

CHAPTER 1

ELIZA HASEGANU'S ANALYSIS OF WRINKLING IN PRESSURIZED MEMBRANES

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Eliza Haseganu intended to publish results from her dissertation research on the numerical solution of highly elastic wrinkled membranes under pressure. These are quite striking and reproduce unusual and unexpected features observed in experiments. Her numerical analysis was sufficiently robust to achieve this without the need for any special measures. However, she delayed publication in the expectation that she would find the time to refine some of her more important analyses to take the effect of self contact of the membrane into account. Eliza's results are presented here on her behalf.

1.1. Introduction

This brief tribute to Dr. Eliza Haseganu collects some results contained in her thesis which would certainly have formed the basis of a substantial and influential paper, had she lived to prepare it. As her thesis adviser, I am both honoured and saddened to have the opportunity to summarize this body of work here on her behalf. Most of the essential details regarding the formulation and solution of mixed traction/displacement membrane boundary-value problems are summarized in her paper which appeared some years ago [Haseganu and Steigmann (1994)]¹ and which has been cited quite frequently in the interim. In it she set a standard for the numerical analysis of wrinkling in finitely deformed membranes which has not been surpassed.

Eliza's achievement was the development and implementation of a numerical scheme for solving the ill-conditioned equilibrium equations generated by Pipkin's *relaxed* strain-energy function [Pipkin (1986)]². The re-

laxed formulation is motivated by the fact that the conventional theory admits compressive stresses in equilibrium. Although admissible in conventional two- and three-dimensional elasticity, in the setting of membrane theory such a state violates strong ellipticity, a well-known necessary condition for the existence of stable (energy-minimizing) equilibria. To rectify this Pipkin showed how to construct energy-minimizing sequences of deformations from the conventional strain energy which contain ever-more finely spaced wrinkles. The limit of such a sequence is a smooth deformation without compressive stress, and the strain energy function attributed to such a state, the so-called relaxed energy, may be used in place of the original when formulating equations of equilibrium. The relaxed energy also possesses certain convexity properties which are central to existence theorems based on the direct methods of the calculus of variations. In this way the problematic existence issue associated with membrane theory is addressed while the merits of its relatively simplicity are retained.

The obvious alternative to relaxation is to regularize the equations by including higher-order gradient (curvature) terms in the strain-energy function. This is tantamount to using a kind of shell theory in place of membrane theory. The advantage is that a length scale is built directly into the differential equations which effectively determines the wavelengths of wrinkles. Further, the energetic penalty associated with curvature sets a lower bound to the wavelength. This is clearly more realistic than the idealizations inherent in relaxed membrane theory; there the absence of bending resistance implies the absence of a lower bound and thus that wrinkles are continuously distributed over the surface. Fortunately, the associated equilibrium equations are statically determinate; therefore the stress distribution in a wrinkled region is insensitive to the details of the deformation and hence to the errors incurred in its description by using relaxed membrane theory. However, offsetting the advantages of relaxed membrane theory is the fact that the absence of stress in certain connected regions of strain space implies an absence of stiffness also, so that conventional stiffness-based solution strategies for the computation of equilibria are not effective. To overcome this difficulty, Eliza embedded the (relaxed) equilibrium problem into an artificial damped dynamical problem, constructed in such a way that the desired equilibria are globally and asymptotically stable relative to arbitrary initial data. One then discretizes this system in time using a simple difference scheme. Temporal accuracy is not an issue since it is only the asymptotic states which are of interest. This allows the use of simple explicit finite difference operators with mass-proportional damping to achieve

an efficient vectorized system for computations.

In this paper I describe the adjustments to Eliza's earlier work [Haseganu and Steigmann (1994)]¹ needed to accommodate pressure loading. The basic formulation of relaxed membrane theory is described in section 2. The numerical method and the Lyapunov function associated with conservative pressure loads are discussed in section 3. Section 4 is devoted to specific simulations taken from Eliza's thesis. These examples, which are quite striking, illustrate her considerable achievement in this challenging branch of computational finite elasticity.

