A METHOD TO SOLVE INVERSE SCATTERING PROBLEMS FOR ELECTROMAGNETIC FIELDS IN CHIRAL MEDIA

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A method is proposed to study inverse scattering problems for electromagnetic fields in chiral media. The direct scattering problem for the perfect conductor and the dielectric is formulated in it's dyadic form considering that the space in the exterior of the chiral scatterer is also an infinite chiral medium in order to present the problem in it's general form. Herglotz dyadic fields in chiral media are defined. Superposition theorems are proved, far-field operators are defined and integral equations are posed. The inversion scheme is based on the unboundedness of the solution of the integral equation and on the properties of Herglotz dyadics.

1. Introduction

In the present work we develop an approximation method for the inverse obstacle scattering problem in chiral media based on Herglotz dyadics. Scattering theorems for dyadic electromagnetic fields in chiral media have been proved in [3]. Herglotz functions in chiral media (Beltrami Herglotz functions and chiral Herglotz pairs) have been defined and studied for the vector case in [4]. In [6] Beltrami Herglotz dyadics and dyadic electromagnetic Herglotz pairs have been defined for the dyadic case. The method we develop forms an extension of the Colton and Kirsch method for acoustics (see [8,9]). In [11] Colton and Kress study inverse problems in acoustics and electromagnetics. Dassios and Rigou in [13] and Gintides and Kiriaki in [14] study inverse problems in elasticity. We develop an inverse scattering method for electromagnetic fields in chiral media when the scatterer is a perfect conductor or a dielectric.

Thus, in Section 2 we formulate the direct perfect conductor problem and the transmission problem for electromagnetic dyadic fields in chiral media using Bohren decomposition of electromagnetic fields into suitable dyadic Beltrami fields. We define the LCP and the RCP Beltrami Herglotz dyadics and the dyadic electromagnetic Herglotz pairs. In Section 3 we prove a superposition theorem when the scatterer is a perfect conductor or a dielectric. Far field operators are defined and an integral equation is posed. Finally, an inversion scheme is posed and a theorem for it's solvability is proved.

2. Dyadic formulation in chiral media

We consider a time-harmonic plane dyadic electromagnetic wave $(\widetilde{E}^i, \widetilde{H}^i)$ propagating in a homogeneous isotropic chiral medium Ω with electric permittivity ε, magnetic permeability μ and chirality measure β. Let Ω^- be a bounded and closed subset of $\mathbb{R}^3$ with C^2-boundary $S = \partial\Omega^-$, filled with a homogeneous isotropic chiral medium with corresponding physical parameters ε^-, μ^- and β^-. The set Ω^- will be referred to as the scatterer and it will be considered to be either a perfect conductor or a dielectric.

During its propagation, the electromagnetic field $(\widetilde{E}^i, \widetilde{H}^i)$ is incident upon the scatterer Ω^- and the scattered field $(\widetilde{E}^s, \widetilde{H}^s)$ is produced. Then, the total electromagnetic field $(\widetilde{E}^t, \widetilde{H}^t)$ in Ω is given by

$$\widetilde{E}^t(\mathbf{r}) = \widetilde{E}^i(\mathbf{r}) + \widetilde{E}^s(\mathbf{r}) \quad , \quad \widetilde{H}^t(\mathbf{r}) = \widetilde{H}^i(\mathbf{r}) + \widetilde{H}^s(\mathbf{r}) \quad \text{in } \Omega. \tag{1}$$

An electromagnetic field $(\widetilde{E}, \widetilde{H})$ in Ω solves the equations (see [3,6])

$$\nabla \times \widetilde{E}(\mathbf{r}) = \beta\gamma^2\widetilde{E}(\mathbf{r}) + i\omega\mu\left(\frac{\gamma}{\kappa}\right)^2 \widetilde{H}(\mathbf{r}) \quad \text{in } \Omega, \tag{2}$$

$$\nabla \times \widetilde{H}(\mathbf{r}) = \beta\gamma^2\widetilde{H}(\mathbf{r}) - i\omega\varepsilon\left(\frac{\gamma}{\kappa}\right)^2 \widetilde{E}(\mathbf{r}) \quad \text{in } \Omega, \tag{3}$$

where ω is the anqular frequency and $\kappa^2 = \omega^2\varepsilon\mu \quad , \quad \gamma^2 = \kappa^2(1-\beta^2\kappa^2)^{-1}$. The total interior electromagnetic field in Ω^- satisfies also (2) and (3) with physical parameters ε^-, μ^- and β^-.

The scattered field is assumed to satisfy the equivalent *Silver – Müller* radiation conditions

$$\widetilde{E}^s(\mathbf{r}) + \eta\hat{\mathbf{r}} \times \widetilde{H}^s(\mathbf{r}) = o\left(\frac{1}{r}\right) \quad , \quad \hat{\mathbf{r}} \times \widetilde{E}^s(\mathbf{r}) - \eta\widetilde{H}^s(\mathbf{r}) = o\left(\frac{1}{r}\right), \quad r \to \infty, \tag{4}$$

uniformly in all directions $\hat{\mathbf{r}} = \mathbf{r}/r$, with $r = |\mathbf{r}|$, where $\eta = (\mu/\varepsilon)^{1/2}$ is the intrinsic impedance of the chiral medium in Ω.

In a homogeneous isotropic chiral medium the electromagnetic fields are composed of Left-Circularly Polarized (LCP) and Right-Circularly Polarized (RCP) components, which have different wavenumbers and independent directions of propagation. We consider the Bohren decomposition of $\widetilde{E}$ and $\widetilde{H}$, [3], [15]

$$\widetilde{E}(\mathbf{r}) = \widetilde{E}_L(\mathbf{r}) + \widetilde{E}_R(\mathbf{r}) \quad , \quad \widetilde{H}(\mathbf{r}) = \frac{1}{i\eta}(\widetilde{E}_L(\mathbf{r}) - \widetilde{E}_R(\mathbf{r})). \tag{5}$$

The LCP and RCP Beltrami fields $\widetilde{E}_L$ and $\widetilde{E}_R$ respectively, satisfy the Beltrami equations

$$\nabla \times \widetilde{E}_L(\mathbf{r}) = \gamma_L \widetilde{E}_L(\mathbf{r}) \quad , \quad \nabla \times \widetilde{E}_R(\mathbf{r}) = -\gamma_R \widetilde{E}_R(\mathbf{r}), \tag{6}$$

where the wavenumbers γ_L and γ_R are given by $\gamma_L = \frac{\kappa}{1-\kappa\beta}$, $\gamma_R = \frac{\kappa}{1+\kappa\beta}$, they are divergence free and also satisfy the dyadic Helmholtz equation

$$\Delta \widetilde{E}_A(\mathbf{r}) + \gamma_A^2 \widetilde{E}_A(\mathbf{r}) = \widetilde{O}, \tag{7}$$

for A=L,R.

Using, now, the Silver-Müller radiation conditions (4) for scattered electromagnetic waves and the relations (6) we derive the following radiation conditions, [3],

$$\hat{\mathbf{r}} \times \widetilde{E}_L^s(\mathbf{r}) + i\widetilde{E}_L^s(\mathbf{r}) = o\left(\frac{1}{r}\right) \quad , \quad \hat{\mathbf{r}} \times \widetilde{E}_R^s(\mathbf{r}) - i\widetilde{E}_R^s(\mathbf{r}) = o\left(\frac{1}{r}\right), \quad r \to \infty, \tag{8}$$

uniformly for all directions $\hat{\mathbf{r}}$.

For a perfect conductor the total Beltrami fields $\widetilde{E}_L^t, \widetilde{E}_R^t$ satisfy the boundary condition

$$\hat{\mathbf{n}} \times \widetilde{E}_L^t(\mathbf{r}) = -\hat{\mathbf{n}} \times \widetilde{E}_R^t(\mathbf{r}) \quad on \quad S. \tag{9}$$

For a dielectric the total exterior and interior Beltrami fields $\widetilde{E}_L^t, \widetilde{E}_R^t$ and $\widetilde{E}_L^-, \widetilde{E}_R^-$ respectively satisfy the transmission conditions

$$\hat{\mathbf{n}} \times (\widetilde{E}_L^t - \widetilde{E}_L^-)(\mathbf{r}) = \hat{\mathbf{n}} \times (\widetilde{E}_R^- - \widetilde{E}_R^t)(\mathbf{r}) \quad on \quad S, \tag{10}$$

$$\eta^- \hat{\mathbf{n}} \times (\widetilde{E}_L^t - \widetilde{E}_L^-)(\mathbf{r}) = \eta^t \hat{\mathbf{n}} \times (\widetilde{E}_R^t - \widetilde{E}_R^-)(\mathbf{r}) \quad on \quad S. \tag{11}$$

The solvability of the perfect conductor and the transmission problem has been studied in [1] and [2] respectively, where existence and uniqueness of solution has been proved.

If, now, the unit vectors $\hat{\mathbf{d}}_L$ and $\hat{\mathbf{d}}_R$ describe the directions of propagation of the LCP and RCP wave, respectively, then the incident plane dyadic electric field assumes the form

$$\tilde{E}^i(\mathbf{r}|\hat{\mathbf{d}}_L, \hat{\mathbf{d}}_R) = \tilde{E}^i_L(\mathbf{r}|\hat{\mathbf{d}}_L) + \tilde{E}^i_R(\mathbf{r}|\hat{\mathbf{d}}_R), \tag{12}$$

where

$$\tilde{E}^i_L(\mathbf{r}|\hat{\mathbf{d}}_L) = \tilde{K}_L(\hat{\mathbf{d}}_L)e^{i\gamma_L \hat{\mathbf{d}}_L \cdot \mathbf{r}} \quad , \quad \tilde{E}^i_R(\mathbf{r}|\hat{\mathbf{d}}_R) = \tilde{K}_R(\hat{\mathbf{d}}_R)e^{i\gamma_R \hat{\mathbf{d}}_R \cdot \mathbf{r}}, \tag{13}$$

are the LCP and RCP plane dyadic electric fields respectively. The dyadics $\tilde{K}_L$ and $\tilde{K}_R$ are given by, [3]

$$\tilde{K}_L(\hat{\mathbf{d}}_L) = \tilde{I} - \hat{\mathbf{d}}_L\hat{\mathbf{d}}_L + i\hat{\mathbf{d}}_L \times \tilde{I} \quad , \quad \tilde{K}_R(\hat{\mathbf{d}}_R) = \tilde{I} - \hat{\mathbf{d}}_R\hat{\mathbf{d}}_R - i\hat{\mathbf{d}}_R \times \tilde{I}, \tag{14}$$

where $\tilde{I}$ is the identity dyadic.

