Preface

The present book was written on the basis of research carried out in recent years under the direction of one of the authors in the laboratory for nonlinear and chaotic dynamics of the Institute for Systems Analysis of the Russian Academy of Science. In this book the authors state their, in many cases distinct from traditional, point of view on principles of formation, scenarios of occurrence and ways of control of chaotic motion in nonlinear dissipative dynamical systems described by autonomous and non-autonomous ordinary differential equations, diffusion type partial differential equations and differential equations with delay argument.

Systems of nonlinear differential equations are a special case of an extensive family of nonlinear dynamical systems into which also enter various nonlinear algebraic, difference, integral, functional and abstract operational equations. In this connection, until recently the uniform geometrical approach to study nonlinear dynamical systems was represented absolutely natural, allowing to consider from common positions the nonlinear systems described by both discrete mappings, and ordinary and partial differential equations. Intensive application of geometrical approach to the analysis of dynamical systems originated with the well-known work of the American mathematician S. Smale who offered a design of mapping which subsequently received the name of Smale's horseshoe. It has been shown, that stable limit set (attractor) of discrete dynamical system cannot be such a smooth manifold of the whole dimension which are, for example, a stable limit cycle or torus, and everywhere holey, self-similar fractal set of fractional dimension. Besides it has been shown, that the behavior of trajectories of dynamical system for such strange attractor in the terminology of D. Ruelle and F. Takens is complex enough, combining global stability (the trajectory does not leave some area of phase space) with local instability of separate close trajectories, exponentially running up in time, that is characterized by the presence on an attractor both negative, and positive Lyapunov exponents. Other chaotic dynamical systems described by discrete mappings and possessing strange attractors have been further found, for example, logistic map, Henon map, Smale–Williams solenoid, *etc.*

As the analysis of properties of continuous dynamical systems described by ordinary differential equations, can be reduced, as it seemed, to the analysis of properties of some mapping — Poincare mapping, it was observed that in continuous dynamical systems irregular, chaotic behaviour of trajectories became connected to the presence of a strange attractor in the system. However, the proof of this fact directly for the well-known Lorenz system of three ordinary differential equations in which the irregular behavior of trajectories for the first time was revealed, has faced with significant difficulties. For a long time, numerous attempts to prove by the methods of geometrical theory of dynamical systems the presence of a strange attractor in neighbourhoods of a saddle–node or a saddle–focus separatrix loops of the Lorenz system have ended with failure. Moreover, the problem to show, whether the behavior of solutions of the Lorenz system coincides with the dynamics of a geometrical Lorenz attractor was formulated by S. Smale as one of 18 most significant mathematical problems of XXI century. And the recent results of the authors have allowed to confirm definitely, that the geometrical approach developed for discrete mappings even if allowed to obtain a number of brilliant results for them, is not absolutely adequate with reference to the continuous dynamical systems described by the differential equations. Now we can insist absolutely, that the definition of chaotic attractor of continuous dynamical system as a strange attractor, and also such traditional sections of chaotic dynamics as calculation of attractor's dimension, scenarios of transition to chaos, criteria of dynamical chaos and spatio-temporal chaos demand significant updating and revision.

As numerous examples show, neither the presence of a positive Lyapunov exponent, nor the presence of a saddle–node or saddle–focus separatrix loops, nor the presence of a saddle–node or saddle–focus itself is a necessary condition for the existence of chaotic dynamics in a nonlinear dissipative autonomous system of ordinary differential equations. Moreover, irregular attractors of a huge class of three-dimensional nonlinear dissipative autonomous systems of ordinary differential equations, containing all classical chaotic systems, are born as a result of the same cascade of soft bifurcations of stable limit cycles. The start is always the Feigenbaum cascade of period doubling bifurcations. It continues by complete or incomplete Sharkovskii subharmonic cascade of bifurcations of stable limit cycles with an arbitrary period and then continues by complete or incomplete homoclinic cascade of bifurcations, opened and described by authors.

In the present book, the theory of such attractors, named as singular attractors, is proposed. Any singular attractor exists only at a separate accumulation point of values of the bifurcation parameter. It contains unstable cycles of various periods in any its neighbourhood. It is proved, that any singular attractor of three-dimensional nonlinear autonomous dissipative system of ordinary differential equations lies on a two-dimensional (in general, many-sheeted) surface of three-dimensional phase space, that is the closure of a two-dimensional invariant unstable manifold (separatrix surface) of a singular saddle cycle which gives rise to the cascade of period doubling bifurcations. In this connection, fractal dimension of any singular attractor of a three-dimensional system cannot exceed two. Chaotic dynamics in all systems of the considered class of differential equations arises owing to phase shift between trajectories forming separatrix surface of the singular cycle. This leads to an appearance of the continuous one-dimensional mappings having multiple-valued inverse mappings in a two-dimensional rotating plane transversal to the cycle. This is impossible in any Poincare section, transition to which leads to loss of phase. Any singular attractor in any system of the considered class of differential equations cannot have positive Lyapunov exponent and it is not a hyperbolic set.

Thus, in all systems of the class of nonlinear autonomous three-dimensional systems of ordinary differential equations considered in the book, only complete or incomplete subharmonic or homoclinic singular attractors are born during the first stages of transition to chaos. The same scenario of transition to chaos takes place for all known classical three-dimensional autonomous dissipative systems of nonlinear ordinary differential equations, including Lorenz, Rossler, Chua, Magnitskii systems, *etc.* Moreover, as shown in the book, the same universal scenario of transition to chaos is realized in many-dimensional systems of ordinary differential equations, in nonlinear partial differential equations and in differential equations with delay arguments. And as other scenarios of transition to chaos in dissipative systems of nonlinear differential equations, except subharmonic or homoclinic cascades of bifurcations, are not observed yet, the hypothesis is rather believable about universality of the method of appearance of chaotic dynamics in dissipative systems of differential equations described in the book.

The book consists of six chapters. The basic concepts, definitions and theorems of the theory of ordinary differential equations are stated in Chapter 1. Chapter 2 is devoted to the description of the basic bifurcations of singular points, limit cycles, tori and irregular attractors of nonlinear systems of ordinary differential equations. Special attention is given to insufficiently known non-local bifurcations of homoclinic and heteroclinic contours, and also to various cascades of bifurcations of both regular, and irregular attractors. Chapter 3 shows on the basis of the conducted numerical calculations and the large illustrative material, that all classical dissipative nonlinear three-dimensional autonomous systems of ordinary differential equations have the unique universal scenario of transition to chaos through the cascade of period doubling bifurcations, subharmonic and then homoclinic cascades of soft bifurcations of stable limit cycles. This scenario is described by the theory of dynamical chaos in nonlinear systems of ordinary differential equations, developed by one of the authors and stated in Chapter 4.

Chapter 5 shows, that the same scenario of transition to chaos takes place also in many-dimensional autonomous dissipative systems of ordinary differential equations, in infinitely-dimensional systems of partial differential equations of the reaction–diffusion type, and also in the ordinary differential equations with delay argument. In such systems, transition to chaos is carried out through the cascade of period doubling bifurcations, subharmonic and then homoclinic cascades of soft bifurcations of stable two-dimensional tori. Chapter 6 considers both classical and original methods of solution of the basic problem of chaos control, consisting of detection and stabilization of unstable cycles of nonlinear systems of differential equations possessing chaotic dynamics.

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The problems raised in the book, methods of their solution and the obtained results go far beyond the frameworks of traditional representations about chaotic attractors of nonlinear dissipative systems of ordinary differential equations. Please send your opinions, remarks and offers to the e-mail address nmag@isa.ru.

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