1.2. Relaxed Membrane Theory

The equilibrium equations for membranes, regarded as two-dimensional continua, have a simple structure identical to those for conventional bulk continua. For convenience, and with no essential loss of generality, I use a plane Ω as reference, which is here regarded as an unstressed configuration of the membrane. Let \mathbf{e}_α , $\alpha = 1, 2$, be orthogonal unit vectors spanning (the translation space of) Ω , and let $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. We use the usual summation convention with Greek subscripts taking values in $\{1, 2\}$ and Latin in $\{1, 2, 3\}$. Then the equilibrium equations in vector form are

$$\operatorname{div} \mathbf{T} + pJ\mathbf{n} = \mathbf{0}, \quad (1)$$

where

$$\mathbf{T} = T_{i\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha \quad (2)$$

is the Piola stress, div is the (two-dimensional) divergence operator on Ω , p is the net lateral pressure across the membrane surface,

$$\mathbf{n} = \frac{\mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2}{\mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2} \quad (3)$$

is the unit normal to the membrane surface after deformation,

$$\mathbf{F} = F_{i\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha \quad (4)$$

is the deformation gradient, where

$$F_{i\alpha} = r_{i,\alpha} \equiv \frac{\partial r_i}{\partial x_\alpha} \quad (5)$$

in which x_α and r_i are the Cartesian coordinates of a material particle before and after deformation, respectively, and

$$J = (\det \mathbf{C})^{1/2} \quad (6)$$

where

$$\mathbf{C} = \mathbf{F}^t \mathbf{F} \quad (7)$$

is the Cauchy–Green strain and the superscript t is used to denote transposition.

An elastic membrane possesses a strain-energy function W per unit area of Ω which depends on the deformation gradient and which delivers the stress through

$$\mathbf{T} = W_{\mathbf{F}}, \quad (8)$$

where the subscript identifies the gradient with respect to \mathbf{F} . Thus,

$$T_{i\alpha} = \frac{\partial W}{\partial F_{i\alpha}}. \quad (9)$$

These equations are augmented by traction data $t_i = T_{i\alpha} \nu_\alpha$ or position data $r_i = r_i^0$ on appropriate parts of the boundary, where ν_α are the components of the exterior unit normal to an edge in the reference configuration.

Isotropy of space requires that W depend on \mathbf{F} through \mathbf{C} . If the membrane *material* is isotropic then the latter dependence occurs through the *principal stretches* λ and μ , the positive square roots of the eigenvalues of the strain \mathbf{C} . Thus,

$$\mathbf{T} = w_\lambda \mathbf{l} \otimes \mathbf{L} + w_\mu \mathbf{m} \otimes \mathbf{M}, \quad (10)$$

where $w_{\lambda,\mu}$ are the partial derivatives of the symmetric function

$$w(\lambda, \mu) = w(\mu, \lambda) = W(\mathbf{F}), \quad (11)$$

and $\{\mathbf{l}, \mathbf{m}\}$, $\{\mathbf{L}, \mathbf{M}\}$ are the orthonormal principal strain axes on the deformed surface and on Ω , respectively [Haseganu and Steigmann (1994)]¹. The latter generate a useful formula for the deformation gradient which is identical in form to (10):

$$\mathbf{F} = \lambda \mathbf{l} \otimes \mathbf{L} + \mu \mathbf{m} \otimes \mathbf{M}. \quad (12)$$

Relaxed membrane theory is based on the composite strain-energy function

$$\begin{aligned} w(\lambda, \mu); & \quad \lambda > v(\mu) \quad \text{and} \quad \mu > v(\lambda) \\ w(\lambda, v(\lambda)); & \quad \lambda > 1 \quad \text{and} \quad \mu \leq v(\lambda) \\ w(v(\mu), \mu); & \quad \mu > 1 \quad \text{and} \quad \lambda \leq v(\mu) \\ 0; & \quad \lambda \leq 1 \quad \text{and} \quad \mu \leq 1, \end{aligned} \quad (13)$$

where $v(x)$ is the (normally unique) solution of the implicit equation

$$w_y(x, y) = 0. \quad (14)$$

Thus the second and third branches of (13) correspond to states of uniaxial tension along the principal strain axes. These are valid even when λ or μ are less than the corresponding value of $v(\mu)$ or $v(\lambda)$, respectively, the difference being due to fine-scale wrinkling without further change of energy and with no compressive stress. The first branch corresponds to biaxial tension and the original strain-energy function is operative in the corresponding region of the plane of principal stretches. The fourth branch corresponds to double wrinkling without stress, this branch being operative in slack regions of the membrane. In general, the determination of the regions on the material surface Ω which correspond to the various branches of (13) constitutes a difficult free-boundary problem. In Eliza's work, such difficulties were avoided altogether by simply computing the stretches at any given stage of deformation and then selecting the appropriate branch for the subsequent calculation.

