabolic subgroup P_1 of type $(2, 1)$, and the reducible representation $I_{P_2}(\nu, \mathrm{sp}(1, \nu^{-1}))$ induced from the maximal parabolic P_2 of type $(1, 2)$. This last representation has composition series consisting of the irreducible $\pi_{P_2}(\nu, \mathrm{sp}(1, \nu^{-1}))$ and the Steinberg representation St. This result is due to Bernstein-Zelevinsky [BZ2]. Now $I_{P_2}(\nu, \mathrm{sp}(1, \nu^{-1}))$ is not σ-invariant, but St, being the unique square-integrable irreducible constituent of $I(\eta) \simeq {}^\sigma I(\eta)$, is σ-invariant. Hence, $\pi_{P_2}(\nu, \mathrm{sp}(1, \nu^{-1}))$, as well as $\pi_{P_1}(\mathrm{sp}(\nu, 1), \nu^{-1})$ (for the same reason), is not σ-invariant. The one-dimensional representation $\mathbf{1}_3$ of G is clearly σ-invariant. Hence

$$\operatorname{tr} I(\eta)(f dg \times \sigma) = \operatorname{tr} \mathrm{St}(f dg \times \sigma) + \operatorname{tr} \mathbf{1}_3(f dg \times \sigma).$$

LEMMA. *The trivial and special representations of H lift to the trivial and Steinberg representations of G, respectively.*

PROOF. As the characters of both $\mathbf{1}_0$ and $\mathbf{1}_3$ are identically one, the lemma follows at once from the definition (3.3) of the lifting. □

REMARK. The only σ-invariant one-dimensional representation π of G is the trivial one. Indeed, π is given by a character β of $F^\times$ (namely, $\pi(g) = \beta(\det g)$) of order 3, thus $\beta^3 = 1$. But π is σ-invariant only if $\beta = \beta^{-1}$. Hence $\beta = 1$ and π is trivial, as asserted.

I.4 Orthogonality

4.1 Orthogonality relations. For any conjugacy class functions χ, χ' on the elliptic set H_e of H put

$$\langle \chi, \chi' \rangle_e = \int_{H_e/\sim} \chi(h) \overline{\chi}'(h) dh$$

$$= \sum_{\{T_0\}} [W(T_0)]^{-1} |T_0|^{-1} \int_{T_0} \Delta_0(\gamma)^2 \chi(\gamma) \overline{\chi}'(\gamma) d\gamma.$$

The sum ranges over a set of representatives T_0 for the conjugacy classes of elliptic tori of H over F. $[W(T_0)]$ is the cardinality of the Weyl group of T_0 (1 or 2). As usual, $|T_0|$ denotes the volume of T_0. We write $\gamma \sim \gamma'$ if γ, γ' are conjugate. The measure dh on $H_e/\sim$ is defined by the last displayed equality. The Hermitian bilinear form $\langle \chi, \chi' \rangle_e$ satisfies the Schwartz inequality

$$\langle \chi, \chi' \rangle_e^2 \leq \langle \chi, \chi \rangle_e \cdot \langle \chi', \chi' \rangle_e.$$

If χ, χ' are stable conjugacy class functions, $\langle \chi, \chi' \rangle_e^2$ is equal to

$$\langle \chi, \chi' \rangle_e = \frac{1}{2} \sum_{\{T_0\}_s} [D(T_0)] |T_0|^{-1} \int_{T_0} \Delta_0(\gamma)^2 \chi(\gamma) \overline{\chi}'(\gamma) d\gamma.$$

Here the sum is taken over a set of representatives T_0 for the stable conjugacy classes of elliptic tori of H over F. $[D(T_0)]$ is the number of conjugacy classes within the stable conjugacy class of T_0; it is 2 if T_0 is elliptic, 1 if T_0 is split.

