Chapter 1
Three-Dimensional Theories

In this chapter we summarize the three-dimensional equations of the nonlinear theory of electroelasticty for large deformations and strong fields [1,2], the linear theory of piezoelectricity for infinitesimal deformation and fields [3,4], the linear theory for small fields superposed on finite biasing or initial fields [5,6,7], and the theory for weak, cubic nonlinearity [8,9]. A systematic presentation of these theories can also be found in [10]. This chapter uses the two-point Cartesian tensor notation, the summation convention for repeated tensor indices, and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index.

1.1. Nonlinear Electroelasticity for Strong Fields

Consider a deformable continuum which, in the reference configuration at time t_0, occupies a region V with a boundary surface S (see Fig. 1.1.1). $\mathbf{N}$ is the unit exterior normal of S. In this state the body is free from deformation and fields. The position of a material point in this state is denoted by a vector $\mathbf{X} = X_K\mathbf{I}_K$ in a rectangular coordinate system X_K where X_K denotes the reference or material coordinates of the material point. They form a continuous labeling of material particles so that they are identifiable. At time t, the body occupies a region v with a boundary surface s and an exterior normal $\mathbf{n}$. The current position of the material point associated with $\mathbf{X}$ is given by $\mathbf{y} = y_k\mathbf{i}_k$, which denotes the present or spatial coordinates of the material point.

Since the coordinate systems are othogonal,

$$\mathbf{i}_k \cdot \mathbf{i}_l = \delta_{kl} \quad \text{and} \quad \mathbf{I}_K \cdot \mathbf{I}_L = \delta_{KL}, \tag{1.1.1}$$

where δ_{kl} and δ_{KL} are the Kronecker delta. In matrix notation,

$$[\delta_{kl}] = [\delta_{KL}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.1.2}$$

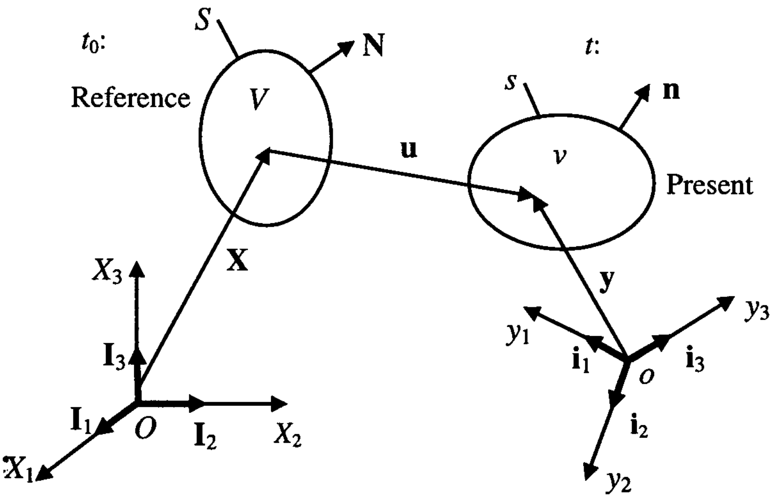

Fig. 1.1.1. Motion of a continuum and coordinate systems.

For the rest of this book the two coordinate systems are chosen to be coincident, i.e.,

$$o = O, \quad \mathbf{i}_1 = \mathbf{I}_1, \quad \mathbf{i}_2 = \mathbf{I}_2, \quad \mathbf{i}_3 = \mathbf{I}_3. \tag{1.1.3}$$

The transformation coefficients (shifters) between the two coordinate systems are denoted by

$$\mathbf{i}_k \cdot \mathbf{I}_L = \delta_{kL}. \tag{1.1.4}$$

When the two coordinate systems are coincident, δ_{kL} is simply the Kronecker delta. It is still needed for notational homogeneity. A vector can be resolved into rectangular components in different coordinate systems. For example, we can also write

$$\mathbf{y} = y_K \mathbf{I}_K, \tag{1.1.5}$$

with

$$y_M = \delta_{Mi} y_i. \tag{1.1.6}$$

The motion of the body is described by $y_i = y_i(\mathbf{X}, t)$. The equations of motion and Gauss's equation of electrostatic (the charge equation) are

$$\begin{aligned} &K_{Lj,L} + \rho_0 f_j = \rho_0 \ddot{y}_j, \\ &\mathcal{D}_{K,K} = \rho_E, \end{aligned} \tag{1.1.7}$$

where K_{Lj} is the two-point total stress tensor, ρ_0 (a scalar) is the reference mass density, f_j is the mechanical body force per unit mass, and $\mathcal{D}_K$ is the reference electric displacement vector. ρ_E, a scalar (E is not an index), is the free charge density per unit reference volume, and a superimposed dot represents the material time derivative

$$\ddot{y}_i = \frac{D^2 y_i}{Dt^2} = \left.\frac{\partial^2 y_i(\mathbf{X},t)}{\partial t^2}\right|_{\mathbf{X}\ \text{fixed}} . \tag{1.1.8}$$

In Eq. (1.1.7), K_{Lj} and $\mathcal{D}_K$ are given by:

$$\begin{aligned} &K_{Lj} = F_{Lj} + M_{Lj}, \\ &F_{Lj} = y_{j,K} T^S_{KL}, \quad M_{Lj} = JX_{L,i}\varepsilon_0\left(E_i E_j - \frac{1}{2}E_k E_k \delta_{ij}\right), \\ &J = \det(y_{i,K}), \quad T^S_{KL} = \rho_0 \frac{\partial \psi}{\partial S_{KL}}, \quad E_i = -\phi_{,i}, \end{aligned} \tag{1.1.9}$$

and

$$\begin{aligned} &\mathcal{D}_K = \varepsilon_0 J X_{K,i} D_i = \varepsilon_0 J C^{-1}_{KL} \mathcal{E}_L + \mathcal{P}_K, \\ &C^{-1}_{KL} = X_{K,i} X_{L,i}, \\ &\mathcal{E}_K = y_{i,K} E_i = -\phi_{,K}, \quad \mathcal{P}_K = JX_{K,i} P_i = -\rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}, \end{aligned} \tag{1.1.10}$$

where ε_0 (a scalar) is the electric permittivity of free space, E_i is the electric field, P_i is the electric polarization per unit present volume, and D_i is the electric displacement vector. $\mathcal{E}_K$ is the reference electric field vector, and $\mathcal{P}_K$ is the reference electric polarization vector. ϕ is the electric potential and C^{-1}_{KL} is the inverse of the deformation tensor. $\psi = \psi(S_{KL}, \mathcal{E}_K)$ is a free energy density per unit mass, which is a function of $\mathcal{E}_K$ and the following finite strain tensor:

$$S_{KL} = (y_{i,K} y_{i,L} - \delta_{KL})/2 . \tag{1.1.11}$$

From Eqs. (1.1.9) and (1.1.10), we have

$$\begin{aligned} &K_{Lj} = y_{j,K}\rho_0 \frac{\partial \psi}{\partial S_{KL}} + JX_{L,i}\varepsilon_0\left(E_i E_j - \frac{1}{2}E_k E_k \delta_{ij}\right), \\ &\mathcal{D}_K = \varepsilon_0 J C^{-1}_{KL} \mathcal{E}_L - \rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}. \end{aligned} \tag{1.1.12}$$

With successive substitutions from Eqs. (1.1.9) through (1.1.11), Eq. (1.1.7) can be written as four equations for the four unknowns $y_i(\mathbf{X},t)$ and $\phi(\mathbf{X},t)$.