In the examples discussed here the specific strain-energy function used in the first branch of (13) is the specialization to membrane theory of Ogden's function [Ogden (1997)]³ for rubberlike solids that are incompressible in bulk, namely

$$w(\lambda, \mu) = \frac{G}{\alpha^r} \sum_{r=1}^3 g_r [\lambda^{\alpha_r} + \mu^{\alpha_r} + (\lambda\mu)^{-\alpha_r} - 3], \quad (15)$$

where

$$\begin{aligned} \alpha_1 &= 1.3, & \alpha_2 &= 5.0, & \alpha_3 &= -2.0; \\ g_1 &= 1.491, & g_2 &= 0.003, & g_3 &= -0.0237 \end{aligned} \quad (16)$$

and the constant $G(> 0)$ may be regarded as the ground-state shear modulus multiplied by the initial membrane thickness. The associated functions $v(x)$ and $\hat{w}(x)[= w(x, v(x)) = w(v(x), x)]$ are

$$v(x) = x^{-1/2} \quad \text{and} \quad \hat{w}(x) = \frac{G}{\alpha^r} \sum_{r=1}^3 g_r (x^{\alpha_r} + 2x^{-\alpha_r/2} - 3). \quad (17)$$

1.3. Numerical Scheme

The numerical method Eliza used is based on the *Green's theorem finite difference* technique [Haseganu and Steigmann (1994)]¹ for the spatial discretization of the domain Ω . In this method the x_1, x_2 -plane is sub-divided

into small quadrilateral cells that are arranged so as to cover the domain completely. Since the cells need not be rectangular, it is possible to approximate a domain with curved or irregular boundaries with high precision by using a sufficient number of cells. In contrast, conventional finite differences require computationally expensive boundary interpolation or grid-mapping to accommodate irregular domains. The procedure is known to be most effective when used in conjunction with spatial finite differences [Underwood (1983)]⁴. It is for this reason that finite differences are used rather than the more widespread finite element method.

In the present method the divergence operator $T_{k\alpha,\alpha}$ appearing in (1) is approximated at node (i, j) by applying Green's theorem to a quadrilateral region enclosed by a contour described by node points in the material region of interest. The area integral in the theorem is estimated as the nodal value of the integrand multiplied by the enclosed area, while the contour integral is approximated by setting its integrand equal to the values it assumes at the midpoints on each of the four edges that make up the boundary. This yields the approximate divergence of stress at node (i, j) [Haseganu and Steigmann (1994)]¹:

$$\begin{aligned}
 f_k^{i,j} &\equiv A^{i,j} (T_{k\alpha,\alpha})^{i,j} \\
 &= e_{\alpha\beta} \left[T_{k\alpha}^{i+1/2,j+1/2} \left(x_\beta^{i,j+1} - x_\beta^{i+1,j} \right) \right. \\
 &\quad + T_{k\alpha}^{i-1/2,j+1/2} \left(x_\beta^{i-1,j} - x_\beta^{i,j+1} \right) \\
 &\quad + T_{k\alpha}^{i-1/2,j-1/2} \left(x_\beta^{i,j-1} - x_\beta^{i-1,j} \right) \\
 &\quad \left. + T_{k\alpha}^{i+1/2,j-1/2} \left(x_\beta^{i+1,j} - x_\beta^{i,j-1} \right) \right], \quad (18)
 \end{aligned}$$

where $e_{\alpha\beta}$ is the unit alternator ($e_{12} = -e_{21} = 1$, $e_{11} = e_{22} = 0$) and $A^{i,j}$ is the area of the quadrilateral.