Tempered (irreducible) representations π, π' of a reductive p-adic group G are called *relatives* if both are direct summands of the representation normalizedly induced from a tempered representation of a parabolic subgroup of G (which is trivial on the unipotent radical). The orthogonality relations for characters (see [K2], Theorems G, K) assert that $\langle \chi_\pi, \chi_{\pi'} \rangle_e$ is zero unless the tempered π, π' are relatives, and if one of them is square integrable then the result is 1 if $\pi \simeq \pi'$ and 0 if not. Then

4.1.1 LEMMA. *Let $\{\pi_0\}$ and $\{\pi_0'\}$ be stable finite sets of admissible irreducible tempered representations of H which are induced or square integrable. Then $\langle \chi_{\{\pi_0\}}, \chi_{\{\pi_0'\}} \rangle_e$ is equal to the number of square-integrable irreducible representations in $\{\pi_0\} \cap \{\pi_0'\}$.* □

4.2 Twisted orthogonality. Let π be a σ-invariant irreducible representation of G. As in (1.2) there is an intertwining operator A from the space of π to itself such that ${}^\sigma\pi(g) = \pi(\sigma(g))$ is equal to $A\pi(g)A^{-1}$. Since π is irreducible and A^2 intertwines π with itself, by Schur's lemma A^2 is a scalar, which we may normalize (by multiplying A with $1/\sqrt{A^2}$) to be 1. Extend π to a representation π' of $G' = G \rtimes \langle \sigma \rangle$ by setting $\pi(\sigma) = A$.

As noted in (3.3), the twisted character $\chi_{\pi'}^\sigma$ of π' is a σ-conjugacy class function which is locally integrable on G and is smooth on the subset of G

which consists of δ with regular $\gamma = N\delta$. Such δ is called σ-regular. Its σ-centralizer $Z_{\mathbf{G}}(\delta\sigma)$ in $\mathbf{G}$ is isomorphic to the centralizer $Z_{\mathbf{H}}(\gamma)$ of γ in $\mathbf{H}$.

For any two σ-conjugacy class functions χ^σ and χ'^σ on the σ-elliptic (δ with elliptic $N(\delta)$) subset G_e^σ of G define $\langle \chi^\sigma, \chi'^\sigma \rangle_e$ to be

$$\frac{1}{2}\sum_E |Z\backslash T_E|^{-1} \int_{T_E/N\mathbf{Z}(E)} \Delta_0(\gamma)^2 \chi(\delta\sigma)\overline{\chi}'(\delta'\sigma)da.$$

We write $\chi(\delta\sigma)$ for $\chi^\sigma(\delta)$. The sum defines a measure dg on $G_e^\sigma/\sim$, where $\delta \sim \delta'$ if δ is σ-conjugate to δ', for which

$$\langle \chi^\sigma, \chi'^\sigma \rangle_e = \int_{G_e^\sigma/\sim} \chi^\sigma(g)\overline{\chi}'^\sigma(g)dg.$$

If $\delta \mapsto \chi(\delta\sigma)$ is a stable σ-conjugacy class function, the inner product can be written as

$$\frac{1}{2}\sum_E |Z\backslash T_E|^{-1} \int_{T_E/Z} \Delta_0(\gamma)^2 \chi(\delta\sigma) \sum_{\delta'} \overline{\chi}'(\delta'\sigma)da.$$

The sum over δ' ranges over a set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of δ. For a in T_E we have $\delta = (ae)_1$, and there are two δ' in our case of δ with compact $Z_G(\delta\sigma) \simeq Z_H(\gamma)$, $\gamma = N(\delta)$.

4.2.1 LEMMA. *Given a stable conjugacy class function χ on H_e define $\chi_G(\delta) = \chi(N(\delta))$. Given a stable σ-conjugacy class function χ^σ on G_e^σ define $\chi_H(\gamma) = \chi(\delta\sigma)$ for $\gamma = N(\delta)$. Then*

$$\langle \chi^\sigma, \chi'_G \rangle_e = \langle \chi_H^\sigma, \chi' \rangle_e.$$

PROOF. This is clear from the definitions. Note that the inner product on the left is on G, while the one on the right is on H. □

Let π be a cuspidal σ-invariant representation. Such π do not exist unless the residual characteristic of F is 2. (This is proven in chapter V using the trace formula.). The orthogonality relations for characters assert in this case the following.

4.2.2 LEMMA. *Let π_2 be a σ-invariant irreducible admissible representation of G and π a σ-invariant cuspidal representation of G. Suppose that the function $\delta \mapsto \chi_\pi(\delta\sigma)$ is a stable σ-conjugacy class function on G_e^σ. Then $\langle \chi_\pi^\sigma, \chi_{\pi_2}^\sigma \rangle_e$ is equal to 0 unless π and π_2 are equivalent, in which case it is equal to 1.*

Thus for π which is cuspidal and σ-stable (by which we mean that $\chi_{\pi'}^\sigma$ is a stable σ-class function), $\langle \chi, \chi \rangle_e$ (inner product on $H_e/\sim$) is equal to 1, where χ is the stable class function on H defined by $\chi(N\delta) = \chi_{\pi'}(\delta\sigma)$.