The free energy ψ that determines the constitutive relations of nonlinear electroelastic materials may be written as

$$\begin{aligned}
&\rho_0\psi(S_{KL},\mathcal{E}_K)\\
&=\frac{1}{2}{}_2c_{ABCD}S_{AB}S_{CD}-e_{ABC}\mathcal{E}_AS_{BC}-\frac{1}{2}{}_2\chi_{AB}\mathcal{E}_A\mathcal{E}_B\\
&+\frac{1}{6}{}_3c_{ABCDEF}S_{AB}S_{CD}S_{EF}+\frac{1}{2}{}_1k_{ABCDE}\mathcal{E}_AS_{BC}S_{DE}\\
&-\frac{1}{2}b_{ABCD}\mathcal{E}_A\mathcal{E}_BS_{CD}-\frac{1}{6}{}_3\chi_{ABC}\mathcal{E}_A\mathcal{E}_B\mathcal{E}_C\\
&+\frac{1}{24}{}_4c_{ABCDEFGH}S_{AB}S_{CD}S_{EF}S_{GH}+\frac{1}{6}{}_2k_{ABCDEFG}\mathcal{E}_AS_{BC}S_{DE}S_{FG}\\
&+\frac{1}{4}{}_1a_{ABCDEF}\mathcal{E}_A\mathcal{E}_BS_{CD}S_{EF}+\frac{1}{6}{}_3k_{ABCDE}\mathcal{E}_A\mathcal{E}_B\mathcal{E}_CS_{DE}\\
&-\frac{1}{24}{}_4\chi_{ABCD}\mathcal{E}_A\mathcal{E}_B\mathcal{E}_C\mathcal{E}_D+\cdots,
\end{aligned}\tag{1.1.13}$$

where the material constants

$$\begin{aligned}
&{}_2c_{ABCD},\quad e_{ABC},\quad {}_2\chi_{AB},\\
&{}_3c_{ABCDEF},\quad {}_1k_{ABCDE},\quad b_{ABCD},\quad {}_3\chi_{ABC},\\
&{}_4c_{ABCDEFGH},\quad {}_2k_{ABCDEFG},\quad {}_1a_{ABCDEF},\quad {}_3k_{ABCDE},\quad {}_4\chi_{ABCD}
\end{aligned}\tag{1.1.14}$$

are called the second-order elastic, piezoelectric, electric susceptibility, third-order elastic, first odd electroelastic, electrostrictive, third-order electric susceptibility, fourth-order elastic, second odd electroelastic, first even electroelastic, third odd electroelastic, and fourth-order electric susceptibility, respectively. The second-order constants are responsible for linear material behaviors. The third- and higher-order material constants are related to nonlinear behaviors of materials.

For mechanical boundary conditions S is partitioned into S_y and S_T, on which motion (or displacement) and traction are prescribed, respectively. Electrically S is partitioned into S_ϕ and S_D with prescribed electric potential and surface free charge, respectively, and

$$\begin{aligned} &S_y \cup S_T = S_\phi \cup S_D = S, \\ &S_y \cap S_T = S_\phi \cap S_D = 0. \end{aligned} \tag{1.1.15}$$

The usual boundary value problem for an electroelastic body consists of Eq. (1.1.7) and the following boundary conditions:

$$\begin{aligned} &y_i = \bar{y}_i \quad \text{on} \quad S_y, \\ &\phi = \bar{\phi} \quad \text{on} \quad S_\phi, \\ &K_{Lk} N_L = \overline{T}_k \quad \text{on} \quad S_T, \\ &\mathcal{D}_K N_K = -\bar{\sigma}_E \quad \text{on} \quad S_D, \end{aligned} \tag{1.1.16}$$

where $\bar{y}_i$ and $\bar{\phi}$ are the prescribed boundary motion and potential, $\overline{T}_i$ is the surface traction per unit undeformed area, and $\bar{\sigma}_E$ (a scalar) is the surface free charge per unit undeformed area.

Consider the following variational functional:

$$\begin{aligned} \Pi(\mathbf{y},\phi) = \int_{t_0}^{t_1} dt \int_V \Big[&\frac{1}{2}\rho_0 \dot{y}_i \dot{y}_i - \rho_0 \psi(S_{KL}, \mathcal{E}_K) \\ &+ \pi(S_{KL}, \mathcal{E}_K) + \rho_0 f_i y_i - \rho_E \phi \Big] dV \\ &+ \int_{t_0}^{t_1} dt \int_{S_T} \overline{T}_i y_i dS - \int_{t_0}^{t_1} dt \int_{S_D} \bar{\sigma}_E \phi \, dS, \end{aligned} \tag{1.1.17}$$

where

$$\pi(S_{KL}, \mathcal{E}_K) = \frac{1}{2}\varepsilon_0 J E_k E_k = \frac{1}{2}\varepsilon_0 J C_{MN}^{-1} \mathcal{E}_M \mathcal{E}_N . \tag{1.1.18}$$

The admissible y_i and ϕ for Π satisfy the following initial conditions in V and boundary conditions on S_y and S_ϕ:

$$\begin{aligned} &\delta y_i \,|_{t=t_0} = 0, \quad \delta y_i \,|_{t=t_1} = 0 \quad \text{in} \quad V, \\ &y_i = \bar{y}_i \quad \text{on} \quad S_y, \quad t_0 < t < t_1, \\ &\phi = \bar{\phi} \quad \text{on} \quad S_\phi, \quad t_0 < t < t_1. \end{aligned} \tag{1.1.19}$$

Then the first variation of Π is

$$\begin{aligned}\delta\Pi = & \int_{t_0}^{t_1} dt \int_V \left[(K_{Li,L} + \rho_0 f_i - \rho_0 \ddot{y}_i)\delta y_i + (\mathcal{D}_{L,L} - \rho_E)\delta\phi\right] dV \\ & - \int_{t_0}^{t_1} dt \int_{S_T} (K_{Li} N_L - \overline{T}_i)\delta y_i dS \\ & - \int_{t_0}^{t_1} dt \int_{S_D} (\mathcal{D}_L N_L + \overline{\sigma}_E)\delta\phi\, dS.\end{aligned} \tag{1.1.20}$$

Therefore, the stationary condition of Π implies the following equations and natural boundary conditions:

$$\begin{aligned} & K_{Lk,L} + \rho_0 f_k = \rho_0 \ddot{y}_k \quad \text{in} \quad V, \\ & \mathcal{D}_{K,K} = \rho_E \quad \text{in} \quad V, \\ & K_{Lk} N_L = \overline{T}_k \quad \text{on} \quad S_T, \\ & \mathcal{D}_K N_K = -\overline{\sigma}_E \quad \text{on} \quad S_D. \end{aligned} \tag{1.1.21}$$

1.2. Linear Piezoelectricity for Infinitesimal Fields

The equations of linear piezoelectricity are obtained by expansions and truncations of the nonlinear theory in the previous section.

1.2.1. Linearization

In linear theory, we introduce the small displacement vector $\mathbf{u} = \mathbf{y} - \mathbf{X}$ and assume infinitesimal displacement gradient and electric potential gradient. The infinitesimal strain tensor is denoted by

$$S_{kl} = \frac{1}{2}(u_{l,k} + u_{k,l}). \tag{1.2.1}$$

The material electric field becomes

$$\mathcal{E}_K = E_i y_{i,K} \cong E_i \delta_{iK} \to E_k. \tag{1.2.2}$$

Similarly,

$$M_{Lj} \cong 0, \quad K_{Lj} \cong F_{Lj}, \quad \mathcal{P}_K \to P_k, \quad \mathcal{D}_K \to D_k. \tag{1.2.3}$$

Since the various stress tensors are either approximately zero (quadratic or of higher order in the infinitesimal gradients) or about the same, we use T_{ij} to denote the stress tensor that is linear in the infinitesimal gradients. This notation follows the IEEE Standard on Piezoelectricity [3]. Our notation for the rest of the linear theory will also follow the IEEE Standard. Then

$$K_{Lj} \cong F_{Lj} \to T_{lj}, \quad T^S_{KL} \to T_{kl}. \tag{1.2.4}$$

For small fields the free energy density can be approximated by

$$\begin{aligned}
&\rho_0 \psi(S_{KL}, \mathcal{E}_K) - \frac{1}{2}\varepsilon_0 J E_k E_k \\
&\cong \frac{1}{2} \underset{2}{c}_{ABCD} S_{AB} S_{CD} - e_{ABC} \mathcal{E}_A S_{BC} - \frac{1}{2} \underset{2}{\chi}_{AB} \mathcal{E}_A \mathcal{E}_B - \frac{1}{2}\varepsilon_0 J E_k E_k \\
&\to \frac{1}{2} c^E_{ijkl} S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \frac{1}{2}\varepsilon^S_{ij} E_i E_j = H(S_{kl}, E_k),
\end{aligned} \tag{1.2.5}$$

where

$$\varepsilon^S_{ij} = \underset{2}{\chi}_{ij} + \varepsilon_0 \delta_{ij}. \tag{1.2.6}$$

The superscript E in c^E_{ijkl} indicates that the independent electric constitutive variable is the electric field **E**. The superscript S in ε^S_{ij} indicates that the mechanical constitutive variable is the strain tensor **S**.