The stresses on the right hand side in (18) are evaluated at the midpoints between nodes. These depend via the constitutive relations on the corresponding values of the deformation gradient, which are in turn computed using a second application of Green's theorem. As before, the associated area integral is estimated as the value of the integrand at the midpoints multiplied by the enclosed area. However, the four edge contributions to the boundary integral are now approximated by replacing the integrand in each with the average of their values at the node points. The resulting

difference formula is [Haseganu and Steigmann (1994)]¹

$$F_{k\alpha}^{i+1/2,j+1/2} = \left(A^{i+1/2,j+1/2} \right)^{-1} e_{\alpha\beta} \left[\left(x_{\beta}^{i,j+1} - x_{\beta}^{i+1,j} \right) \left(r_k^{i+1,j+1} - r_k^{i,j} \right) + \left(x_{\beta}^{i+1,j+1} - x_{\beta}^{i,j} \right) \left(r_k^{i,j+1} - r_k^{i+1,j} \right) \right], \quad (19)$$

where $A^{i+1/2,j+1/2}$ is the area of the associated quadrilateral.

Given the expression for the deformation gradient, the pressure term $pJ\mathbf{n}$ in the equilibrium equations integrates to yield the force

$$p_k^{i+1/2,j+1/2} = p e_{kmn} F_{m1}^{i+1/2,j+1/2} F_{n2}^{i+1/2,j+1/2} A^{i+1/2,j+1/2}, \quad (20)$$

where e_{ijk} is the three-dimensional unit alternator. Eliza [Haseganu (1994)]⁵ used this in the formula

$$p_k^{i,j} = \frac{1}{4} \left(p_k^{i+1/2,j+1/2} + p_k^{i-1/2,j+1/2} + p_k^{i-1/2,j-1/2} + p_k^{i+1/2,j-1/2} \right) \quad (21)$$

to generate the effective nodal force due to pressure. The total nodal force is then given by

$$g_k^{i,j} = f_k^{i,j} + p_k^{i,j}. \quad (22)$$

To solve the nonlinear algebraic system generated by the finite difference discretization, the equilibrium problem is embedded into an artificial dynamical problem with suitably chosen mass and viscosity. Since it is only the equilibrium configurations that are of interest, the dynamical system may be chosen arbitrarily, subject to the requirement that the desired equilibria be asymptotically stable within the class of dynamics considered. Further, the system should be robust in the sense that its equilibria are not sensitive to the detailed features of the system on dynamical solution trajectories. This procedure is a variant of the method of relaxation *dynamic!relaxation*. In the present context, a convenient dynamical system, *not* equivalent to the actual equations of motion for the membrane, is

$$T_{i\alpha,\alpha} + pJn_i = \rho\dot{v}_i + c\rho v_i, \quad v_i = \dot{r}_i, \quad (23)$$

subject to the kinematically admissible initial conditions

$$r_i(x_\alpha, 0) = R_i(x_\alpha), \quad v_i(x_\alpha, 0) = 0. \quad (24)$$

Here $\rho(x_\alpha)$ is the mass per unit area of the reference plane and c is a positive damping coefficient.

Alternative formulations of dynamic relaxation have been based on the actual equations of motion for the system at hand. In such formulations

damping is introduced via the constitutive relations. The resulting discretization yields a coupled system for updating the configuration from one time step to the next. This is inconvenient in practice, but is generally necessary in problems for which the equilibria obtained are strongly influenced by the details of the dynamics in a given initial value problem. However, in the present application to membrane theory, the equations of equilibrium are derived from a strain energy that is modified to incorporate the effects of wrinkling automatically. In mixed position/traction problems without pressure, the resulting energy is often (for some strain-energy functions) a convex function of the deformation gradient in regions where the membrane is under stress [Haseganu and Steigmann (1994)]¹. Accordingly, these regions are uniquely determined if the membrane is in equilibrium, and are thus insensitive to the details of a dynamical system that exhibits such configurations as equilibria. The non-uniqueness of slack regions of the membrane is not a deficiency of the model, but instead reflects the fact that membrane equilibria are entirely arbitrary in the absence of stress (such states do not occur in equilibrium if pressure is present). The uniqueness of the stressed regions of equilibrium configurations together with the path-independence of the elastic constitutive relations were exploited by Eliza to construct a dynamical problem that is computationally efficient.