Finally, we define Herglotz dyadics in chiral media, [6]. A Beltrami Herglotz dyadic is a dyadic field of the form

$$\widetilde{\mathcal{E}}_A(\mathbf{r}) = \mathbf{q}_{A1}(\mathbf{r}) \otimes \hat{\mathbf{e}}_1 + \mathbf{q}_{A2}(\mathbf{r}) \otimes \hat{\mathbf{e}}_2 + \mathbf{q}_{A3}(\mathbf{r}) \otimes \hat{\mathbf{e}}_3, \tag{15}$$

where $\mathbf{q}_{Ai}$, $A = L, R$, $i = 1, 2, 3$ are three Beltrami Herglotz vector functions and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ denotes the orthonormal base in R^3. A Beltrami Herglotz dyadic satisfies the well known Herglotz condition and has the following representation form

$$\widetilde{\mathcal{E}}_A(\mathbf{r}) = \int_{S^2} \widetilde{b}_A(\hat{\mathbf{d}}_A)e^{i\gamma_A \hat{\mathbf{d}}_A \cdot \mathbf{r}} ds(\hat{\mathbf{d}}_A), \quad A = L, R, \tag{16}$$

where $\widetilde{b}_A \in \widetilde{T}^2_A(S^2)$, $A = L, R$ and

$$\widetilde{T}^2_L(S^2) = \{\widetilde{b}_L : S^2 \to \mathbb{C}^9 : \widetilde{b}_L \in L^2(S^2), \hat{\mathbf{n}} \cdot \widetilde{b}_L = \mathbf{0}, \hat{\mathbf{n}} \times \widetilde{b}_L = -i\widetilde{b}_L\}, \tag{17}$$

$$\widetilde{T}^2_R(S^2) = \{\widetilde{b}_R : S^2 \to \mathbb{C}^9 : \widetilde{b}_R \in L^2(S^2), \hat{\mathbf{n}} \cdot \widetilde{b}_R = \mathbf{0}, \hat{\mathbf{n}} \times \widetilde{b}_R = i\widetilde{b}_R\}. \tag{18}$$

A dyadic electromagnetic Herglotz pair is defined by

$$\widetilde{\mathcal{E}}(\mathbf{r}) = \widetilde{\mathcal{E}}_L(\mathbf{r}) + \widetilde{\mathcal{E}}_R(\mathbf{r}) \quad , \quad \widetilde{\mathcal{H}}(\mathbf{r}) = \frac{1}{i\eta}(\widetilde{\mathcal{E}}_L(\mathbf{r}) - \widetilde{\mathcal{E}}_R(\mathbf{r})), \tag{19}$$

and represents entire solution to equations (2) and (3).

We call the dyadic field $\widetilde{b} = \widetilde{b}_L + \widetilde{b}_R$ for $\widetilde{b}_A \in \widetilde{T}^2_A(S^2)$, $A = L, R$, the electric Herglotz kernel for the electric Herglotz dyadic $\widetilde{\mathcal{E}}$ and we denote the set of all electric Herglotz kernels for the electric Herglotz dyadic $\widetilde{\mathcal{E}}$ by $\widetilde{T}^2_{LR}(S^2)$. In [6] it has been proved that the set of dyadic electromagnetic Herglotz pairs is dense within the set of the solutions of equations (2), (3), that is, for every solution $\widetilde{E}$ and $\widetilde{H}$ of equations (2), (3) and for every $\epsilon > 0$ there exists a dyadic electromagnetic Herglotz pair $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{H}})$, such that

$$\max_{\mathbf{r}\in\overline{\Omega^-}} \|\widetilde{E}(\mathbf{r}) - \widetilde{\mathcal{E}}(\mathbf{r})\| \leq \epsilon \quad , \quad \max_{\mathbf{r}\in\overline{\Omega^c}} \|\widetilde{H}(\mathbf{r}) - \widetilde{\mathcal{H}}(\mathbf{r})\| \leq \epsilon. \tag{20}$$

3. An inverse scattering method

Now that we have defined Herglotz dyadics we will make use of them to prove a superposition theorem for electromagnetic dyadic fields in chiral media when the scatterer is a perfect conductor or a dielectric.

Theorem 3.1. *Given dyadic densities $\widetilde{b}_A \in \widetilde{T}^2_A(S^2)$, $A = L, R$, when the incident electric field is of the form*

$$\widetilde{\mathcal{E}}^i(\boldsymbol{r}|\hat{\boldsymbol{d}}_L, \hat{\boldsymbol{d}}_R) = \int_{S^2} \widetilde{b}_L(\hat{\boldsymbol{d}}_L) e^{i\gamma_L \hat{\boldsymbol{d}}_L\cdot\boldsymbol{r}} ds(\hat{\boldsymbol{d}}_L) + \int_{S^2} \widetilde{b}_R(\hat{\boldsymbol{d}}_R) e^{i\gamma_R \hat{\boldsymbol{d}}_R\cdot\boldsymbol{r}} ds(\hat{\boldsymbol{d}}_R), \tag{21}$$

then, the scattered field is given by the formula

$$\begin{aligned} \widetilde{\mathcal{E}}^s(\boldsymbol{r}|\hat{\boldsymbol{d}}_L, \hat{\boldsymbol{d}}_R) = &\int_{S^2} \widetilde{b}_L(\hat{\boldsymbol{d}}_L) \cdot \{\widetilde{E}^s_L(\boldsymbol{r}|\hat{\boldsymbol{d}}_L) + \widetilde{E}^s_R(\boldsymbol{r}|\hat{\boldsymbol{d}}_L)\} ds(\hat{\boldsymbol{d}}_L) \\ &+ \int_{S^2} \widetilde{b}_R(\hat{\boldsymbol{d}}_R) \cdot \{\widetilde{E}^s_L(\boldsymbol{r}|\hat{\boldsymbol{d}}_R) + \widetilde{E}^s_R(\boldsymbol{r}|\hat{\boldsymbol{d}}_R)\} ds(\hat{\boldsymbol{d}}_R), \end{aligned} \tag{22}$$

and has the far-field pattern

$$\begin{aligned} \widetilde{\mathcal{E}}^\infty(\hat{\boldsymbol{r}}|\hat{\boldsymbol{d}}_L, \hat{\boldsymbol{d}}_R) = &\int_{S^2} \widetilde{b}_L(\hat{\boldsymbol{d}}_L) \cdot \{\widetilde{E}^\infty_L(\hat{\boldsymbol{r}}|\hat{\boldsymbol{d}}_L) + \widetilde{E}^\infty_R(\hat{\boldsymbol{r}}|\hat{\boldsymbol{d}}_L)\} ds(\hat{\boldsymbol{d}}_L) \\ &+ \int_{S^2} \widetilde{b}_R(\hat{\boldsymbol{d}}_R) \cdot \{\widetilde{E}^\infty_L(\hat{\boldsymbol{r}}|\hat{\boldsymbol{d}}_R) + \widetilde{E}^\infty_R(\hat{\boldsymbol{r}}|\hat{\boldsymbol{d}}_R)\} ds(\hat{\boldsymbol{d}}_R), \end{aligned} \tag{23}$$

where $\widetilde{E}^\infty_A$, $A = L, R$ are the corresponding to $\widetilde{E}^s_A$ far-field patterns.

Proof. We use the dyadic potentials

$$[S^-(k)\widetilde{\alpha}](\mathbf{r}) = \int_S \widetilde{\alpha}(\acute{\mathbf{r}}) \Phi(\mathbf{r}, \mathbf{r}'; k) ds(\mathbf{r}'), \quad \mathbf{r} \in \Omega^-, \tag{24}$$

$$[C^-(k)\widetilde{\alpha}](\mathbf{r}) = \nabla\times[S^-(k)\widetilde{\alpha}](\mathbf{r}) \quad , \quad [F^-(k)\widetilde{\alpha}](\mathbf{r}) = \nabla\times[C^-(k)\widetilde{\alpha}](\mathbf{r}), \mathbf{r} \in \Omega^-, \tag{25}$$

where $\Phi(\mathbf{r}, \mathbf{r}'; k)$ is the fundamental solution of the Helmholtz equation $\Delta u + ku^2 = 0$ which is given by $\Phi(\mathbf{r}, \mathbf{r}'; k) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad , \quad \mathbf{r} \neq \mathbf{r}'$.

The proof now comes following the steps of the corresponding vector case, [5].

We define the far field operator

$$F_{LR} : \widetilde{T}^2_{LR}(S^2) \longrightarrow \widetilde{T}^2_{LR}(S^2), \tag{26}$$

by

$$F_{LR}(\widetilde{b}_L + \widetilde{b}_R)(\hat{\mathbf{r}}) = \int_{S^2} \widetilde{b}_L(\hat{\mathbf{d}}_L) \cdot \{\widetilde{E}^{\infty}_L(\hat{\mathbf{r}}|\hat{\mathbf{d}}_L) + \widetilde{E}^{\infty}_R(\hat{\mathbf{r}}|\hat{\mathbf{d}}_L)\} ds(\hat{\mathbf{d}}_L) + \int_{S^2} \widetilde{b}_R(\hat{\mathbf{d}}_R) \cdot \{\widetilde{E}^{\infty}_L(\hat{\mathbf{r}}|\hat{\mathbf{d}}_R) + \widetilde{E}^{\infty}_R(\hat{\mathbf{r}}|\hat{\mathbf{d}}_R)\} ds(\hat{\mathbf{d}}_R). \tag{27}$$

It can be proved that the far field operator F_{LR} is injective and has dense range if and only if there does not exist a solution $\widetilde{E}, \widetilde{H}$ of equations (2), (3) which is a nontrivial electromagnetic Herglotz pair.