In Eq. (1.2.5) we have also introduced the electric enthalpy H. The linear constitutive relations generated by H are:

$$\begin{aligned}
T_{ij} &= \frac{\partial H}{\partial S_{ij}} = c^E_{ijkl} S_{kl} - e_{kij} E_k, \\
D_i &= -\frac{\partial H}{\partial E_i} = e_{ikl} S_{kl} + \varepsilon^S_{ik} E_k.
\end{aligned} \tag{1.2.7}$$

The material constants in Eq. (1.2.7) have the following symmetries:

$$\begin{aligned}
c^E_{ijkl} &= c^E_{jikl} = c^E_{klij}, \\
e_{kij} &= e_{kji}, \\
\varepsilon^S_{ij} &= \varepsilon^S_{ji}.
\end{aligned} \tag{1.2.8}$$

We also assume that the elastic and dielectric material tensors are positive-definite in the following sense:

$$\begin{aligned}
&c^E_{ijkl} S_{ij} S_{kl} \ge 0 \quad \text{for any} \quad S_{ij} = S_{ji}, \\
&\quad \text{and} \quad c^E_{ijkl} S_{ij} S_{kl} = 0 \;\Rightarrow\; S_{ij} = 0, \\
&\varepsilon^S_{ij} E_i E_j \ge 0 \quad \text{for any} \quad E_i, \\
&\quad \text{and} \quad \varepsilon^S_{ij} E_i E_j = 0 \;\Rightarrow\; E_i = 0.
\end{aligned} \tag{1.2.9}$$

Similar to Eq. (1.2.7), linear constitutive relations can also be written as [3]

$$\begin{aligned} T_{ij} &= c^D_{ijkl} S_{kl} - h_{kij} D_k, \\ E_i &= -h_{ikl} S_{kl} + \beta^S_{ik} D_k, \end{aligned} \tag{1.2.10}$$

$$\begin{aligned} S_{ij} &= s^E_{ijkl} T_{kl} + d_{kij} E_k, \\ D_i &= d_{ikl} T_{kl} + \varepsilon^T_{ik} E_k, \end{aligned} \tag{1.2.11}$$

and

$$\begin{aligned} S_{ij} &= s^D_{ijkl} T_{kl} + g_{kij} D_k, \\ E_i &= -g_{ikl} T_{kl} + \beta^T_{ik} D_k. \end{aligned} \tag{1.2.12}$$

The equations of motion and the charge equation become

$$\begin{aligned} T_{ji,j} + \rho f_i &= \rho \ddot{u}_i, \\ D_{i,i} &= \rho_e, \end{aligned} \tag{1.2.13}$$

where ρ is the present mass density, and ρ_e (a scalar) is the free charge density per unit present volume. The difference between ρ and ρ_0, and that between ρ_E and ρ_e are neglected in Eq. (1.2.13).

In summary, the linear theory of piezoelectricity consists of the equations of motion and charge in Eq. (1.2.13), the constitutive relations

$$T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ijk} S_{jk} + \varepsilon_{ij} E_j, \tag{1.2.14}$$

where the superscripts in the material constants in Eq. (1.2.7) have been dropped, and the strain-displacement and electric field-potential relations

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}\,. \tag{1.2.15}$$

With successive substitutions from Eqs. (1.2.14) and (1.2.15), Eq. (1.2.13) can be written as four equations for **u** and ϕ:

$$\begin{aligned} c_{ijkl} u_{k,lj} + e_{kij} \phi_{,kj} + \rho f_i &= \rho \ddot{u}_i, \\ e_{ikl} u_{k,li} - \varepsilon_{ij} \phi_{,ij} &= \rho_e. \end{aligned} \tag{1.2.16}$$

Let the region occupied by the piezoelectric body be V and its boundary surface be S as shown in Fig. 1.2.1. For linear piezoelectricity we use **x** as the independent spatial coordinates. Let the unit outward normal of S be **n**.

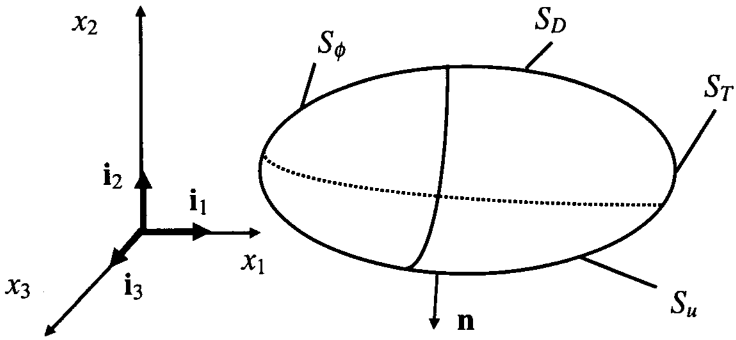

Fig. 1.2.1. A piezoelectric body and partitions of its surface.

For boundary conditions we consider the following partitions of S:

$$S_u \cup S_T = S_\phi \cup S_D = S, \quad S_u \cap S_T = S_\phi \cap S_D = 0, \tag{1.2.17}$$

where S_u is the part of S on which the mechanical displacement is prescribed, and S_T is the part of S where the traction vector is prescribed. S_ϕ represents the part of S which is electroded where the electric potential is no more than a function of time, and S_D is the unelectroded part. We consider very thin electrodes whose mechanical effects can be neglected. For mechanical boundary conditions we have prescribed displacement $\bar{u}_i$

$$u_i = \bar{u}_i \quad \text{on} \quad S_u, \tag{1.2.18}$$

and prescribed traction $\bar{t}_j$

$$T_{ij} n_i = \bar{t}_j \quad \text{on} \quad S_T. \tag{1.2.19}$$

Electrically, on the electroded portion of S,

$$\phi = \bar{\phi} \quad \text{on} \quad S_\phi, \tag{1.2.20}$$

where $\bar{\phi}$ does not vary spatially. On the unelectroded part of S, the charge condition can be written as

$$D_j n_j = -\bar{\sigma}_e \quad \text{on} \quad S_D, \tag{1.2.21}$$

where $\bar{\sigma}_e$ (a scalar) is the free charge density per unit surface area.

On an electrode S_ϕ, the total free electric charge Q_e (a scalar) can be represented by

$$Q_e = \int_{S_\phi} -n_i D_i dS. \qquad (1.2.22)$$

The electric current flowing out of the electrode is given by

$$I = -\dot{Q}_e. \qquad (1.2.23)$$

Sometimes there are two (or more) electrodes on a body that are connected to an electric circuit. In this case, circuit equation(s) will need to be considered.

The equations and boundary conditions of linear piezoelectricity can be derived from a variational principle. Consider [4]

$$\begin{aligned}\Pi(\mathbf{u},\phi) = &\int_{t_0}^{t_1} dt \int_V \left[\frac{1}{2}\rho\dot{u}_i\dot{u}_i - H(\mathbf{S},\mathbf{E}) + \rho f_i u_i - \rho_e\phi\right]dV \\ &+ \int_{t_0}^{t_1} dt \int_{S_T} \bar{t}_i u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} \bar{\sigma}_e \phi dS.\end{aligned} \qquad (1.2.24)$$

$\mathbf{u}$ and ϕ are variationally admissible if they are smooth enough and satisfy

$$\begin{aligned}&\delta u_i\,|_{t_0} = \delta u_i\,|_{t_1} = 0 \quad \text{in} \quad V, \\ &u_i = \bar{u}_i \quad \text{on} \quad S_u, \quad t_0 < t < t_1, \\ &\phi = \bar{\phi} \quad \text{on} \quad S_\phi, \quad t_0 < t < t_1.\end{aligned} \qquad (1.2.25)$$