PROOF. First suppose that π_2 is equivalent to π. Put $\pi_i' = \omega^i \pi'$ ($i = 0, 1$), where ω is the character of G' which attains the value 1 on G and the value -1 at σ. The representations π_0', π_1' are inequivalent. Put

$$\overline{\phi}(g) = d(\pi)(\pi'(g)u, \tilde{u}), \qquad \pi_i'(\phi dg) = \int_{G'} \phi(g)\pi_i'(g)\, dg.$$

Here $d(\pi)$ denotes the formal degree of π; u, $\tilde{u}$ are vectors in the space of π and the contragredient of π, with $(u, \tilde{u}) = 1$. By the Schur orthogonality relations for the square-integrable representations π_i' we have

$$\operatorname{tr} \pi_0'(\phi dg) = 1, \qquad \operatorname{tr} \pi_1'(\phi dg) = 0.$$

Then

$$1 = \operatorname{tr} \pi_0'(\phi dg) - \operatorname{tr} \pi_1'(\phi dg) = 2 \int_G \phi(g\sigma)\chi_\pi(g\sigma) dg.$$

By the Weyl integration formula (3.2) this is equal to

$$2 \cdot \frac{1}{2} \sum_E \int_{N\mathbf{Z}(E)\backslash T_E} \Delta_0(\gamma)^2 \chi_\pi(\delta\sigma) da \int_{G/Z_G(\delta\sigma)} \phi(g\delta\sigma(g)^{-1}) \frac{dg}{da}$$
$$= 2 \cdot \frac{1}{2} \sum_E \int_{Z\backslash T_E} \Delta_0(\gamma)^2 \chi_\pi(\delta\sigma) da \sum_{\delta'} \int_{G/Z_G(\delta'\sigma)} \phi(g\delta'\sigma(g)^{-1}) \frac{dg}{da}.$$

Harish-Chandra's "Selberg principle" [HC1], Theorem 29 implies the vanishing of the inner integral if $Z_G(\delta\sigma) \simeq Z_H(\gamma)$ is a torus of H which splits over F. If it is a compact torus of $\mathbf{H} = \mathrm{SL}(2)$ over F then the proof of [JL], Lemma 7.4.1, shows that

$$\chi_\pi(\delta\sigma) = d(\pi) \int_G [(\pi'(g \cdot \delta\sigma \cdot g^{-1})u, \tilde{u}) + (\pi'(g\sigma \cdot \delta\sigma \cdot (g\sigma)^{-1})u, \tilde{u})] dg$$

$$= 2\,d(\pi)|Z_G(\delta\sigma)| \int_{G/Z_G(\delta\sigma)} (\pi'(g\delta\sigma(g)^{-1}\cdot\sigma)u,\tilde{u})\frac{dg}{da}.$$

Note that $\delta\sigma(\delta)\sigma(\delta)^{-1} = \delta$ for the last equality. We obtain

$$\frac{1}{2}\sum_E |Z_G(\delta\sigma)|^{-1} \int_{Z_H(\gamma)} \Delta_0(\gamma)^2\chi_\pi(\delta\sigma)\sum_{\delta'}\overline{\chi}_\pi(\delta'\sigma)d\gamma.$$

We used the isomorphism $Z\backslash T_E \simeq Z_G(\delta\sigma) \simeq Z_H(\gamma)$, and the relation $d\delta$ $(= da) = d\gamma$ of measures on the groups $Z_G(\delta\sigma)$, $Z_H(\gamma)$.