If, now we want the scattered field to be a known radiating solution $\widetilde{\mathcal{E}}^s$ to equations (2), (3) with corresponding known far-field pattern $\widetilde{\mathcal{E}}^{\infty}$, we have to find the solution to the integral equation

$$F_{LR}(\widetilde{b}_L + \widetilde{b}_R) = \widetilde{\mathcal{E}}^{\infty}. \tag{28}$$

Finally we use the properties of Herglotz dyadics to derive an approximation of the integral equation (28) and therefore to be able to determine the scatterer from the knowledge of the far field data. The following theorem holds

Theorem 3.2. *Let $(\widetilde{E}^s, \widetilde{H}^s)$ be a radiating solution to equations (2), (3) with electric far-field patterns $\widetilde{E}^{\infty}_A$, $A = L, R$. Then, the linear integral equation of the first kind (28) posses a solution $\widetilde{b} = (\widetilde{b}_L + \widetilde{b}_R) \in \widetilde{T}^2_{LR}(S^2)$ if and only if $(\widetilde{E}^s, \widetilde{H}^s)$ is defined in $\mathbb{C}^3 \setminus \overline{\Omega}$, it is continuous in $\mathbb{C}^3 \setminus \Omega$ and the interior problem for a perfect conductor or a dielectric is solvable and a solution $(\widetilde{\mathcal{E}}^i, \widetilde{\mathcal{H}}^i)$ is a dyadic electromagnetic Herglotz pair.*

In the following theorem we conclude to an approximation of the integral equation (28)

Theorem 3.3. *Let Ω^- be simply connected and suppose that $(\widetilde{E}^s, \widetilde{H}^s)$ is a solution to equations (2), (3) which satisfies the Silver – Müller radiation conditions (4). Then, for every $\varepsilon > 0$ and $r_0 \in \Omega^-$ there exist dyadics*

$\widetilde{b}_L(\cdot, r_0) \in \widetilde{T}_L^2(S^2)$ and $\widetilde{b}_R(\cdot, r_0) \in \widetilde{T}_R^2(S^2)$, with $\widetilde{b} = \widetilde{b}_L + \widetilde{b}_R$, such that the integral equation (28) can be approximated by

$$\|\widetilde{\mathcal{E}}^\infty(\hat{r}|\hat{d}_L, \hat{d}_R) - \widetilde{B}^\infty(\hat{r}, r_0)\| < \varepsilon, \tag{29}$$

where the far field pattern $\widetilde{\mathcal{E}}^\infty(\hat{r}|\hat{d}_L, \hat{d}_R)$ is given by (26), $\widetilde{B}^\infty(\hat{r}, r_0)$ is the far-field of the infinite medium Green's dyadic, $\widetilde{B}(r, r_0)$, [3] and

$$\lim_{r_0 \longrightarrow S} \|\widetilde{b}_L(\cdot, r_0)\|_{L^2(S^2)} = +\infty \quad , \quad \lim_{r_0 \longrightarrow S} \|\widetilde{b}_R(\cdot, r_0)\|_{L^2(S^2)} = +\infty. \tag{30}$$

We note that by $\widetilde{b}_L(\cdot, r_0)$ and $\widetilde{b}_R(\cdot, r_0)$ we denote the dependance of the dyadic Herglotz kernels from the fixed point r_0.

Proof. From inequalities (20), the representation of the scattered field by means of the infinite medium Green's dyadic, $\widetilde{B}(\mathbf{r}, \mathbf{r}_0)$, [3] and from the continuity of the operator in (27) we conclude that there exists a dyadic Herglotz field $\widetilde{\mathcal{B}}^s$ with Herglotz kernels $\widetilde{b}_L(\cdot, \mathbf{r}_0)$ and $\widetilde{b}_R(\cdot, \mathbf{r}_0)$ such that

$$\max_{\mathbf{r} \in \overline{\Omega^c}} \|\widetilde{B}^s(\mathbf{r}) - \widetilde{\mathcal{B}}^s(\mathbf{r})\| \leq \epsilon. \tag{31}$$

Applying the superposition theorem and taking into account the asymptotic behavior of the scattered field we conclude to (29).

Let, now, $\mathbf{r}_0$ tend to the boundary of the scatterer S. From the fact that $\widetilde{B}^s$ is unbounded on S and the relation (31) we get that $\widetilde{\mathcal{B}}^s$ is also unbounded on S. So, the Herglotz kernels $\widetilde{b}_L$ and $\widetilde{b}_R$ of $\widetilde{\mathcal{B}}^s$ must be unbounded on S and therefore $\widetilde{b} = \widetilde{b}_L + \widetilde{b}_R$ is also unbounded on S. Thus, (30) must hold true.

Now that we have found a density $\widetilde{b} = \widetilde{b}_L + \widetilde{b}_R$ which is unbounded on S and satisfies the approximate integral equation (28) we could find the boundary of the scatterer at the points where the L^2 - norms of the dyadics $\widetilde{b}_L$ and $\widetilde{b}_R$ take extremely large values. So, finding first the densities $\widetilde{b}_L(\cdot, \mathbf{r}_0)$ and $\widetilde{b}_R(\cdot, \mathbf{r}_0)$ for a fixed point $\mathbf{r}_0$ and letting $\mathbf{r}_0$ go to the boundary of the scatterer, we expect to find a point of the boundary of the scatterer when the norms of the dyadic kernels $\widetilde{b}_L$ and $\widetilde{b}_R$ are infinite.

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ELECTROMAGNETIC SCATTERING BY A METALLIC SPHEROID

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ABSTRACT

The scattering of a plane electromagnetic wave by a perfectly conducting prolate or oblate spheroid is considered. Two different methods are used for the evaluation. In the first, the electromagnetic field is expressed in terms of spheroidal vector wave functions. In the second, a shape perturbation method, the field is expressed in terms of spherical vector wave functions only, while the equation of the spheroidal boundary is given in spherical coordinates. Analytical expressions are obtained for the scattered electromagnetic field and the scattering cross-sections, when the solution is specialized to small values of h=d/(2a), (h<<1), with d the interfocal distance of the spheroid and 2a the length of its rotation axis. In this case exact, closed-form expressions, valid for each small h, are obtained for the expansion coefficients $g^{(2)}$ and $g^{(4)}$ in the relation $S(h)=S(0)[1+g^{(2)}h^2+g^{(4)}h^4+O(h^6)]$ expressing the scattered field and the various scattering cross-sections. Numerical results are given for various values of the parameters.

1. Introduction

Study of electromagnetic scattering by spheroids is an old problem with numerous applications. Many researchers have been involved with its solution in the past. Among a great number of papers, treating the problem by various methods, we refer a few here [1-5].

In this paper we consider the scattering of a plane electromagnetic wave from a metallic, prolate or oblate spheroid. In Fig. 1 we give the geometry of the prolate spheroid. Its interfocal distance is d, while a and b are the lengths of its major and minor semiaxes, respectively. Two kinds of incident waves are considered, both coming from the same direction, with different polarizations as seen in Fig.1, although only the first (TE case) will be discussed in detail. Any case of plane electromagnetic wave can be considered as a combination of these two polarizations. The prolate spheroid is the only one to be considered explicitly, but corresponding formulas for the oblate one are obtained immediately.

We use two different methods for the solution. In the first, a shape perturbation method, the field is expressed in terms of spherical vector wave functions only and the equation of the spheroidal boundary is given in spherical coordinates, while in the second the electromagnetic field is expressed in terms of spheroidal vector wave functions. When the solution is specialized to small values of the parameter $h = d/(2a), (h << 1)$ analytical expressions of the form $S(h)=S(0)[1+g^{(2)}h^2+g^{(4)}h^4+O(h^6)]$ are obtained for the scattered field and the various scattering cross-sections. The expansion coefficients $g^{(2)}$ and $g^{(4)}$ are given by exact, closed-form expressions, independent of h, while S(0) corresponds to a sphere with radius a (h=0).

The main advantage of such an analytical solution lies in its general validity for each small value of h, while all numerical techniques require repetition of the evaluation for each different h.

The first method is used in section 2 and the second in section 3. Finally, in section 4 some numerical results are given.

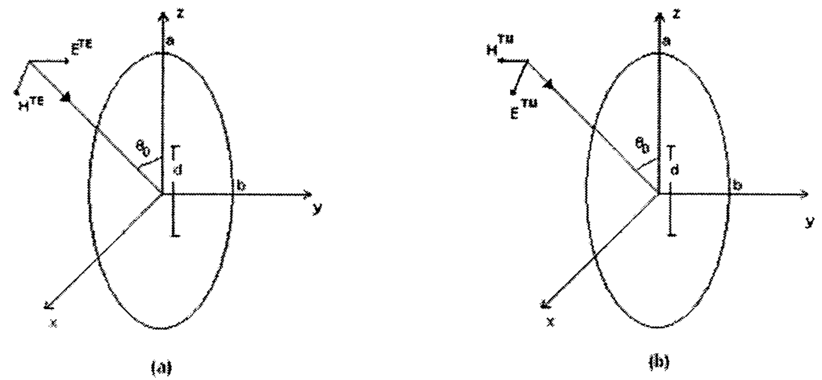

Fig.1 Geometry of the spheroidal scatterer a)TE case b)TM case

2. Solution in Terms of Spherical Vector Wave Functions

We start with the method using spherical vector wave functions. The plane of incidence is the xz plane. We consider two incident plane electromagnetic waves impinging on the scatterer. In the first the incident electric field is normal to the plane of incidence (TE case) while in the second the incident magnetic field is normal to this plane (TM case). We examine first the TE case. The incident electric field is (the time factor exp(-jωt) is suppressed) [6]

$$\begin{aligned}\mathbf{E}_i &= \mathbf{y}\exp\left(x\sin\theta_0 + z\cos\theta_0\right) = \\ &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[c_{mn}\left(\theta_0\right)\mathbf{m}_{emn}^{(1)}\left(r,\theta,\phi\right) + d_{mn}\left(\theta_0\right)\mathbf{n}_{omn}^{(1)}\left(r,\theta,\phi\right)\right]\end{aligned} \tag{1}$$

where r,θ,φ are the spherical coordinates , θ_0 defines the direction of incidence, $\mathbf{m}_{emn}^{(1)}$ and $\mathbf{n}_{omn}^{(1)}$ are spherical vector wave functions of the first kind defined by[7]

$$\begin{aligned}\mathbf{m}_{smn}^{(1)} &= \sqrt{n(n+1)}\,j_n(kr)\,\mathbf{C}_{mn}^{s}(\theta,\phi) \\ \mathbf{n}_{smn}^{(1)} &= n(n+1)\frac{j_n(kr)}{kr}\mathbf{P}_{mn}^{s}(\theta,\phi) + \sqrt{n(n+1)}\frac{j_n^d(kr)}{kr}\mathbf{B}_{mn}^{s}(\theta,\phi) \qquad s = e,o\end{aligned} \tag{2}.$$

where the subscript e stands for even while o stands for odd functions, k is the wavenumber, $j_n(kr)$ are the spherical Bessel functions of the first kind and

$$j_n^d(x) = d\left[xj_n(x)\right]/dx \tag{3}$$

$\mathbf{B}^s_{mn}, \mathbf{C}^s_{mn}, \mathbf{P}^s_{mn}$ are the vector spherical harmonics defined by

$$\mathbf{C}^{e}_{\substack{o\\mn}} = \mp\frac{1}{\sqrt{n(n+1)}}\left[\frac{m}{\sin\theta}P_n^m(\cos\theta)\begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}\boldsymbol{\theta} \pm \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\boldsymbol{\varphi}\right]$$

$$\mathbf{B}^{e}_{\substack{o\\mn}} = \frac{1}{\sqrt{n(n+1)}}\left[\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\boldsymbol{\theta} \mp \frac{m}{\sin\theta}P_n^m(\cos\theta)\begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}\boldsymbol{\varphi}\right] \tag{4}$$

$$\mathbf{P}^{e}_{\substack{o\\mn}} = P_n^m(\cos\theta)\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\mathbf{r}$$

where P_n^m are the associated Legendre functions.