The first variation of Π is

$$\begin{aligned}\delta\Pi = &\int_{t_0}^{t_1} dt \int_V \left[(T_{ji,j} + \rho f_i - \rho\ddot{u}_i)\delta u_i + (D_{i,i} - \rho_e)\delta\phi\right]dV \\ &- \int_{t_0}^{t_1} dt \int_{S_T} (T_{ji}n_j - \bar{t}_i)\delta u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} (D_i n_i + \bar{\sigma}_e)\delta\phi dS.\end{aligned} \qquad (1.2.26)$$

Therefore, the stationary condition of Π is

$$\begin{aligned}&T_{ji,j} + \rho f_i = \rho\ddot{u}_i \quad \text{in} \quad V, \quad t_0 < t < t_1, \\ &D_{i,i} = \rho_e \quad \text{in} \quad V, \quad t_0 < t < t_1, \\ &T_{ji}n_j = \bar{t}_i \quad \text{on} \quad S_T, \quad t_0 < t < t_1, \\ &D_i n_i = -\bar{\sigma}_e \quad \text{on} \quad S_D, \quad t_0 < t < t_1.\end{aligned} \qquad (1.2.27)$$

We now introduce a compact matrix notation [3,4]. This notation consists of replacing pairs of indices ij or kl by single indices p or q, where i, j, k and l take the values of 1, 2, and 3, and p and q take the values of 1, 2, 3, 4, 5, and 6 according to

$$\begin{array}{lcccccc} ij \text{ or } kl: & 11 & 22 & 33 & 23 \text{ or } 32 & 31 \text{ or } 13 & 12 \text{ or } 21 \\ p \text{ or } q: & 1 & 2 & 3 & 4 & 5 & 6 \end{array} . \qquad (1.2.28)$$

Thus

$$c_{ijkl} \to c_{pq}, \quad e_{ikl} \to e_{ip}, \quad T_{ij} \to T_p . \qquad (1.2.29)$$

For the strain tensor, we introduce S_p such that

$$\begin{aligned} &S_1 = S_{11}, \quad S_2 = S_{22}, \quad S_3 = S_{33}, \\ &S_4 = 2S_{23}, \quad S_5 = 2S_{31}, \quad S_6 = 2S_{12}. \end{aligned} \qquad (1.2.30)$$

The constitutive relations in Eq. (1.2.7) can then be written as

$$\begin{aligned} T_p &= c_{pq}^E S_q - e_{kp} E_k , \\ D_i &= e_{iq} S_q + \varepsilon_{ik}^S E_k . \end{aligned} \qquad (1.2.31)$$

In matrix form, Eq. (1.2.31) becomes

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{pmatrix} c_{11}^E & c_{12}^E & c_{13}^E & c_{14}^E & c_{15}^E & c_{16}^E \\ c_{21}^E & c_{22}^E & c_{23}^E & c_{24}^E & c_{25}^E & c_{26}^E \\ c_{31}^E & c_{32}^E & c_{33}^E & c_{34}^E & c_{35}^E & c_{36}^E \\ c_{41}^E & c_{42}^E & c_{43}^E & c_{44}^E & c_{45}^E & c_{46}^E \\ c_{51}^E & c_{52}^E & c_{53}^E & c_{54}^E & c_{55}^E & c_{56}^E \\ c_{61}^E & c_{62}^E & c_{63}^E & c_{64}^E & c_{65}^E & c_{66}^E \end{pmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} - \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \\ e_{14} & e_{24} & e_{34} \\ e_{15} & e_{25} & e_{35} \\ e_{16} & e_{26} & e_{36} \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix},$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} + \begin{pmatrix} \varepsilon_{11}^S & \varepsilon_{12}^S & \varepsilon_{13}^S \\ \varepsilon_{21}^S & \varepsilon_{22}^S & \varepsilon_{22}^S \\ \varepsilon_{31}^S & \varepsilon_{32}^S & \varepsilon_{33}^S \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}.$$

(1.2.32)

1.2.2. Polarized ceramics

Polarized ceramics are transversely isotropic. When the poling direction is along the x_3-axis, we have

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{31} & c_{31} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}, \tag{1.2.33}$$

where $c_{66} = (c_{11} - c_{12})/2$. The matrices in Eq. (1.2.33) have the same structures as those of crystals class C_{6v} (or 6mm). The constitutive relations take the following form:

$$\begin{aligned} T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + e_{31}\phi_{,3}, \\ T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{13}u_{3,3} + e_{31}\phi_{,3}, \\ T_{33} &= c_{13}u_{1,1} + c_{13}u_{2,2} + c_{33}u_{3,3} + e_{33}\phi_{,3}, \\ T_{23} &= c_{44}(u_{2,3} + u_{3,2}) + e_{15}\phi_{,2}, \\ T_{31} &= c_{44}(u_{3,1} + u_{1,3}) + e_{15}\phi_{,1}, \\ T_{12} &= c_{66}(u_{1,2} + u_{2,1}), \end{aligned} \tag{1.2.34}$$

and

$$\begin{aligned} D_1 &= e_{15}(u_{3,1} + u_{1,3}) - \varepsilon_{11}\phi_{,1}, \\ D_2 &= e_{15}(u_{2,3} + u_{3,2}) - \varepsilon_{11}\phi_{,2}, \\ D_3 &= e_{31}(u_{1,1} + u_{2,2}) + e_{33}u_{3,3} - \varepsilon_{33}\phi_{,3}. \end{aligned} \tag{1.2.35}$$

The equations of motion and charge are

$$\begin{aligned}
&c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{44})u_{3,13} + c_{66}u_{1,22} \\
&\quad + c_{44}u_{1,33} + (e_{31} + e_{15})\phi_{,13} = \rho\ddot{u}_1, \\
&c_{66}u_{2,11} + (c_{12} + c_{66})u_{1,12} + c_{11}u_{2,22} + (c_{13} + c_{44})u_{3,23} \\
&\quad + c_{44}u_{2,33} + (e_{31} + e_{15})\phi_{,23} = \rho\ddot{u}_2, \\
&c_{44}u_{3,11} + (c_{44} + c_{13})u_{1,31} + c_{44}u_{3,22} + (c_{13} + c_{44})u_{2,23} \\
&\quad + c_{33}u_{3,33} + e_{15}(\phi_{,11} + \phi_{,22}) + e_{33}\phi_{,33} = \rho\ddot{u}_3, \\
&e_{15}u_{3,11} + (e_{15} + e_{31})u_{1,13} + e_{15}u_{3,22} + (e_{15} + e_{31})u_{2,32} \\
&\quad + e_{31}u_{3,33} - \varepsilon_{11}(\phi_{,11} + \phi_{,22}) - \varepsilon_{33}\phi_{,33} = 0.
\end{aligned} \tag{1.2.36}$$

1.2.3. Quartz and langasite

Quartz is another widely used piezoelectric crystal. It belongs to crystal class 32 (or D_3). Langasite and some of its isomorphs (langanite and langatate) are emerging piezoelectric crystals which have stronger piezoelectric coupling than quartz and also belong to crystal class 32. For such a crystal with x_3 as a trigonal axis and x_1 as a digonal axis, the material matrices are

$$\begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\
c_{21} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & c_{14} \\
0 & 0 & 0 & 0 & c_{14} & c_{66}
\end{pmatrix},$$

$$\begin{pmatrix}
e_{11} & -e_{11} & 0 & e_{14} & 0 & 0 \\
0 & 0 & 0 & 0 & -e_{14} & -e_{11} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
\varepsilon_{11} & 0 & 0 \\
0 & \varepsilon_{11} & 0 \\
0 & 0 & \varepsilon_{33}
\end{pmatrix}. \tag{1.2.37}$$

The independent material constants are 6 + 2 + 2 = 10.

Quartz plates are often used to make devices. Plates taken from a bulk crystal at different orientations are referred to as plates of different cuts. A particular cut is specified by two angles, φ and θ, with respect to the crystal axes (X, Y, Z).

