PREFACE

This volume concerns two related but independent topics in the theory of liftings of automorphic representations. These are the symmetric square lifting from the group SL(2) to the group PGL(3), and the basechange lifting from the unitary group $U(3, E/F)$ to $GL(3, E)$, where E/F is a quadratic extension of number fields. I initially considered these topics in preprints dated 1981 and 1982, and since then found reasonably simple proofs for many of the technical details, such as the fundamental lemma and the unrestricted equality of the trace formulae. The fruits of these efforts are the subject matter of the first two parts of this volume, which are independent of each other, while the third part concerns applications of the basechange theory for U(3) to the theory of Galois representations which occur in the cohomology of the Shimura variety associated with U(3).

The method used relies on a comparison of trace formulae, the same as in my *Automorphic Forms and Shimura Varieties of* PGSp(2), which concerns a rank-two situation. Both topics considered in this volume are lower, rank-one cases. They can be viewed as more elementary, certainly more complete. The last part of the volume on PGSp(2), entitled *Background*, contains many of the (standard) definitions used in this volume too. It is a brief exposition to the principle of functoriality, which predicts the liftings which concern us here, on a conjectural level, in terms of homomorphisms of dual groups. Thus here we consider two rank-one examples of this principle.

To describe the first topic, let F be a number field. Denote by $\mathbb{A}$ its ring of adèles. Let λ be the symmetric square (or adjoint) three-dimensional representation of the dual group $\widehat{H} = PGL(2, \mathbb{C})$ of the F-group $\mathbf{H} = SL(2)$ in the dual group $\widehat{G} = SL(3, \mathbb{C})$ of $\mathbf{G} = PGL(3)$. We study the lifting (or correspondence) of automorphic forms on $SL(2, \mathbb{A})$ to those of $PGL(3, \mathbb{A})$ which is compatible with λ. This lifting is defined by means of character relations. It is studied using a trace formula twisted by the outer automorphism σ of $\mathbf{G}$, which takes a representation to its contragredient. Complete results are obtained. We not only demonstrate the existence of the lifting but also describe its image and fibers. Main results include an intrinsic definition of packets of admissible and automorphic representations of $SL(2, F_v)$ and $SL(2, \mathbb{A})$, a proof of multiplicity one theorem for the cuspidal representations of $SL(2, \mathbb{A})$ and of the rigidity theorem for packets of such cuspidal representations, and a determination of the selfadjoint automorphic representations of $PGL(3, \mathbb{A})$.

Technical novelties include an elementary proof of the Fundamental Lemma, a simplification of the trace formula by means of regular functions, and a twisted analogue of Rodier's theorem capturing the number of Whittaker models of a (local) representation in the germ expansion of its character.

In the second part, locally we introduce packets and quasi-packets of admissible representations of the quasi-split unitary group $\mathrm{U}(3, E/F)$ in three variables, where E/F is a quadratic extension of local fields, and determine their structure. We determine the admissible representations of $\mathrm{GL}(3, E)$ which are invariant under the involution transpose-inverse-bar. These (quasi) packets are defined by means of both the basechange lifting from $\mathrm{U}(3, E/F)$ to $\mathrm{GL}(3, E)$ and the endoscopic lifting from $\mathrm{U}(2, E/F)$ to $\mathrm{U}(3, E/F)$. Globally, we introduce packets and quasi-packets of the discrete -spectrum automorphic representations of $\mathbf{U}(3, E/F)(\mathbb{A})$ where E/F is a quadratic extension of number fields, determine their structure, and determine the discrete-spectrum automorphic representations of $\mathrm{GL}(3, \mathbb{A}_E)$ fixed by the same involution. In particular we prove multiplicity one theorem for $\mathrm{U}(3, E/F)$, determine which members of a (quasi-) packet are automorphic, establish a rigidity theorem for (quasi-) packets of $\mathrm{U}(3, E/F)$, prove the existence of the global basechange and endoscopic liftings, as well as another twisted endoscopic lifting from $\mathrm{U}(2, E/F)$ to $\mathrm{GL}(3, E)$, and show that each packet of $\mathrm{U}(3, E/F)$ which lifts to a generic representation of $\mathrm{GL}(3, E)$ contains a unique generic member. Technical novelties include a proof of multiplicity one theorem and counting the generic members in packets, two elementary proofs of the Fundamental Lemma, and a simple proof of the unrestricted equality of trace formulae for all test functions by means of regular functions.

To emphasize, multiplicity one theorem was claimed as proved since 1982, but we noticed that the global proof was lacking and completed our local proof (for all noneven places) only a few years before this local proof appeared in 2004. For more details on the development of this area see the concluding remarks section at the end of part 2.

The third part concerns the cohomology $H_c^*(\mathcal{S}_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$ with compact supports and coefficients in any local system (ρ, V), of a Shimura variety $\mathcal{S}_{K_f}$ defined over its reflex field $\mathbb{E}$, associated with the quasi-split unitary group of similitudes $G = \mathrm{GU}(3, E/F)$, where E is a totally imaginary quadratic extension E of a totally real field F. It is a Hecke $\times$ Galois bi-module. We determine its decomposition. The Hecke modules which appear are the finite parts π_f of the discrete-spectrum representation $\pi_f \otimes \pi_\infty$ of $G(\mathbb{A}_F)$ such that π_∞ has nonzero Lie algebra cohomology. We determine the π_f-isotypic part $H_c^*(\pi_f)$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$-module in terms of the Hecke eigenvalues of π_f. In the stable case $\dim[H_c^*(\pi_f)]$ is $3^{[F:\mathbb{Q}]}$. The dimension is smaller in the unstable case. The cuspidal part of $H_c^*(\mathcal{S}_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$ coincides with the cuspidal part of the intersection cohomology $IH^*(\mathcal{S}'_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$ of the Satake Baily-Borel compactification $\mathcal{S}'_{K_f}$. Purity for the eigenvalues of the Frobenius acting on IH^*, using a computation of the Lie algebra cohomology of the π_∞, implies the Ramanujan conjecture for the π_f (with the exception of the obvious counter examples "$\pi(\mu)$"). More precisely we show that the Satake parameters of each local component π_v of π_f are algebraic, and if $\pi \neq \pi(\mu)$ that all of their conjugates lie on the unit circle in the complex plane. A description of the Zeta function of H_c^* formally follows.

This third part uses the results of the second part, and compares the trace formula with the Lefschetz-Grothendieck fixed point formula. This comparison is greatly simplified on using the (proven) Deligne conjecture on the form of the fixed point formula for a correspondence twisted by a sufficiently high power of the Frobenius. The underlying idea is used in the representation theoretic parts in the avatar of regular, Iwahori biinvariant functions. It leads to a drastic simplification of the proof of the comparison of trace formulae, on which the work of parts 1 and 2 is based. It was found while working with D. Kazhdan on applications of Drinfeld moduli schemes to the reciprocity law relating cuspidal representations of $\mathrm{GL}(n)$ over a function field (which have a cuspidal component) with n-dimensional Galois representations of this field (whose restriction to a decomposition group is irreducible). This work relied on Deligne's conjecture. First representation theoretic applications, inspired by Deligne's insight, were found in the proof with Kazhdan of the metaplectic correspondence, and then to prove basechange for $\mathrm{GL}(n)$. However, the higher-rank applications concern only cuspidal representations with a cuspidal component, while in the low-rank case considered here there are no restrictions. I then feel this idea has not yet been fully exploited. It may lead to significant simplifications in the use of the trace formula.

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