The coefficients c_{mn} and d_{mn} in (1) are given by [6]

$$\begin{aligned} c_{mn}(\theta_0) &= -\frac{\epsilon_m i^n}{n(n+1)}\frac{(2n+1)(n-m)!}{(n+m)!}\frac{dP_n^m(\cos\theta_0)}{d\theta_0} \\ d_{mn}(\theta_0) &= \frac{2mi^{n-1}}{n(n+1)}\frac{(2n+1)(n-m)!}{(n+m)!}\frac{P_n^m(\cos\theta_0)}{\sin\theta_0} \end{aligned} \tag{5}$$

with $\epsilon_0 = 1$ and $\epsilon_m = 2,\ m \geq 1$ the Neumann factor.

The scattered wave can be expressed as

$$\mathbf{E}_i = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[A_{mn}\mathbf{m}^{(3)}_{emn}(r,\theta,\phi) + B_{mn}\mathbf{n}^{(3)}_{omn}(r,\theta,\phi)\right] \tag{6}$$

where $\mathbf{m}^{(3)}_{omn}$ and $\mathbf{n}'^{(3)}_{omn}$ are spherical vector wave functions of the third kind with expressions similar to those in (2), with the spherical Bessel functions of the first kind replaced by $h_n(kr)$, the spherical Hankel functions of the first kind.

We express the equation of the spheroidal surface in terms of r and θ and use its expansion into power series in h for h<<1 [8]

$$r = \frac{\mathrm{a}}{\sqrt{1-\nu\sin^2\theta}} = \mathrm{a}\left[1 - \frac{h^2}{2}\sin^2\theta - \frac{h^4}{2}\left(\sin^2\theta - \frac{3}{4}\sin^4\theta\right) + O(h^6)\right] \tag{7}$$

where
$$\nu = 1 - \frac{\mathrm{a}^2}{b^2} = -h^2 - h^4 + O(h^6) \tag{8}$$

By using (7) we obtain the expansion

$$j_n(kr) = j_n(x) - \frac{h^2}{2}xj_n'(x)\sin^2\theta +$$

$$h^4\left\{-\frac{x}{2}j_n'(x)\sin^2\theta + \frac{1}{8}\left[3xj_n'(x) + x^2 j_n''(x)\right]\sin^4\theta\right\} + O(h^6),\ x = ka \tag{9}$$

and similar ones for $h_n(kr)$, where the primes denote derivatives with respect to the argument Analogous expressions are obtained for $j_n'(kr)$ and $h_n'(kr)$, respectively, with the only difference that one more prime is added in each of the Bessel functions met there. Also we obtain the expansion

$$\frac{j_n(kr)}{kr}=\frac{j_n(x)}{x}-\frac{h^2}{2}\left[-\frac{j_n(x)}{x}+j_n'(x)\right]\sin^2\theta+\frac{h^4}{2}\left\{\left[\frac{j_n(x)}{x}-j_n'(x)\right]\sin^2\theta+\right.$$

$$\left.+\frac{1}{4}\left[-\frac{j_n(x)}{x}+j_n'(x)+xj_n''(x)\right]\sin^4\theta\right\}+O\left(h^6\right),x=ka \qquad (10)$$

and a similar one for $h_n(kr)/kr$.

In order to calculate the unknown expansion coefficients A_{mn} and B_{mn} the boundary condition on the surface of the conducting spheroid $\hat{n}\times(\mathbf{E}_i+\mathbf{E}_s)=0$ ($\hat{n}$ is the normal unit vector there), must be satisfied. Using the orthogonal properties of the vector spherical harmonics an infinite set of linear inhomogeneous equations for the expansion coefficients A_{mn} and B_{mn} of the following form (up to the order h^4) is obtained:

$$\sum_{s=n,n\pm2,n\pm4} a_{n,s}A_s+\sum_{s=n\pm1,n\pm3} b_{n,s}B_s=K_n \;,\; \sum_{s=n\pm1,n\pm3} a'_{n,s}A_s+\sum_{s=n,n\pm2,n\pm4} b'_{n,s}B_s=K'_n \qquad (11)$$

where $a_{n,s}, a'_{n,s}, b_{n,s}, b'_{n,s}, K_n, K'_n$ can be written in the form

$$\begin{aligned} a_{n,n}&=D^{(0)}_{n,n}+h^2D^{(2)}_{n,n}+h^4D^{(4)}_{n,n}+O\left(h^6\right),\\ a_{n,n\pm2}&=h^2D^{(2)}_{n,n\pm2}+h^4D^{(4)}_{n,n\pm2}+O\left(h^6\right),\quad a_{n,n\pm4}=h^4D^{(4)}_{n,n\pm4}+O\left(h^6\right)\end{aligned} \qquad (12)$$

$$a'_{n,n\pm1}=h^2D'^{(2)}_{n,n\pm1}+h^4D'^{(4)}_{n,n\pm1}+O\left(h^6\right),\quad a'_{n,n\pm3}=h^4D'^{(4)}_{n,n\pm3}+O\left(h^6\right) \qquad (13)$$

$$b_{n,n\pm1}=h^2E^{(2)}_{n,n\pm1}+h^4E^{(4)}_{n,n\pm1}+O\left(h^6\right),\quad b_{n,n\pm3}=h^4E^{(4)}_{n,n\pm3}+O\left(h^6\right) \qquad (14)$$

$$\begin{aligned} b'_{n,n}&=E'^{(0)}_{n,n}+h^2E'^{(2)}_{n,n}+h^4E'^{(4)}_{n,n}+O\left(h^6\right),\\ b'_{n,n\pm2}&=h^2E'^{(2)}_{n,n\pm2}+h^4E'^{(4)}_{n,n\pm2}+O\left(h^6\right),\quad b'_{n,n\pm4}=h^4E'^{(4)}_{n,n\pm4}+O\left(h^6\right)\end{aligned} \qquad (15)$$

$$K_n=K^{(0)}_n+h^2K^{(2)}_n+h^4K^{(4)}_n+O\left(h^6\right),K'_n=K'^{(0)}_n+h^2K'^{(2)}_n+h^4K'^{(4)}_n+O\left(h^6\right) \qquad (16)$$

A's and B are obtained from the solution of (11) by Cramer's rule, following steps similar with the ones in [9].

Using the asymptotic expansion for h_n we obtain the scattered far field expression

$$\mathbf{E}_s=\frac{e^{ikr}}{r}\mathbf{f}(\theta,\phi) \qquad (17)$$

where

$$\mathbf{f}(\theta,\phi)_i=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sqrt{n(n+1)}\frac{(-i)^{n+1}}{k}\left[A_{mn}\mathbf{C}^{e}_{emn}(\theta,\phi)+B_{mn}\mathbf{B}^{o}_{mn}(\theta,\phi)\right]= \tag{18}$$
$$=f_\theta(\theta,\phi)\boldsymbol{\theta}+f_\phi(\theta,\phi)\boldsymbol{\varphi}$$

The backscattering (σ_b), the forward (σ_f) and the total (Q_t) scattering cross-sections are defined as follows:

$$\frac{\sigma_b}{\lambda^2}=4\pi\left(\left|f_\theta(\pi-\theta_0,\pi)\right|^2+\left|f_\phi(\pi-\theta_0,\pi)\right|^2\right),$$
$$\frac{\sigma_f}{\lambda^2}=4\pi\left(\left|f_\theta(\theta_0,0)\right|^2+\left|f_\phi(\theta_0,0)\right|^2\right),\frac{Q_t}{\lambda^2}=\frac{1}{4\pi^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\left|f_\theta\right|^2+\left|f_\phi\right|^2\right)\sin\theta d\theta d\phi \tag{19}$$

with λ the wavelength of the electromagnetic waves.

For h small analytical closed-form expressions of the form

$$S(h)=S(0)[1+g^{(2)}h^2+g^{(4)}h^4+O(h^6)] \tag{20}$$

can be found for the various scattering cross-sections.

The TM case is examined in a similar manner. For the oblate spheroid we simply replace h^2 with $-h^2$ in each case.

3. Solution in Terms of Spheroidal Vector Wave Functions

We next apply the method using spheroidal vector wave functions. The incident plane electromagnetic wave impinging on the scatterer is expressed as (TE case) [6]

$$\mathbf{E}_i=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[c'_{mn}(c,\theta_0)\mathbf{M}^{r(1)}_{emn}(c;\eta,\xi,\phi)+d'_{mn}(c,\theta_0)\mathbf{N}^{r(1)}_{omn}(c;\eta,\xi,\phi)\right] \tag{21}$$

where θ_0 defines the direction of the incidence, η,ξ,φ are the spheroidal coordinates , $\mathbf{M}^{r(1)}_{emn}$ and $\mathbf{N}^{r(1)}_{omn}$ are spheroidal vector wave functions of the first kind (where the subscript e stands for even while o stands for odd)defined in [6] , c=kd/2 (k is the wavenumber).The coefficients c'_{mn} and d'_{mn} are given by

$$c'_{mn}(c,\theta_0)=-\frac{2\in_m i^n}{N_{mn}}\sum_{r=0,1}^{\infty}{}'\frac{d_r^{mn}}{(r+m)(r+m+1)}\frac{dP^m_{m+r}(\cos\theta_0)}{d\theta_0}$$
$$d'_{mn}(c,\theta_0)=\frac{4mi^{n-1}}{N_{mn}}\sum_{r=0,1}^{\infty}{}'\frac{d_r^{mn}}{(r+m)(r+m+1)}\frac{P^m_{m+r}(\cos\theta_0)}{\sin\theta_0} \tag{22}$$

where N_{mn} is the normalization constant define by

$$N_{mn}=2\sum_{r=0,1}^{\infty}{}'\frac{\left(d_r^{mn}\right)^2(r+2m)!}{(2r+2m+1)r!} \tag{23}$$

The prime over the summation symbols indicates that when n-m is even/odd this summation starts with the first/second value of r and continues only with values with the same parity, while the expansion coefficients d_r^{mn} are defined in [6].