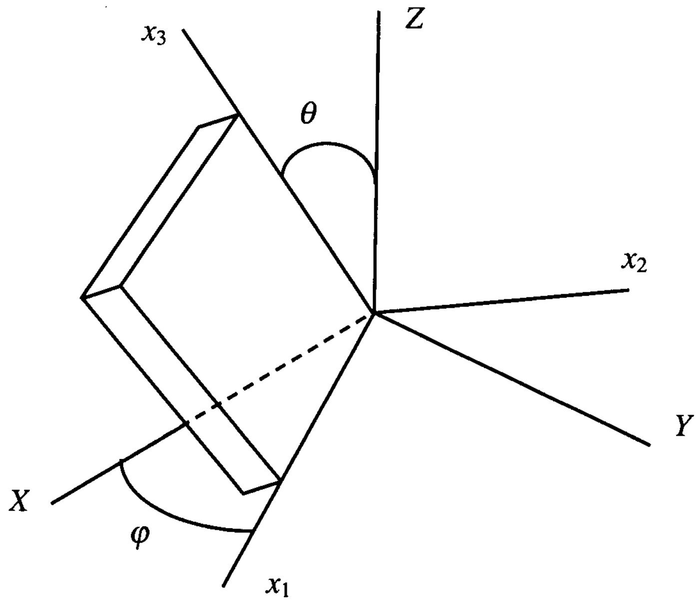

Fig. 1.2.2. A quartz plate cut from a bulk crystal.

Plates of different cuts have different material matrices with respect to coordinate axes in and normal to the plane of the plates. One class of cuts of quartz plates, called rotated Y-cuts, has $\varphi = 0$ and is particularly useful in device applications. Rotated Y-cut quartz exhibits monoclinic symmetry of class 2 (or C_2) in a coordinate system (x_1, x_2) in and normal to the plane of the plate. Therefore we list the equations for monoclinic crystals below which are useful for studying quartz devices. For monoclinic crystals, with the digonal axis along the x_1-axis, we have

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{65} & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{25} & e_{26} \\ 0 & 0 & 0 & 0 & e_{35} & e_{36} \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{23} \\ 0 & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}. \tag{1.2.38}$$

The constitutive relations are

$$\begin{aligned} T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}(u_{2,3} + u_{3,2}) + e_{11}\phi_{,1}, \\ T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}(u_{2,3} + u_{3,2}) + e_{12}\phi_{,1}, \\ T_{33} &= c_{13}u_{1,1} + c_{23}u_{2,2} + c_{33}u_{3,3} + c_{34}(u_{2,3} + u_{3,2}) + e_{13}\phi_{,1}, \\ T_{23} &= c_{14}u_{1,1} + c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{2,3} + u_{3,2}) + e_{14}\phi_{,1}, \\ T_{31} &= c_{55}(u_{3,1} + u_{1,3}) + c_{56}(u_{1,2} + u_{2,1}) + e_{25}\phi_{,2} + e_{35}\phi_{,3}, \\ T_{12} &= c_{56}(u_{3,1} + u_{1,3}) + c_{66}(u_{1,2} + u_{2,1}) + e_{26}\phi_{,2} + e_{36}\phi_{,3}, \end{aligned} \tag{1.2.39}$$

and

$$\begin{aligned} D_1 &= e_{11}u_{1,1} + e_{12}u_{2,2} + e_{13}u_{3,3} + e_{14}(u_{2,3} + u_{3,2}) - \varepsilon_{11}\phi_{,1}, \\ D_2 &= e_{25}(u_{3,1} + u_{1,3}) + e_{26}(u_{1,2} + u_{2,1}) - \varepsilon_{22}\phi_{,2} - \varepsilon_{23}\phi_{,3}, \\ D_3 &= e_{35}(u_{3,1} + u_{1,3}) + e_{36}(u_{1,2} + u_{2,1}) - \varepsilon_{23}\phi_{,2} - \varepsilon_{33}\phi_{,3}. \end{aligned} \tag{1.2.40}$$

The equations of motion and charge are

$$\begin{aligned} & c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{55})u_{3,13} + (c_{14} + c_{56})u_{2,13} \\ &\quad + (c_{14} + c_{56})u_{3,12} + 2c_{56}u_{1,23} + c_{66}u_{1,22} + c_{55}u_{1,33} \\ &\quad + e_{11}\phi_{,11} + e_{26}\phi_{,22} + (e_{36} + e_{25})\phi_{,23} + e_{35}\phi_{,33} = \rho\ddot{u}_1, \\ & c_{56}u_{3,11} + (c_{56} + c_{14})u_{1,13} + (c_{66} + c_{12})u_{1,12} + c_{66}u_{2,11} \\ &\quad + c_{22}u_{2,22} + (c_{23} + c_{44})u_{3,23} + 2c_{24}u_{2,23} + c_{24}u_{3,22} \\ &\quad + c_{34}u_{3,33} + c_{44}u_{2,33} + (e_{26} + e_{12})\phi_{,12} + (e_{36} + e_{14})\phi_{,13} = \rho\ddot{u}_2, \\ & c_{55}u_{3,11} + (c_{55} + c_{13})u_{1,13} + (c_{56} + c_{14})u_{1,12} + c_{56}u_{2,11} \\ &\quad + c_{24}u_{2,22} + 2c_{34}u_{3,23} + (c_{44} + c_{23})u_{2,23} + c_{44}u_{3,22} \\ &\quad + c_{33}u_{3,33} + c_{34}u_{2,33} + (e_{25} + e_{14})\phi_{,12} + (e_{35} + e_{13})\phi_{,13} = \rho\ddot{u}_3, \\ & e_{11}u_{1,11} + (e_{12} + e_{26})u_{2,12} + (e_{13} + e_{35})u_{3,13} + (e_{14} + e_{36})u_{2,13} \\ &\quad + (e_{14} + e_{25})u_{3,12} + (e_{25} + e_{36})u_{1,23} + e_{26}u_{1,22} \\ &\quad + e_{35}u_{1,33} - \varepsilon_{11}\phi_{,11} - \varepsilon_{22}\phi_{,22} - 2\varepsilon_{23}\phi_{,23} - \varepsilon_{33}\phi_{,33} = 0. \end{aligned} \tag{1.2.41}$$

1.3. Linear Theory for Small Fields Superposed on a Finite Bias

The theory of linear piezoelectricity assumes infinitesimal deviations from an ideal reference state of the material in which there are no pre-existing mechanical and/or electrical fields (initial or biasing fields). The presence of biasing fields makes a material apparently behave like a different material, and renders the linear theory of piezoelectricity invalid. The behavior of electroelastic bodies under biasing fields can be described by the theory for infinitesimal incremental fields superposed on finite biasing fields [5,6], which is a consequence of the nonlinear theory of electroelasticity. This section presents the theory for small fields superposed on finite biasing fields in an electroelastic body.

Consider the following three states of an electroelastic body (see Fig. 1.3.1):

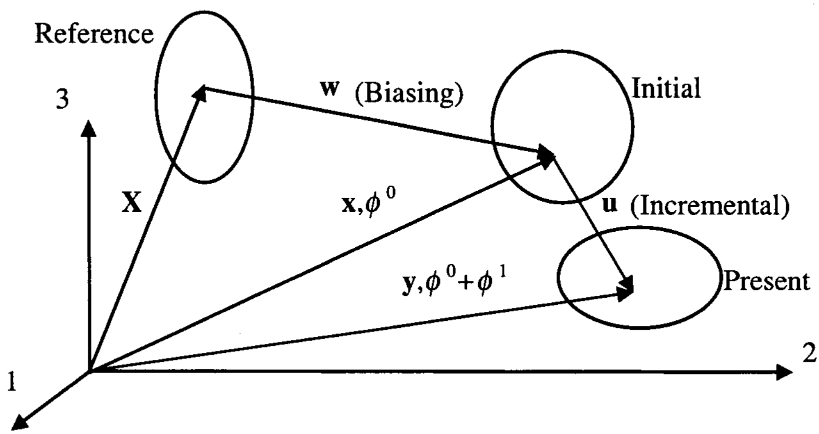

Fig. 1.3.1. Reference, initial, and present configurations of an electroelastic body.

1.3.1. The reference state

In this state the body is undeformed and is free of electric fields. A generic point at this state is denoted by **X** with Cartesian coordinates X_K. The mass density is ρ_0.