It remains to deal with the case where π and π_2 are inequivalent. But then $(\omega^i\pi_2')(\phi) = 0$ for both i, and the lemma follows using the same argument. □

4.2.3 LEMMA. *We have that $\langle\chi_\pi^\sigma,\chi_\pi^\sigma\rangle_e$ is 1 if π is the σ-invariant Steinberg representation.*

PROOF. This follows from (4.1) and Lemma 3.9. The orthogonality relation (4.1) for sp follows from the orthogonality relation for the trivial representation of the group of elements of reduced norm 1 in the quaternion division algebra, and the correspondence of [JL]. □

To deal with π which are not cuspidal or Steinberg, we record a special case of a twisted analogue of [K2], Theorem G. The proof in the twisted case, for an arbitrary reductive not necessarily connected p-adic group, follows closely that of [K2], and will not be given here. Thus, let π, π' be σ-invariant, tempered representations with characters χ_π^σ, $\chi_{\pi'}^\sigma$. Each of π, π' defines a unique (up to association) parabolic subgroup and a square-integrable representation ρ, ρ' of its Levi factor, such that π is a subrepresentation of $I(\rho)$ and π' of $I(\rho')$. Then π, π' are called *relatives* if ρ is equivalent to ρ'. Recall that we have the inner product

$$\langle\chi^\sigma,\chi'^\sigma\rangle_e = \sum_E |T_0|^{-1}\int_{T_E/N\mathbf{Z}(E)} \Delta_0(\gamma)^2\chi(\delta\sigma)\overline{\chi}'(\delta\sigma)\,da.$$

4.2.4 LEMMA ([K2]). *If π, π' are not relatives then $\langle\chi_\pi^\sigma,\chi_{\pi'}^\sigma\rangle_e = 0$.*

The same result holds also when F is the field of real numbers.

In our case of $\mathbf{G} = \mathrm{PGL}(3)$, a G-module normalizedly induced from a tempered one is irreducible, and we need only the following special case of the lemma.

4.2.5 COROLLARY. *If π, π' are inequivalent σ-invariant tempered G-modules, then $\langle\chi_\pi^\sigma, \chi_{\pi'}^\sigma\rangle_e$ is zero.*

The methods of [K2] do not afford computing the value $\langle\chi_\pi^\sigma, \chi_{\pi'}^\sigma\rangle_e$. But in the case of any (σ-stable) cuspidal π, we have $\langle\chi_\pi^\sigma, \chi_\pi^\sigma\rangle_e = 1$ by (4.2.2). In the local lifting theorem of chapter V we list all σ-stable elliptic π, and compute $\langle\chi_\pi^\sigma, \chi_\pi^\sigma\rangle_e$. It is equal to the cardinality of the set $\{\pi_0\}$ which lifts to π.

4.3 DEFINITION. Let **J** be a reductive group over a local field, π a square-integrable irreducible J-module, and fdg a smooth compactly supported (modulo center) measure on J. Then fdg is called a *pseudo-coefficient* of π if $\operatorname{tr}\pi(fdg) = 1$ and $\operatorname{tr}\pi'(fdg) = 0$ for any irreducible tempered J-module π' inequivalent to π.

The existence of pseudo-coefficients for $H = \mathrm{SL}(2, F)$ is well known. Their existence for any p-adic group is proven in Kazhdan [K2], Theorem K. The orbital integral of fdg is equal to $|Z_J(\gamma)|^{-1}\overline{\chi}_\pi(\gamma)$ at an elliptic regular γ (whose centralizer $Z_J(\gamma)$ is a torus), and to zero on the regular nonelliptic set.

Pseudo-coefficients of σ-invariant representations are analogously defined: fdg is called a *pseudo-coefficient* of a σ-invariant (irreducible) representation π if $\operatorname{tr}\pi(fdg \times \sigma) = 1$ and $\operatorname{tr}\pi'(fdg \times \sigma) = 0$ for any irreducible tempered representation π' of G which is not a relative of π. In fact the name σ-pseudo-coefficient is more accurate, but too long, so we omit the prefix σ in the context of representations of G. The σ-orbital integral of fdg is equal to a nonzero multiple of $|Z_G(\gamma\sigma)|^{-1}\overline{\chi}_\pi(\delta\sigma)$ at any σ-elliptic σ-regular δ (whose σ-centralizer $Z_G(\gamma\sigma)$ is a torus), and to zero on the regular nonelliptic set.

4.3.1. Suppose that F is local, G is a reductive group over F, π is an admissible representation of G, C is a compact open subgroup of $F^\times$, fdg is the measure of volume 1 on G which is supported on C and is constant there.

LEMMA. *The number $\operatorname{tr}\pi(fdg)$ is equal to the dimension of the space of C-fixed vectors in π, namely it is a nonnegative integer.*

PROOF. The operator $\pi(fdg)$ is the projection on the space of C-fixed vectors in π. □