The scattered wave is

$$\mathbf{E}_i = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[A'_{mn}\mathbf{M}'^{(3)}_{emn}\left(c;\eta,\xi,\phi\right)+B'_{mn}\mathbf{N}'^{(3)}_{omn}\left(c;\eta,\xi,\phi\right)\right] \tag{24}$$

where $\mathbf{M}'^{(3)}_{emn}$ and $\mathbf{N}'^{(3)}_{omn}$ are spheroidal vector wave functions of the third kind[6]. In order to obtain the unknown expansion coefficients A'_{mn} and B'_{mn} the boundary conditions on the surface of the conducting spheroid are satisfied.

$$\left(E_i + E_s\right)_\eta = 0 \text{ and } \left(E_i + E_s\right)_\phi = 0 \tag{25}$$

We next substitute the expansion formula [6]

$$R^{(1)}_{mn}(c,\cosh\mu)S_{mn}(c,\eta) = \sum_{s=0,1}^{\infty}{}' j^{m-n+s} d_s^{mn}(c)P^m_{m+s}(\cos\theta)j_{m+s}(kr) \tag{26}$$

where $R^{(1)}_{mn}$ and S_{mn} are, respectively, the radial and the angular, spheroidal functions of the first kind, as well as a similar one for $R^{(3)}_{mn}$ (the radial spheroidal function of the third kind) with h_{m+r} instead of j_{m+r}. By using, finally, the orthogonal properties of the associated Legendre and the trigonometric functions, an infinite set of linear inhomogeneous equations for the expansion coefficients A'_{mn} and B'_{mn} with the same form with the one in (11), (12)-(16) .Thus A'_{mn}'sand B'_{mn}'s are obtained by its solution using the same method as before.

Using the relation [10]

$$\mathbf{M}'^{(3)}_{\sigma mn} = \sum_{s=m,m+1}^{\infty}{}' j^{s-n} d^{mn}_{s-m}\mathbf{m}^{(3)}_{ms},\ \mathbf{N}'^{(3)}_{\sigma mn} = \sum_{s=m,m+1}^{\infty}{}' j^{s-n} d^{mn}_{s-m}\mathbf{n}^{(3)}_{ms} \qquad ,\sigma = e,o \tag{27}$$

the scattered electromagnetic field can be written in the form (6) where

$$A_{mn} = \sum_{n=s,s\pm2,s\pm4}\left(A'_{ms}i^{n-s}d^{ms}_{n-m}\right), \qquad B_{mn} = \sum_{n=s,s\pm2,s\pm4}\left(B'_{ms}i^{n-s}d^{ms}_{n-m}\right) \tag{28}$$

and thus we follow the same steps as in section 2.

A similar procedure can be followed for the TM case. For the oblate spheroid we replace h^2 with - h^2.

4. Numerical Results

In Figs. 2-4 the scattering cross-sections are plotted, versus θ_0, for both TE and TM cases, for a prolate and an oblate metallic spheroid for $a/\lambda=0.7$ and $h=0.2$. In each figure the corresponding scattering cross-section for $h=0$ (sphere with radius a) is also plotted. The results are symmetric about $\theta_0=90^0$, as it is imposed by the geometry of the scatterer. Another check for their correctness, moreover to the use of two different methods for the solution, is the validity of the forward scattering theorem, which in the present case has the form $Q_t / \lambda^2 = \mathrm{Im}[G(\theta_0,0)]/\pi$

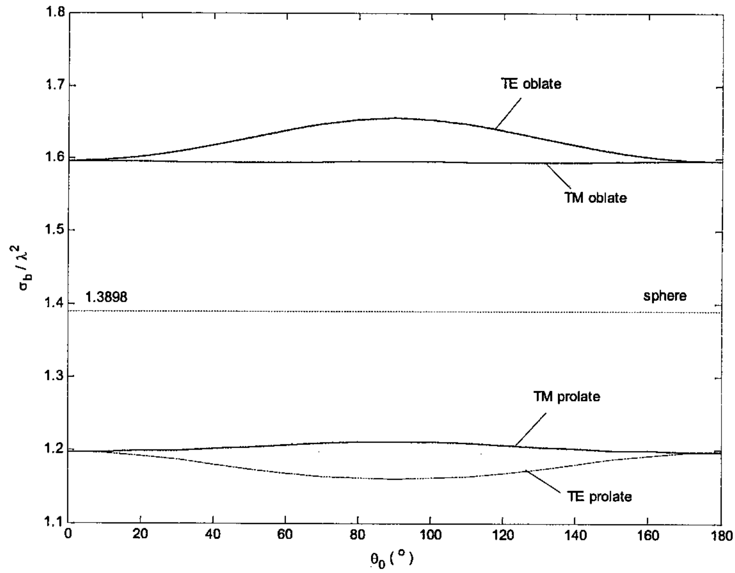

Fig.2 Backcattering cross-sections for oblate and prolate spheroids for $a/\lambda=0.7$ and h=0.2 (TE and TM case)

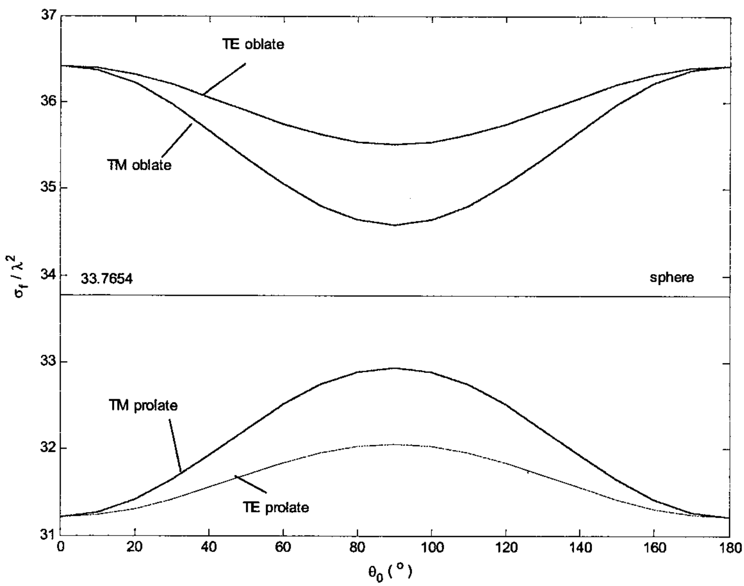

Fig.3 Forward-scattering cross-sections for oblate and prolate spheroids for $a/\lambda=0.7$ and h=0.2 (TE and TM case).

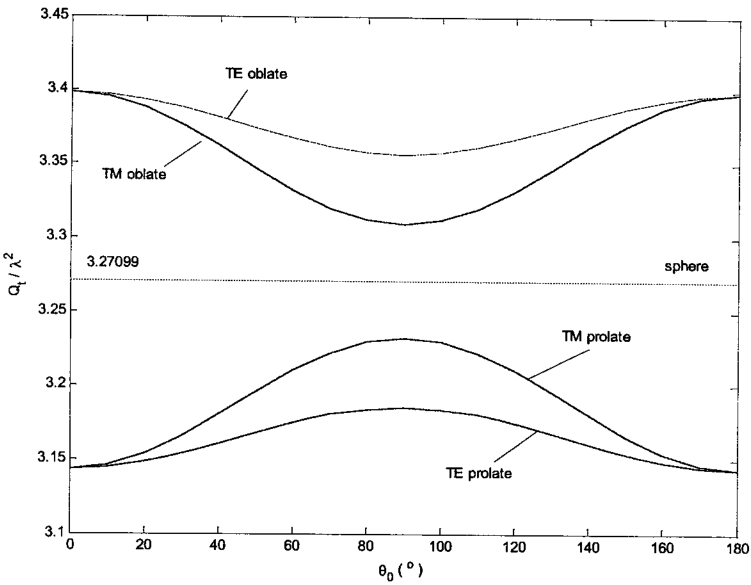

Fig.4 Total-scattering cross-sections for oblate and prolate spheroids for $a/\lambda=0.7$ and h=0.2 (TE and TM case)

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A NEW LINEAR SAMPLING METHOD FOR THE ELECTROMAGNETIC IMAGINING OF BURIED OBJECTS

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We present a new linear sampling method for determining the shape of scattering objects imbedded in a known inhomogeneous medium from a knowledge of the scattered electromagnetic field due to a point source incident field at fixed frequency. The method does not require any a prior information on the physical properties of the scattering object and, under some restrictions, avoids the need to compute the Green's tensor for the background medium.

1. Introduction

The mathematical modelling of the application of scattering of electromagnetic waves in mine detection, medical imagining, nondestructive testing etc. leads to the inverse scattering problem of determining the shape of the scattering object imbedded in a known inhomogeneous background. Typically, in such applications, neither the physical properties of the scatterer object nor geometrical features such as the number of components etc. are known a priori. In particular the scatterer can be a perfect conductor of dielectric, partially coated etc and this information in general is not available. The aim of this paper is to develop a method for solving the inverse problem which does not depend on the physical properties of the scatterer and

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is easy to implement. The solution method we have in mind is a new version of the linear sampling method based on the reciprocity gap functional which, in certain cases, avoids the need to compute the Green's function for the background medium. For the sake of theoretical justification of the method and in order to present the basic ideas we confine ourselves to the case of scattering by a perfect conductor buried in a known piecewise homogeneous background. However, the method can be applied to other type of scatterer and we refer the reader to [1], [4] for the mathematical justification of the method in the case of anisotropic penetrable objects.