1.3.2. The initial state

In this state the body is deformed finitely and statically, and carries finite static electric fields. The body is under the action of body force f_α^0, body charge ρ_E^0, prescribed surface position $\bar{x}_\alpha$, surface traction $\bar{T}_\alpha^0$, surface potential $\bar{\phi}^0$ and surface charge $\bar{\sigma}_E^0$. The deformation and fields at this configuration are the initial or biasing fields. The position of the material point associated with **X** is given by **x** = **x**(**X**) or $x_\gamma = x_\gamma(\mathbf{X})$, with strain S_{KL}^0. Greek indices are used for the initial configuration. The electric potential in this state is denoted by $\phi^0(\mathbf{X})$, with electric field E_α^0. **x**(**X**) and $\phi^0(\mathbf{X})$ satisfy the following static equations of nonlinear electroelasticity:

$$
\begin{aligned}
& S_{KL}^0 = (x_{\alpha,K}x_{\alpha,L} - \delta_{KL})/2, \quad \mathcal{E}_K^0 = -\phi_{,K}^0, \quad E_\alpha^0 = -\phi_{,\alpha}^0, \\
& T_{KL}^0 = \rho_0 \left.\frac{\partial \psi}{\partial S_{KL}}\right|_{S_{KL}^0, \mathcal{E}_K^0}, \quad \mathcal{P}_K^0 = -\rho_0 \left.\frac{\partial \psi}{\partial \mathcal{E}_K}\right|_{S_{KL}^0, \mathcal{E}_K^0}, \\
& J^0 = \det(x_{\alpha,K}), \\
& K_{K\alpha}^0 = x_{\alpha,L}T_{KL}^0 + M_{K\alpha}^0, \quad \mathcal{D}_K^0 = \varepsilon_0 J^0 X_{K,\alpha} X_{L,\alpha} \mathcal{E}_L^0 + \mathcal{P}_K^0, \\
& M_{K\alpha}^0 = J^0 X_{K,\beta}\varepsilon_0 \left(E_\beta^0 E_\alpha^0 - \frac{1}{2} E_\gamma^0 E_\gamma^0 \delta_{\beta\alpha} \right), \\
& K_{K\alpha,K}^0 + \rho_0 f_\alpha^0 = 0, \quad \mathcal{D}_{K,K}^0 = \rho_E^0.
\end{aligned}
\tag{1.3.1}
$$

1.3.3. The present state

In this state, time-dependent, small, incremental deformations and electric fields are applied to the deformed body at the initial state. The body is under the action of f_i, ρ_E, $\bar{y}_i$, $\bar{T}_i$, $\bar{\phi}$ and $\bar{\sigma}_E$. The final position of **X** is given by **y** = **y**(**X**,*t*), and the final electric potential is $\phi(\mathbf{X},t)$. **y**(**X**,*t*) and $\phi(\mathbf{X},t)$ satisfy the dynamic equations of nonlinear electroelasticity:

$$S_{KL} = (y_{i,K}y_{i,L} - \delta_{KL})/2, \quad \mathcal{E}_K = -\phi_{,K}, \quad E_i = -\phi_{,i},$$
$$T^S_{KL} = \rho_0 \frac{\partial \psi}{\partial S_{KL}}\bigg|_{S_{KL},\mathcal{E}_K}, \quad \mathcal{P}_K = -\rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}\bigg|_{S_{KL},\mathcal{E}_K},$$
$$K_{Lj} = y_{j,K}T^S_{KL} + M_{Lj}, \quad \mathcal{D}_K = \varepsilon_0 J C^{-1}_{KL}\mathcal{E}_L + \mathcal{P}_K, \tag{1.3.2}$$
$$M_{Lj} = JX_{L,i}\varepsilon_0\left(E_iE_j - \frac{1}{2}E_kE_k\delta_{ij}\right),$$
$$K_{Lj,L} + \rho_0 f_j = \rho_0 \ddot{y}_j, \quad \mathcal{D}_{K,K} = \rho_E.$$

1.3.4. Equations for the incremental fields

Let the incremental displacement be $\mathbf{u}(\mathbf{X},t)$ and the incremental potential be $\phi^1(\mathbf{X},t)$ (see Fig. 1.3.1). $\mathbf{u}$ and ϕ^1 are assumed to be infinitesimal. We write $\mathbf{y}$ and ϕ as

$$y_i(\mathbf{X},t) = \delta_{i\alpha}[x_\alpha(\mathbf{X},t) + u_\alpha(\mathbf{X},t)],$$
$$\phi(\mathbf{X},t) = \phi^0(\mathbf{X},t) + \phi^1(\mathbf{X},t). \tag{1.3.3}$$

Then it can be shown that the equations governing the incremental fields $\mathbf{u}$ and ϕ^1 are

$$K^1_{K\alpha,K} + \rho_0 f^1_\alpha = \rho_0 \ddot{u}_\alpha,$$
$$\mathcal{D}^1_{K,K} = \rho^1_E, \tag{1.3.4}$$

where f^1_α and ρ^1_E are determined from

$$f_i = \delta_{i\alpha}(f^0_\alpha + f^1_\alpha), \quad \rho_E = \rho^0_E + \rho^1_E. \tag{1.3.5}$$

The incremental stress tensor and electric displacement vector are given by the following constitutive relations:

$$K^1_{L\gamma} = G_{L\gamma M\alpha}u_{\alpha,M} - R_{MLγ}\mathcal{E}^1_M,$$
$$\mathcal{D}^1_K = R_{KL\gamma}u_{\gamma,L} + L_{KL}\mathcal{E}^1_L, \tag{1.3.6}$$

where $\mathcal{E}^1_K = -\phi^1_{,K}$. Equation (1.3.6) shows that the incremental stress tensor and electric displacement vector depend linearly on the incremental displacement gradient and potential gradient. In Eq. (1.3.6),

$$G_{K\alpha L\gamma} = x_{\alpha,M}\,\rho_0 \frac{\partial^2 \psi}{\partial S_{KM}\partial S_{LN}}\bigg|_{S^0_{KL},\mathcal{E}^0_K} x_{\gamma,N} + T^0_{KL}\delta_{\alpha\gamma} + g_{K\alpha L\gamma} = G_{L\gamma K\alpha},$$

$$R_{KL\gamma} = -\rho_0 \frac{\partial^2 \psi}{\partial \mathcal{E}_K \partial S_{ML}}\bigg|_{S^0_{KL},\mathcal{E}^0_K} x_{\gamma,M} + r_{KL\gamma}, \tag{1.3.7}$$

$$L_{KL} = -\rho_0 \frac{\partial^2 \psi}{\partial \mathcal{E}_K \partial \mathcal{E}_L}\bigg|_{S^0_{KL},\mathcal{E}^0_K} + l_{KL} = L_{LK},$$

where

$$\begin{aligned} g_{K\alpha L\gamma} &= \varepsilon_0 J^0 [E^0_\alpha E^0_\beta (X_{K,\beta}X_{L,\gamma} - X_{K,\gamma}X_{L,\beta}) \\ &\quad - E^0_\alpha E^0_\gamma X_{K,\beta}X_{L,\beta} \\ &\quad + E^0_\beta E^0_\gamma (X_{K,\alpha}X_{L,\beta} - X_{K,\beta}X_{L,\alpha}) \\ &\quad + \frac{1}{2} E^0_\beta E^0_\beta (X_{K,\gamma}X_{L,\alpha} - X_{K,\alpha}X_{L,\gamma})], \\ r_{KL\gamma} &= \varepsilon_0 J^0 (E^0_\alpha X_{K,\alpha}X_{L,\gamma} - E^0_\alpha X_{K,\gamma}X_{L,\alpha} - E^0_\gamma X_{K,\alpha}X_{L,\alpha}), \\ l_{KL} &= \varepsilon_0 J^0 X_{K,\alpha}X_{L,\alpha}. \end{aligned} \tag{1.3.8}$$

$G_{K\alpha L\gamma}$, $R_{KL\gamma}$, and L_{KL} are called the effective or apparent elastic, piezoelectric, and dielectric constants. They depend on the initial deformation $x_\alpha(\mathbf{X})$ and electric potential $\phi^0(\mathbf{X})$.

In summary, the boundary value problem for the incremental fields $\mathbf{u}$ and ϕ^1 consists of the following equations and boundary conditions:

$$\begin{aligned} &K^1_{K\alpha,K} + \rho_0 f^1_\alpha = \rho_0 \ddot{u}_\alpha \quad \text{in} \quad V, \\ &\mathcal{D}^1_{K,K} = \rho^1_E \quad \text{in} \quad V, \\ &K^1_{L\gamma} = G_{L\gamma M\alpha} u_{\alpha,M} + R_{ML\gamma}\phi^1_{,M} \quad \text{in} \quad V, \\ &\mathcal{D}^1_K = R_{KL\gamma} u_{\gamma,L} - L_{KL}\phi^1_{,L} \quad \text{in} \quad V, \\ &u_\alpha = \bar{u}_\alpha \quad \text{on} \quad S_y, \\ &\phi^1 = \bar{\phi}^1 \quad \text{on} \quad S_\phi, \\ &K^1_{L\alpha} N_L = \bar{T}^1_\alpha \quad \text{on} \quad S_T, \\ &\mathcal{D}^1_K N_K = -\bar{\sigma}^1_E \quad \text{on} \quad S_D. \end{aligned} \tag{1.3.9}$$

Consider the following variational functional:

$$\Pi(\mathbf{u},\phi^1)=\int_{t_0}^{t_1}dt\int_V\left(\frac{1}{2}\rho_0\dot{u}_\alpha\dot{u}_\alpha-\frac{1}{2}G_{K\alpha L\gamma}u_{K,\alpha}u_{L,\gamma}\right.$$
$$\left.-R_{KL\gamma}\phi^1_{,K}u_{L,\gamma}+\frac{1}{2}L_{KL}\phi^1_{,K}\phi^1_{,L}+\rho_0 f^1_\alpha u_\alpha-\rho^1_E\phi^1\right)dV \tag{1.3.10}$$
$$+\int_{t_0}^{t_1}dt\int_{S_T}\overline{T}^1_\alpha u_\alpha dS-\int_{t_0}^{t_1}dt\int_{S_D}\overline{\sigma}^1_E\phi^1 dS.$$

The admissible $\mathbf{u}$ and ϕ^1 must satisfy

$$\begin{aligned}&\delta u_\alpha\,|_{t_0}=\delta u_\alpha\,|_{t_1}=0\quad\text{in}\quad V,\\&u_\alpha=\overline{u}_\alpha\quad\text{on}\quad S_u,\quad t_0<t<t_1,\\&\phi^1=\overline{\phi}^1\quad\text{on}\quad S_\phi,\quad t_0<t<t_1.\end{aligned} \tag{1.3.11}$$

The first variation is found to be

$$\begin{aligned}\delta\Pi(\mathbf{u},\phi^1)=&\int_{t_0}^{t_1}dt\int_V[(K^1_{L\alpha,L}+\rho_0 f^1_\alpha-\rho_0\ddot{u}_\alpha)\delta u_\alpha\\&+(\mathcal{D}^1_{K,K}-\rho^1_E)\delta\phi^1]dV\\&-\int_{t_0}^{t_1}dt\int_{S_T}(K^1_{L\alpha}N_L-\overline{T}^1_\alpha)\delta u_\alpha dS\\&-\int_{t_0}^{t_1}dt\int_{S_D}(\mathcal{D}^1_K N_K+\overline{\sigma}^1_E)\delta\phi^1 dS.\end{aligned} \tag{1.3.12}$$

Therefore, the stationary condition of the functional gives the following governing equations and boundary conditions:

$$\begin{aligned}&K^1_{K\alpha,K}+\rho_0 f^1_\alpha=\rho_0\ddot{u}_\alpha\quad\text{in}\quad V,\\&\mathcal{D}^1_{K,K}=\rho^1_E\quad\text{in}\quad V,\\&K^1_{L\alpha}N_L=\overline{T}^1_\alpha\quad\text{on}\quad S_T,\\&\mathcal{D}^1_K N_K=-\overline{\sigma}^1_E\quad\text{on}\quad S_D.\end{aligned} \tag{1.3.13}$$

1.3.5. Small bias

In many applications, the biasing deformations and fields are also infinitesimal. In this case, usually only their first-order effects on the incremental fields need to be considered. Then the following energy density of a cubic polynomial is sufficient:

$$\rho_0\psi(S_{KL},\mathcal{E}_K)=\frac{1}{2}c_{ABCD}S_{AB}S_{CD}-e_{ABC}\mathcal{E}_AS_{BC}-\frac{1}{2}\chi_{AB}\mathcal{E}_A\mathcal{E}_B$$
$$+\frac{1}{6}c_{ABCDEF}S_{AB}S_{CD}S_{EF}+\frac{1}{2}k_{ABCDE}\mathcal{E}_AS_{BC}S_{DE} \tag{1.3.14}$$
$$-\frac{1}{2}b_{ABCD}\mathcal{E}_A\mathcal{E}_BS_{CD}-\frac{1}{6}\chi_{ABC}\mathcal{E}_A\mathcal{E}_B\mathcal{E}_C,$$

where the subscripts indicating the orders of the material constants have been dropped. For small biasing fields it is convenient to introduce the small displacement vector **w** of the initial deformation (see Fig. 1.3.1), given as

$$x_\alpha=\delta_{\alpha K}X_K+w_\alpha. \tag{1.3.15}$$

Then, neglecting the quadratic terms of the gradients of **w** and ϕ^0, the effective material constants take the following form [5,6]:

$$G_{K\alpha L\gamma}=c_{K\alpha L\gamma}+\hat{c}_{K\alpha L\gamma},$$
$$R_{KL\gamma}=e_{KL\gamma}+\hat{e}_{KL\gamma}, \tag{1.3.16}$$
$$L_{KL}=\varepsilon_{KL}+\hat{\varepsilon}_{KL},$$

where

$$\hat{c}_{K\alpha L\gamma}=T^0_{KL}\delta_{\alpha\gamma}+c_{K\alpha LN}w_{\gamma,N}+c_{KNL\gamma}w_{\alpha,N}$$
$$+c_{K\alpha L\gamma AB}S^0_{AB}+k_{AK\alpha L\gamma}\mathcal{E}^0_A,$$
$$\hat{e}_{KL\gamma}=e_{KLM}w_{\gamma,M}-k_{KL\gamma AB}S^0_{AB}+b_{AKL\gamma}\mathcal{E}^0_A$$
$$+\varepsilon_0(\mathcal{E}^0_K\delta_{L\gamma}-\mathcal{E}^0_L\delta_{K\gamma}-\mathcal{E}^0_M\delta_{M\gamma}\delta_{KL}), \tag{1.3.17}$$
$$\hat{\varepsilon}_{KL}=b_{KLAB}S^0_{AB}+\chi_{KLA}\mathcal{E}^0_A+\varepsilon_0(S^0_{MM}\delta_{KL}-2S^0_{KL}),$$
$$T^0_{KL}=c_{KLMN}S^0_{MN}-e_{AKL}\mathcal{E}^0_A$$
$$S^0_{AB}\cong(w_{A,B}+w_{B,A})/2,$$
$$\mathcal{E}^0_K=-\phi^0_{,K}.$$

1.3.6. Frequency perturbation due to a small bias

For many applications to be discussed later, we are interested in the effect of biasing fields on resonant frequencies of an electroelastic body.