We consider the scattering of a time-harmonic electromagnetic field of frequency ω by a scattering object embedded in a piecewise homogeneous background in $\mathbb{R}^3$. We assume that the magnetic permeability $\mu_0 > 0$ of the background medium is a positive constant whereas the electric permitivity $\epsilon(x)$ and conductivity $\sigma(x)$ are piecewise constant. Moreover we assume that for $|x| = r > R$, for R sufficiently large, $\sigma = 0$ and $\epsilon(x) = \epsilon_0$. After an appropriate scaling [7] and elimination of the magnetic field we now obtain the following equation for the electric field E in the background medium

$$\operatorname{curl}\operatorname{curl} E - k^2 n(x) E = 0,$$

$k = \epsilon_0 \mu_0 \omega^2$ and $n(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right)$. Note that the piecewise constant function $n(x)$ satisfies $n(x) = 1$ for $r > R$, $\Re(n) > 0$ and $\Im(n) \geq 0$. The surfaces across which $n(x)$ is discontinuous are assumed to be piecewise smooth.

Now let D be the support of a perfect conductor embedded in the above piecewise homogeneous background. We suppose that $\mathbb{R}^3 \setminus \overline{D}$ is connected the boundary ∂D of D is piecewise smooth and denote by ν the outward unit normal. Furthermore, we suppose that the incident field is an electric dipole located at $x_0 \in \Lambda$ with polarization $p \in \mathbb{R}^3$, where Λ is a smooth open surface situated in a layer with constant index of refraction n_s, given by

$$E_e(x, x_0, p, k_s) := \frac{i}{k_s}\operatorname{curl}_x \operatorname{curl}_x p \, \frac{e^{ik_s|x-x_0|}}{4\pi|x-x_0|} \tag{1}$$

where $k_s^2 = k^2 n_s$. We denote by $\mathbb{G}(x, x_0)$ the free space Green's tensor of the background medium and define $E^i(x) := E^i(x, x_0, p) = \mathbb{G}(x, x_0)p$ which satisfies

$$\operatorname{curl}\operatorname{curl} E^i(x) - k^2 n(x) E^i(x) = p\,\delta(x - x_0) \qquad \text{in } \mathbb{R}^3, \tag{2}$$

where δ denotes the Dirac distribution. Note that E^i can be written as

$$E^i(x) = E_e(x, x_0, p, k_s) + E_b^s(x) \tag{3}$$

where $E_b^s = E_b^s(\cdot,\, x_0, p)$ is the electric scattered field due to the background medium. The scattering of the dipole $E_e(x, x_0, p, k_s)$ by the perfect conductor D is described by the following boundary value problem: Given $E^i = E^i(\cdot\,, x_0, p) = \mathbb{G}(\cdot\,, x_0)p$, find $E \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D} \cup \{x_0\})$ satisfying

$$\text{curl}\,\text{curl}\, E - k^2 n(x) E = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \cup \{x_0\}, \tag{4}$$

$$\nu \times E = 0 \qquad \text{on} \quad \partial D, \tag{5}$$

$$E^s := (E - E^i) \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D}), \tag{6}$$

$$\lim_{r\to\infty} (\text{curl}\, E^s \times x - ikrE^s) = 0. \tag{7}$$

where

$$H(\text{curl}, D) := \{u \in (L^2(D))^3 \,:\, \nabla \times u \in (L^2(D))^3\}$$

and $H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{D})$ the space of functions $u \in H(\text{curl}, K)$ for all compact sets $K \subset \mathbb{R}^3 \setminus \overline{D}$.

Remark 1.1. *It is also possible to consider the problem of objects buried in an unbounded multi-layer medium. In this case, the radiation condition and mathematical analysis of the forward become more complicated (see [8] for the case of two layered medium). However the following analysis of the inverse scattering problems remains the same.*

Let Ω be such that $\overline{D}$ is contained in Ω and the open surface Λ is contained in $\mathbb{R}^3 \setminus \overline{\Omega}$. Let Γ denote the piecewise smooth boundary of Ω. Note that Λ may be a subset of Γ. The *inverse scattering problem* we are interested in is to determine D from a knowledge of the tangential components $\nu \times E$ and $\nu \times H$ of the total electric field $E = E(\cdot\,, x_0, p)$ and magnetic field $H = \frac{1}{ik}\text{curl}\, E$ measured on Γ for all point sources $x_0 \in \Lambda$ and two linearly independent polarizations p tangent to Λ at x_0. Here ν denotes the outward unit normal to Γ. The *linear sampling method* can be use to solve the inverse scattering problem. [1] (for a scholarly review of the this method we direct the reader to [2], [3], [5]). In particular, the linear sampling method is based on finding a tangential field $\varphi_z \in L_t^2(\Lambda)$ that satisfies the following integral equation of the first kind referred to as the *near field equation*:

$$(\mathcal{F}\varphi_z)(x) := \int_\Lambda \nu(x) \times E^s(x, y, \varphi_z(y))\, ds(y) = \nu(x) \times \mathbb{G}(x, z)\, q \tag{8}$$

for all $x \in \Gamma$, where $z \in \Omega$ and $q \in \mathbb{R}^3$ is an artificial polarization. Note that since E^s depends linearly on the polarization p, the *near field operator* $\mathcal{F} : L_t^2(\Lambda) \to L_t^2(\Gamma)$ is linear. Assuming that k is not a Maxwell eigenvalue, i.e. the interior boundary value problem

$$\operatorname{curl}\operatorname{curl} E - k^2 n(x) E = 0 \quad \text{in } D \tag{9}$$

$$\nu \times E = -\nu \times \mathbb{G}(\cdot, z) q \quad \text{on } \partial D \tag{10}$$

has a unique solution, one can prove that

(1) For $z \in D$ and a given $\epsilon > 0$, there exists a $\varphi_z^\epsilon \in L_t^2(\Lambda)$ such that

$$\|\mathcal{F}\varphi_z^\epsilon - \nu \times \mathbb{G}(\cdot, z) q\|_{L_t^2(\Gamma)} < \epsilon$$

and the corresponding potential $\mathcal{S}\varphi_z^\epsilon$ converges to the solution of (9)-(10) in $H(curl, D)$ as $\epsilon \to 0$.

(2) For a fixed $\epsilon > 0$, we have that

$$\lim_{z \to \partial D} \|\mathcal{S}\varphi_z^\epsilon\|_{H(curl,D)} = \infty \quad \text{and} \quad \lim_{z \to \partial D} \|\varphi_z^\epsilon\|_{L_t^2(\Lambda)} = \infty.$$

(3) For $z \in \mathbb{R}^3 \setminus \overline{D}$ and a given $\epsilon > 0$, every $\varphi_z^\epsilon \in L_t^2(\Lambda)$ that satisfies

$$\|\mathcal{F}\varphi_z^\epsilon - \nu \times \mathbb{G}(\cdot, z) q\|_{L_t^2(\Gamma)} < \epsilon$$

is such that

$$\lim_{\epsilon \to 0} \|\mathcal{S}\varphi_z^\epsilon\|_{H(curl,D)} = \infty \quad \text{and} \quad \lim_{\epsilon \to 0} \|\varphi_z^\epsilon\|_{L_t^2(\Lambda)} = \infty.$$

The above result provides a characterization for the boundary ∂D of the scattering object D. Unfortunately, since the behavior of $S\varphi_z^\epsilon$ is described in terms of a norm depending on the unknown region D, $S\varphi_z^\epsilon$ can not be used to characterize D. Instead the linear sampling method characterizes the obstacle by the behavior of φ_z^ϵ. In particular, given a discrepancy $\epsilon > 0$ and φ_z^ϵ the ϵ-approximate solution of the near field equation, the boundary of the scatterer is reconstructed as the set of points z where the $L_t^2(\Lambda)$ norm of φ_z^ϵ becomes large.

Even though the linear sampling method, in principle, can be used in the case of a quite general inhomogeneous background, the main drawback of the method in this case is the need to compute the background Green's function. This job can be numerically very costly for complex background geometries. The main goal of this paper is to show how, at the expense of additional data and restrictions, one can avoid the need to compute the background Green's function.

2. The Reciprocity Gap Functional

We make two additional assumptions. First, we assume that the medium inside the domain Ω containing the scattering object D is homogeneous with constant index of refraction n_b and define $k_b^2 = k^2 n_b$. Second, we assume that *both* the tangential components $\nu \times E$ and $\nu \times H$ of the total electric field $E = E(\cdot, x_0, p)$ and magnetic field $H = \frac{1}{ik}\operatorname{curl} E$, respectively, are known on Γ for all point sources $x_0 \in \Lambda$. In other words we assume that we know $\nu \times E|_\Gamma$ and $\nu \times \operatorname{curl} E|_\Gamma$ for all $x_0 \in \Lambda$. Furthermore, without loss of generality, we assume that Λ is a closed surface surrounding Ω situated in a layer with index of refraction n_s. By an analyticity argument the following analysis also holds true if the point sources are located on an open analytic surface provided it can be extended to a closed (analytic) surface as above.

We need to recall the definition of the following trace spaces

$$H_{div}^{-\frac{1}{2}}(\partial D) := \left\{ u \in (H^{-\frac{1}{2}}(\partial D))^3, \quad \nu \cdot u = 0, \quad \operatorname{div}_{\partial D} u \in H^{-\frac{1}{2}}(\partial D) \right\}$$
$$H_{curl}^{-\frac{1}{2}}(\partial D) := \left\{ u \in (H^{-\frac{1}{2}}(\partial D))^3, \quad \nu \cdot u = 0, \quad \operatorname{curl}_{\partial D} u \in H^{-\frac{1}{2}}(\partial D) \right\}$$

with $\operatorname{curl}_{\partial D}$ denoting the surface curl. It is known that traces $\nu \times u|_{\partial D}$ and $\nu \times (u \times \nu)|_{\partial D}$ of $u \in H(\operatorname{curl}, D)$ (or $u \in H_{loc}(\operatorname{curl}, D)$) are in $H_{div}^{-\frac{1}{2}}(\partial D)$ and $H_{curl}^{-\frac{1}{2}}(\partial D)$ respectively. Note that by an integration by parts we can define a duality relation between $H_{div}^{-\frac{1}{2}}(\partial D)$ and $H_{curl}^{-\frac{1}{2}}(\partial D)$.
Let $E = E(\cdot, x_0, p) = E^s(\cdot, x_0, p) + \mathbb{G}(\cdot, x_0)p$ and $H = 1/ik \operatorname{curl} E$ be the total electric and magnetic fields, respectively, corresponding to the scattering problem (4)-(7). Then for any function $W \in H(\operatorname{curl}, \Omega)$, we can define the *gap reciprocity functional* by

$$\mathcal{R}(E, W) = \int_\Gamma (\nu \times E) \cdot \operatorname{curl} W - (\nu \times W) \cdot \operatorname{curl} E \, ds. \tag{11}$$