Consider the following eigenvalue problem for free vibrations of an electroelastic body under small biasing fields:

$$-[(c_{L\gamma M\alpha}+\hat{c}_{L\gamma M\alpha})u_{\alpha,M}+(e_{ML\gamma}+\hat{e}_{ML\gamma})\phi^1_{,M}]_{,L}=\rho_0\omega^2 u_\gamma,$$
$$[(e_{KL\gamma}+\hat{e}_{KL\gamma})u_{\gamma,L}-(\varepsilon_{KL}+\hat{\varepsilon}_{KL})\phi^1_{,L}]_{,K}=0, \tag{1.3.18}$$

with appropriate boundary conditions. In Eq. (1.3.18), ω and u_α are the resonant frequency and the corresponding mode, respectively, when the biasing fields are present and may be called a perturbed frequency and mode. Then it can be shown [7] that, to the first-order of the biasing fields,

$$\Delta\omega=\frac{1}{2\omega_0}\frac{\int_V[\hat{c}_{L\gamma M\alpha}U_{\gamma,L}U_{\alpha,M}+2\hat{e}_{ML\gamma}\Phi_{,M}U_{\gamma,L}-\hat{\varepsilon}_{ML}\Phi_{,M}\Phi_{,L}]dV}{\int_V\rho_0 U_\alpha U_\alpha dV}, \tag{1.3.19}$$

where $\Delta\omega=\omega-\omega_0$. ω_0, U_α and Φ are the frequency and mode when the biasing fields are not present, or the unperturbed frequency and mode. They are governed by

$$-(c_{L\gamma M\alpha}U_{\alpha,M}+e_{ML\gamma}\Phi_{,M})_{,L}=\rho_0\omega_0^2 U_\gamma,$$
$$(e_{KL\gamma}U_{\gamma,L}-\varepsilon_{KL}\Phi_{,L})_{,K}=0, \tag{1.3.20}$$

with appropriate boundary conditions. Equation (1.3.19) is the result of a first-order perturbation analysis. It breaks a complicated eigenvalue problem of Eq. (1.3.18) for vibrations under a bias into two simpler problems of Eq. (1.3.19) for vibrations without a bias, and the biasing field problem.

1.4. Cubic Theory for Weak Nonlinearity

By cubic theory we mean that effects of all terms up to the third power of the displacement and potential gradients or their products are included [8]. Cubic theory is an approximate theory for relatively weak nonlinearities, and can be obtained by expansions and truncations from the nonlinear theory in the first section of this chapter. The resulting equations are:

$$
\begin{aligned}
F_{Lj} \cong \delta_{jM} \Big[& \underset{2}{c}_{LMAB} u_{A,B} + e_{ALM} \phi_{,A} + \frac{1}{2} \underset{2}{c}_{LMAB} u_{K,A} u_{K,B} \\
& + \underset{2}{c}_{LKAB} u_{M,K} u_{A,B} + \frac{1}{2} \underset{3}{c}_{LMABCD} u_{A,B} u_{C,D} \\
& + e_{ALK} u_{M,K} \phi_{,A} - \underset{1}{k}_{ABCLM} u_{B,C} \phi_{,A} - \frac{1}{2} b_{ABLM} \phi_{,A} \phi_{,B} \\
& + \frac{1}{2} \underset{2}{c}_{LRAB} u_{M,R} u_{K,A} u_{K,B} + \frac{1}{2} \underset{3}{c}_{LKABCD} u_{M,K} u_{A,B} u_{C,D} \\
& + \frac{1}{2} \underset{3}{c}_{LMABcD} u_{A,B} u_{K,C} u_{K,D} + \frac{1}{6} \underset{4}{c}_{LMABCDEF} u_{A,B} u_{C,D} u_{E,F} \\
& - \underset{1}{k}_{ABCLK} u_{B,C} u_{M,K} \phi_{,A} - \frac{1}{2} \underset{1}{k}_{ABCLM} u_{K,B} u_{K,C} \phi_{,A} \\
& - \frac{1}{2} \underset{2}{k}_{ABCDELM} u_{B,C} u_{D,E} \phi_{,A} - \frac{1}{2} b_{ABLK} u_{M,K} \phi_{,A} \phi_{,B} \\
& + \frac{1}{2} \underset{1}{a}_{ABCDLM} u_{c,D} \phi_{,A} \phi_{,B} + \frac{1}{6} \underset{3}{k}_{ABCLM} \phi_{,A} \phi_{,B} \phi_{,C} \Big],
\end{aligned}
\qquad (1.4.1)
$$

$$
\begin{aligned}
\mathcal{P}_L \cong\ & e_{LBC} u_{B,C} - \underset{2}{\chi}_{AL} \phi_{,A} + \frac{1}{2} e_{LBC} u_{K,B} u_{K,C} - \frac{1}{2} \underset{1}{k}_{LBCDE} u_{B,C} u_{D,E} - b_{ALCD} u_{C,D} \phi_{,A} \\
& + \frac{1}{2} \underset{3}{\chi}_{ABL} \phi_{,A} \phi_{,B} - \frac{1}{2} \underset{1}{k}_{LBCDE} u_{B,C} u_{K,D} u_{K,E} \\
& - \frac{1}{6} \underset{2}{k}_{LBCDEFG} u_{B,C} u_{D,E} u_{F,G} - \frac{1}{2} b_{ALCD} u_{K,C} u_{K,D} \phi_{,A} \\
& + \frac{1}{2} \underset{1}{a}_{ALCDEF} u_{C,D} u_{E,F} \phi_{,A} + \frac{1}{2} \underset{3}{k}_{ABLDE} u_{D,E} \phi_{,A} \phi_{,B} - \frac{1}{6} \underset{4}{\chi}_{ABCL} \phi_{,A} \phi_{,B} \phi_{,C},
\end{aligned}
\qquad (1.4.2)
$$

$$
\begin{aligned}
M_{Lj} \cong \varepsilon_0 \delta_{jM} \Big[& \phi_{,L} \phi_{,M} - \frac{1}{2} \phi_{,K} \phi_{,K} \delta_{LM} - \phi_{,K} \phi_{,M} u_{K,L} \\
& - \phi_{,K} \phi_{,M} u_{L,K} + \phi_{,L} \phi_{,M} u_{K,K} - \phi_{,L} \phi_{,K} u_{K,M} \\
& + \phi_{,K} \phi_{,R} u_{R,K} \delta_{LM} + \frac{1}{2} \phi_{,K} \phi_{,K} u_{L,M} - \frac{1}{2} \phi_{,R} \phi_{,R} u_{K,K} \delta_{LM} \Big],
\end{aligned}
\qquad (1.4.3)
$$

$$\varepsilon_0 J C_{KL}^{-1} \mathcal{E}_K \cong \varepsilon_0 \Big[-\phi_{,L} + \phi_{,K} u_{L,K} - \phi_{,L} u_{K,K} + \phi_{,K} u_{K,L} - \phi_{,M} u_{L,K} u_{K,M} + \phi_{,K} u_{M,M} u_{L,K} - \frac{1}{2} \phi_{,L} u_{K,K} u_{M,M} + \frac{1}{2} \phi_{,L} u_{K,M} u_{M,K} - \phi_{,M} u_{L,K} u_{M,K} + \phi_{,M} u_{M,L} u_{K,K} - \phi_{,M} u_{M,K} u_{K,L} \Big]. \tag{1.44}$$

A special case of the cubic theory is the case of relatively large mechanical deformations and weak electric fields [9]. In this case all electrical nonlinearities can be neglected. The following energy density is sufficient:

$$\rho_0 \psi = \frac{1}{2} c_{ABCD} S_{AB} S_{CD} - e_{ABC} \mathcal{E}_A S_{BC} - \frac{1}{2} \chi_{AB} \mathcal{E}_A \mathcal{E}_B + \frac{1}{6} c_{ABCDEF} S_{AB} S_{CD} S_{EF} + \frac{1}{24} c_{ABCDEFGH} S_{AB} S_{CD} S_{EF} S_{GH}. \tag{1.4.5}$$

Keeping the linear terms of the electric potential gradient and up to cubic terms of the displacement gradient, we obtain

$$\begin{aligned} K_{LM} &= c_{LMRS} u_{R,S} + e_{KLM} \phi_{,K} \\ &\quad + c^e_{LMRSKN} u_{R,S} u_{K,N} + c^e_{LMRSKNIJ} u_{R,S} u_{K,N} u_{I,J}, \\ \mathcal{D}_K &= e_{KRS} u_{R,S} - \varepsilon_{KL} \phi_{,L}, \end{aligned} \tag{1.4.6}$$

where

$$\begin{aligned} c^e_{LMRSKN} &= \frac{1}{2}(c_{LMRSKN} + c_{LMNS}\delta_{KR} + c_{LNRS}\delta_{KM}), \\ c^e_{LMRSKNIJ} &= \frac{1}{6} c_{LMRSKNIJ} \\ &\quad + \frac{1}{2}(c_{LMKNSJ}\delta_{RI} + c_{LNSJ}\delta_{MK}\delta_{RI} + c_{LNRSIJ}\delta_{MK}). \end{aligned} \tag{1.4.7}$$