Since $E \in H(\operatorname{curl}, \Omega)$, the integral is interpreted in the sense of the duality between $H_{div}^{-\frac{1}{2}}(\Gamma)$ and $H_{curl}^{-\frac{1}{2}}(\Gamma)$. Note that E depends on x_0 and hence so does $\mathcal{R}$. Next we define the subspace $\mathbb{H}(\Omega) \subset H(\operatorname{curl}, \Omega)$ by

$$\mathbb{H}(\Omega) := \left\{ W \in H(\operatorname{curl}, \Omega) : \ \operatorname{curl}\operatorname{curl} W - k_b^2 W = 0 \right\}.$$

The reciprocity gap functional restricted to $\mathbb{H}(\Omega)$ can be seen as an operator $R : \mathbb{H}(\Omega) \to L_t^2(\Lambda)$ defined by

$$R(W)(x_0) \cdot p(x_0) = \mathcal{R}(E(\cdot, x_0, p(x_0)), W) \tag{12}$$

for all $x_0 \in \Lambda$ and $p(x_0)$ a tangent vector to Λ at x_0. Now, we consider a subset $\{A\varphi \in \mathbb{H}(\Omega) : \varphi \in L^2_{div}(\tilde{\Lambda})\}$ of $\mathbb{H}(\Omega)$, where $A\varphi$ is the single layer potential defined by

$$(A\varphi)(x) := \operatorname{curl}\operatorname{curl}\int_{\tilde{\Lambda}} \varphi(y)\Phi(x,y,k_b)\,ds, \qquad \varphi \in L^2_{div}(\tilde{\Lambda}) \tag{13}$$

where

$$\Phi(x,y,k_b) := \frac{1}{4\pi}\frac{e^{ik_b|x-y|}}{|x-y|}, \quad x \neq y,$$

$\tilde{\Lambda}$ is a regular part of the boundary of some simply connected domain containing Ω in its interior, and $L^2_{div}(\tilde{\Lambda})$ is the space of vector functions $u \in (L^2(\tilde{\Lambda}))^3$ such that $\nu \cdot u = 0$ and $\operatorname{div}_{\partial D} u \in L^2(\tilde{\Lambda})$. Next, letting

$$E_e(x,z,q,k_b) = \frac{i}{k}\operatorname{curl}_x\operatorname{curl}_x q\,\Phi(x,z,k_b), \qquad q \in \mathbb{R}^3 \tag{14}$$

denote the electric dipole corresponding to k_b we look for a solution $\varphi \in L^2_{div}(\tilde{\Lambda})$ of

$$\mathcal{R}(E, A\varphi) = \mathcal{R}(E, E_e(\cdot, z, q, k_b)). \tag{15}$$

Then, the linear sampling method based on the reciprocity gap functional characterizes D from the behavior of φ for different sampling points $z \in \Omega$. In the rest of the paper we investigate the solvability of (15) with respect to φ. To this end we prove the following important lemmas.

Lemma 2.1. *Assume that k_b is not a Maxwell eigenvalue for D. Then the operator $R : \mathbb{H}(\Omega) \to L^2_t(\Lambda)$ defined by (12) is injective.*

Proof. $RW = 0$ means $\mathcal{R}(E(\cdot, x_0, p(x_0)), W) = 0$ for all $(x_0, p(x_0))$ as in (12). Since both E and W satisfy Maxwell's equation in $\Omega \setminus \overline{D}$, we have, using the boundary condition on ∂D,

$$0 = \textstyle\int_\Gamma (\nu \times E)\cdot \operatorname{curl} W - (\nu \times W)\cdot \operatorname{curl} E\,ds = -\displaystyle\int_{\partial D} (\nu \times W)\cdot \operatorname{curl} E\,ds.$$

It suffices to show that the set $L := \{(\operatorname{curl} E(\cdot,\, x_0, p(x_0)))_T : x_0 \in \Lambda\}$ is dense in $H^{-\frac{1}{2}}_{curl}(\partial D)$. Indeed, this fact implies that $\nu \times W = 0$ on ∂D and from the uniqueness of the solution to (9)-(10) we have that $W = 0$ in D, whence from the unique continuation principle we obtain $W = 0$ in Ω.
To prove the denseness property, let $f \in H^{-\frac{1}{2}}_{div}(\partial D)$ and assume that

$$\int_{\partial D} f \cdot (\nu \times \operatorname{curl} E)\,ds = 0$$

for all total fields E such that $(\operatorname{curl} E)_T \in L$. Let $\tilde{E}$ be the unique solution to

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\tilde{E} - k^2 n(x)\tilde{E} = 0 \qquad & \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \\ \nu \times \tilde{E} = f \qquad & \text{on} \quad \partial D \\ \lim_{r\to\infty}\left(\operatorname{curl}\tilde{E} \times x - ikr\tilde{E}\right) = 0. & \end{aligned}$$

By a duality argument we have that

$$\begin{aligned} 0 &= \int_{\partial D} f \cdot (\nu \times \operatorname{curl} E)\, ds = \int_{\partial D} \tilde{E} \cdot [\nu \times \operatorname{curl}(E^s + \mathbb{G}(\cdot, x_0)p)]\, ds \\ &= \int_{\partial D} \tilde{E} \cdot (\nu \times \operatorname{curl} E^s)\, ds + \int_{\partial D} \tilde{E} \cdot (\nu \times \operatorname{curl}\mathbb{G}(\cdot, x_0)p)\, ds. \end{aligned} \tag{16}$$

Since both E^s and $\tilde{E}$ are radiating solutions to $\operatorname{curl}\operatorname{curl} E - k^2 n(x)E = 0$ outside D, by applying the vector Green's formula we have that

$$\int_{\partial D} \tilde{E} \cdot (\nu \times \operatorname{curl} E^s)\, ds = \int_{\partial D} E^s \cdot (\nu \times \operatorname{curl}\tilde{E})\, ds. \tag{17}$$

Substituting (17) into (16) and using the boundary condition $\nu \times E^s = -\nu \times \mathbb{G}(\cdot, x_0)p$ on ∂D we have that

$$0 = \int_{\partial D} \tilde{E} \cdot (\nu \times \operatorname{curl}\mathbb{G}(\cdot, x_0)p)\, ds + \int_{\partial D} E^s \cdot (\nu \times \operatorname{curl}\tilde{E})\, ds = p \cdot \tilde{E}(x_0).$$

Since p is an arbitrary polarization in the tangent plane to Λ at x_0, we obtain $\nu \times \tilde{E}(x_0) = 0$ for $x_0 \in \Lambda$. Furthermore, since $\tilde{E}$ is a radiating solution to Maxwell's equations outside the domain bounded by Λ, we conclude by the uniqueness theorem for the scattering problem for a perfect conductor (c.f. [7]) that $\tilde{E} = 0$ outside the domain bounded by Λ. Then the unique continuation principle implies that $\tilde{E} = 0$ outside D, whence $f = 0$, which proves the lemma. □

Lemma 2.2. *Assume that k_b is not a Maxwell eigenvalue for D. Then the operator $R : \mathbb{H}(\Omega) \to L^2_t(\Lambda)$ defined by (12) has dense range.*

Proof. Consider $\alpha \in L^2_t(\Lambda)$ and assume that

$$(RW, \alpha)_{L^2_t(\Lambda)} = 0 \ \text{ for all } W \in \mathbb{H}(\Omega).$$

From (12) and the bi-linearity of $\mathcal{R}$ one has

$$(RW, \alpha)_{L^2_t(\Lambda)} = \int_\Lambda \mathcal{R}(E(\cdot, x_0, \alpha(x_0)), W)\, ds(x_0) = \mathcal{R}(\mathcal{E}, W),$$

where

$$\mathcal{E}(x) = \int_{\Lambda} E(x, x_0, \alpha(x_0))\, ds(x_0). \tag{18}$$

Using Green's vector formulas and the boundary condition on ∂D one concludes that

$$0 = \mathcal{R}(\mathcal{E}, W) = -\int_{\partial D} \operatorname{curl} \mathcal{E} \cdot (\nu \times W)\, ds \tag{19}$$

for all $W \in \mathbb{H}(\Omega)$. Since $\mathbb{H}(\Omega)$ contains the Herglotz wave functions, from [6] one has that the set of $(\nu \times W)|_{\partial D}$ is dense in $H_{div}^{-\frac{1}{2}}(\partial D)$. Therefore

$$\operatorname{curl} \mathcal{E} \times \nu = 0 \quad \text{on } \partial D.$$

Since $\mathcal{E} \times \nu = 0$ on ∂D as well, the extension of $\mathcal{E}$ by 0 inside D satisfies Maxwell's equations inside the domain bounded by Λ with the index n set equal to n_b inside D. From the unique continuation principle one has that $\mathcal{E}$ is 0 inside the domain bounded by Λ and outside D. Noting that

$$\mathcal{E}(x) = \int_{\Lambda} (E^s(x, x_0, \alpha(x_0)) + \mathbb{G}(x, x_0)\alpha(x_0))\, ds(x_0)$$

one concludes that $\mathcal{E} \times \nu$ is continuous across Λ. The uniqueness theorem of the exterior problem for Maxwell's equations with boundary data $\nu \times \mathcal{E} = 0$ on Λ implies that $\mathcal{E} = 0$ outside the domain bounded by Λ as well. Finally, from the jump relations of the vector potential across Λ [7] we have that

$$0 = \operatorname{curl} \mathcal{E}|_{\Lambda^+} - \operatorname{curl} \mathcal{E}|_{\Lambda^-} = -\alpha \quad \text{on} \quad \Lambda$$

which ends the proof. □

Lemma 2.3. *Assume that k is not a Maxwell eigenvalue for D. Then the set $\{A\varphi,\ \varphi \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})$ is dense in $H(curl, D)$.*

Proof. Making use of the well-posedness of

$$\operatorname{curl}\operatorname{curl} W - k^2 n_b W = 0 \quad \text{in } D \tag{20}$$
$$\nu \times W = f \quad \text{on } \partial D \tag{21}$$

with $f \in H_{div}^{-\frac{1}{2}}(\partial D)$, it suffices to show that $\nu \times A\varphi|_{\partial D}$ for all $\varphi \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})$ is dense in $H_{div}^{-\frac{1}{2}}(\partial D)$. To this end, let $\psi \in H_{curl}^{-\frac{1}{2}}(\partial D)$ and look at the dual operator $A^* : H_{curl}^{-\frac{1}{2}}(\partial D) \to H_{div}^{-\frac{1}{2}}(\Gamma)$ such that

$$\langle \nu \times A\varphi,\ \psi \rangle_{\partial D} = \langle \varphi,\ A^* \psi \rangle_{\tilde{\Lambda}}$$

where $\langle\cdot,\,\cdot\rangle$ denotes the $H_{div}^{-\frac{1}{2}}$, $H_{curl}^{-\frac{1}{2}}$ duality pairing. By changing the order of integration one can show that for $\psi \in H_{div}^{-\frac{1}{2}}(\partial D)$

$$(A^*\psi)(y) = \nu(y) \times \left(\mathrm{curl}_y\,\mathrm{curl}_y \int_{\partial D} \psi(x)\Phi(x,y)\,ds(x)\right) \times \nu(y), \qquad y \in \tilde{\Lambda}$$

where ν is the unit outward normal to $\tilde{\Lambda}$. Now, since k_b is not a Maxwell eigenvalue for D, we conclude that A^* is injective, whence $\left\{\nu \times A\varphi|_{\partial D} :\ \varphi \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})\right\}$ is dense in $H_{div}^{-\frac{1}{2}}(\partial D)$. $\square$ Now we are at the position to prove the main result of this paper.

Theorem 2.1. *Assume that k is not a Maxwell eigenvalue for D and let $E = E(\cdot\,,x_0,p)$ and $H = 1/ik\,curl E$ be the total electric and magnetic fields, respectively, corresponding to the scattering problem (4)-(7). Then*

(1) For $z \in D$ and a given $\epsilon > 0$, there exists a $\varphi_z^\epsilon \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})$ such that

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b))\|_{L_t^2(\Lambda)} < \epsilon$$

and the corresponding potential $A\varphi_z^\epsilon$ converges to the solution of

$$curl\,curl W - k^2 n_b W = 0 \quad in\ D \tag{22}$$

$$\nu \times W = E_e(\cdot,\, z, q, k_b) \quad on\ \partial D \tag{23}$$

in $H(\mathrm{curl}, D)$ as $\epsilon \to 0$.

(2) For a fixed $\epsilon > 0$, we have that

$$\lim_{z\to\partial D} \|A\varphi_z^\epsilon\|_{H(curl,D)} = \infty \quad and \quad \lim_{z\to\partial D} \|\varphi_z^\epsilon\|_{H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})} = \infty.$$

(3) For $z \in \mathbb{R}^3 \setminus \overline{D}$ and a given $\epsilon > 0$, every $\varphi_z^\epsilon \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})$ that satisfies

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b))\|_{L_t^2(\Lambda)} < \epsilon$$

is such that

$$\lim_{\epsilon\to 0} \|A\varphi_z^\epsilon\|_{H(curl,D)} = \infty \quad and \quad \lim_{\epsilon\to 0} \|\varphi_z^\epsilon\|_{H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})} = \infty.$$

Proof. Let $z \in D$. Since $W \in \mathbb{H}(\Omega)$ and $E_e(\cdot,\, z, q, k_b)$ satisfy $\mathrm{curl}\,\mathrm{curl}\,W - k_b W = 0$ in $\Omega \setminus \overline{D}$, integrating by parts and using the boundary condition for the total field we have that

$$\mathcal{R}(E, W) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b)) = -\int_{\partial D} (\nu\times W - \nu\times E_e(\cdot,\, z, q, k_b))\cdot \mathrm{curl}\,E\,ds.$$

From the proof of Lemma 2.1 we see that $\mathcal{R}(E, W) = \mathcal{R}(E, E_e(\cdot,\, z, q, k_b))$ has a unique solution W if and only if there exists a $W \in \mathbb{H}(\Omega)$ such that $\nu\times$

$W = \nu\times E_e(\cdot,\ z, q, k_b) = 0$ on ∂D which is in general not true. However from Lemma 2.3 we have that the family $\{A\varphi, \in H_{div}^{-\frac{1}{2}}(\tilde{\Lambda})\}$ is dense in $H(\text{curl}, \Omega)$. Hence, from the trace theorem, for every $\epsilon > 0$ there exists a potential $A\varphi_z^\epsilon$ such that $\nu\times A\varphi_z^\epsilon$ approximates $\nu\times E_e(\cdot,\ z, q)$ with respect to the $H_{div}^{-\frac{1}{2}}(\partial D)$ norm. In particular, φ_z^ϵ is an approximate solution to (15) and $\nu \times A\varphi_z^\epsilon$ converges to the solution of (22)-(23) in the $H(\text{curl}, D)$ norm as $\epsilon \to 0$. Next, since $\|\nu\times E_e(\cdot,\ z, q)\|_{H_{div}^{-\frac{1}{2}}(\partial D)}$ blows up as z approaches the boundary, we obtain that, for a fixed $\epsilon > 0$, $\lim_{z\to\partial D}\|\nu \times A\varphi_z^\epsilon\|_{H_{div}^{-\frac{1}{2}}(\partial D)} = \infty$ and consequently $\lim_{z\to\partial D}\|A\varphi_z^\epsilon\|_{H(curl,D)} = \infty$ and $\lim_{z\to\partial D}\|g_z^\epsilon\|_{L_t^2(S^2)} = \infty$. Now we consider $z \in \Omega\setminus\overline{D}$ and let g_z^ϵ and its corresponding Herglotz function $A\varphi_z^\epsilon$ be such that

$$\|\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot,\ z, q, k_b))\|_{L^2(\Lambda)} < \epsilon. \tag{24}$$

Note that from Lemma 2.2 we can always find such a $A\varphi_z^\epsilon$. Assume to the contrary that $\|A\varphi_z^\epsilon\|_{H(curl,D)} < C$ where the positive constant C is independent of ϵ. From the trace theorem we have that $\nu \times A\varphi_z^\epsilon$ is also bounded in the $H_{div}^{-\frac{1}{2}}(\partial D)$ norm. Noting that the total field can be written as $E(\cdot,\ x_0, p) = E^s(\cdot,\ x_0, p) + \mathbb{G}(\cdot, x_0)p$ and integrating by parts, we obtain that

$$\begin{aligned}\mathcal{R}(E, E_e(x, z, q, k_b)) = &\int_\Gamma (\nu \times E^s(x, x_0, p)) \cdot \operatorname{curl} E_e(x, z, q, k_b)\, ds_x \\ &- \int_\Gamma (\nu \times E_e(x, z, q, k_b)) \cdot \operatorname{curl} E^s(x, x_0, p)\, ds_x \\ &+ \int_\Gamma (\nu \times \mathbb{G}(x, x_0)p) \cdot \operatorname{curl} E_e(x, z, q, k_b)\, ds_x \\ &- \int_\Gamma (\nu \times E_e(x, z, q, k_b)) \cdot \operatorname{curl} \mathbb{G}(x, x_0)p\, ds_x.\end{aligned}$$

Due to the symmetry of the background Green's function, $E^s(x, x_0, p)$ as a function of x_0 solves $\operatorname{curl}_{x_0}\operatorname{curl}_{x_0} E^s(x, x_0, p) - k^2 n(x_0)E^s(x, x_0, p) = 0$ in the domain bounded by Λ and ∂D. Hence the first two integrals in the above equation give a solution $W(x_0)$ to the same equation as $E^s(\cdot,\ x_0, p)$, while the last two integrals add up to $-\mathbb{G}(z, x_0)p$ by the Stratton-Chu formula and the fact that $E_e(x, z, q, k_b)$ is the fundamental solution of $\operatorname{curl}\operatorname{curl} E - k_b^2 E = 0$. On the other hand it is easy to see that

$$\mathcal{R}(E, A\varphi_z^\epsilon) = -\int_{\partial D} (\nu \times A\varphi_z^\epsilon) \cdot \operatorname{curl} E\, ds.$$

Combining the above results we finally have that

$$\mathcal{R}(E, A\varphi_z^\epsilon) - \mathcal{R}(E, E_e(\cdot,\, z, q, k_b)) \tag{25}$$
$$= -\int_{\partial D} (\nu \times A\varphi_z^\epsilon) \cdot \operatorname{curl} E\, ds - W(x_0) + \mathbb{G}(z, x_0)p.$$

Now since $\|A\varphi_z^\epsilon\|_{H_{div}^{-\frac{1}{2}}(\partial D)} < C$ there exists a subfamily, still denoted by $A\varphi_z^\epsilon$, that converges weakly to a $V \in H_{div}^{-\frac{1}{2}}(\partial D)$ in the duality pairing between $H_{div}^{-\frac{1}{2}}(\partial D)$ and $H_{curl}^{-\frac{1}{2}}(\partial D)$ as $\epsilon \to 0$. Let us set

$$\tilde{W}(x_0) = \lim_{\epsilon \to 0} \mathcal{R}(E, A\varphi_z^\epsilon) = -\int_{\partial D} (\nu \times V) \cdot \operatorname{curl} E(\cdot,\, x_0, p)\, ds, \qquad x_0 \in \Lambda.$$

From (24) we now have that

$$\tilde{W}(x_0) = W(x_0) + \mathbb{G}(z, x_0)p \qquad x_0 \in \Lambda. \tag{26}$$

Since $\tilde{W}(x_0)$ and $W(x_0)$ can be continued as radiating solutions to

$$\operatorname{curl}_{x_0} \operatorname{curl}_{x_0} E^s(x, x_0, p) - k^2 n(x_0) E^s(x, x_0, p) = 0$$

outside the domain bounded by Λ we deduce by uniqueness and the unique continuation principle that (26) holds true in $\mathbb{R}^3 \setminus (\overline{D} \cup \{z_0\})$. We now arrive at a contradiction by letting $x_0 \to z$. Hence $A\varphi_z^\epsilon$ is unbounded in the $H(D, \operatorname{curl})$ norm as $\epsilon \to 0$, which proves the theorem. $\square$

The above theorem, provides a characterization of the boundary ∂D of the scattering objects. In particular, ∂D is the set of points where the $L_t^2(\tilde{\Lambda})$-norm of the regularized approximate solution φ_z^ϵ of the equation (15). becomes large. For a detailed discussion on the numerical implementation of both the classical linear sampling method and the linear sampling method based on the reciprocity gap functional we refer the reader to [1]. Numerical examples comparing the performance of the linear sampling method based on the reciprocity gap functional to the classical linear method for solving the inverse scattering problem for objects imbedded in a two layered medium are also presented in [1].

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