It remains to show that equilibria are asymptotically stable in the class of dynamics described by (23) and (24). Position/traction problems are discussed first. In view of the minimizing property of equilibria, it is sufficient to show that the total mechanical energy decreases on solution trajectories of (23). Asymptotic stability then follows by a theorem of Lyapunov. To this end (23) is scalar-multiplied by the velocity and the resulting equation is integrated over the reference plane Ω to derive

$$\frac{d}{dt} \left[K + \int_{\Omega} W(\mathbf{F}) da \right] + c \int_{\Omega} \rho |\mathbf{v}|^2 da = \int_{\partial\Omega} \mathbf{T}\nu \cdot \mathbf{v} da, \quad (25)$$

where

$$K = \frac{1}{2} \int_{\Omega} \rho |\mathbf{v}|^2 da \quad (26)$$

is the kinetic energy and the relation $\dot{W} = T_{i\alpha} v_{i,\alpha}$ has been used together with Green's theorem. In a standard mixed boundary value problem with position assigned along a part of the boundary $\partial\Omega$ and traction assigned on the remainder, (25) reduces to

$$\frac{d}{dt} [K + E] = -c \int_{\Omega} \rho |\mathbf{v}|^2 da \leq 0, \quad (27)$$

where

$$E = \int_{\Omega} W(\mathbf{F}) da - \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{r} da \quad (28)$$

is the total potential energy in which $\partial\Omega_t$ is the part of the boundary where the fixed traction \mathbf{t} is assigned, position being assigned on the complementary part. Thus the total energy $K + E$ decays on the solution trajectories of (23) and the asymptotic stability of equilibria follows. Strictly, these statements are valid for the discretized equations. Their proof for the analytical model remains open due to certain difficulties associated with Lyapunov's theorem for continuous systems. Again, these issues are of no concern in the present context since the discrete dynamical problem merely serves to expedite the computation of equilibria.

Equation (27) remains valid for the pure displacement problem with prescribed pressure, provided that E is replaced by [Haseganu (1994)]⁵

$$E' = \int_{\Omega} [W(\mathbf{F}) - (p/3)J\mathbf{n} \cdot \mathbf{r}] da. \quad (29)$$

The solution of eqs. (23) and (24) is based on the combination of the spatial difference formula (22) with central difference operators in time. This yields the explicit decoupled scheme [Haseganu and Steigmann (1994)]¹

$$\begin{aligned} \left(h^{-1} + \frac{c}{2}\right) m^{i,j} \dot{\mathbf{r}}^{i,j,n+1/2} &= \left(h^{-1} + \frac{c}{2}\right) m^{i,j} \dot{\mathbf{r}}^{i,j,n-1/2} + \mathbf{g}^{i,j,n}, \\ \mathbf{r}^{i,j,n+1} &= \mathbf{r}^{i,j,n} + h \dot{\mathbf{r}}^{i,j,n+1/2}, \end{aligned} \quad (30)$$

which is used to advance the solution in time node by node. In terms of a fixed orthonormal basis $\{\mathbf{e}_k\}$, $\mathbf{r} = r_k \mathbf{e}_k$ is the nodal position and $\mathbf{g} = g_k \mathbf{e}_k$ is the nodal force. In addition, $m^{i,j} = A^{i,j} \rho(x_{\alpha}^{i,j})$ is the nodal mass, t_n is the time after n cycles and h is the time step. This system is obtained by integrating (23) over the quadrilateral region formed by the four nearest neighbors of the node and setting the deformation in the interior equal to the nodal value. The starting procedure for (30) is derived from (24). Thus $\mathbf{r}^{i,j,0} = \mathbf{R}(\mathbf{x}^{i,j})$, $\dot{\mathbf{r}}^{i,j,0} = \mathbf{0}$ and the central difference formulas together with (30) yield

$$\frac{2}{h} m^{i,j} \dot{\mathbf{r}}^{i,j,1/2} = \mathbf{g}^{i,j,0}, \quad (31)$$

wherein the right hand side is determined by $\mathbf{R}(\mathbf{x})$. The system of equations is then non-dimensionalized and the solution advanced to the first time t_n such that $\max_{i,j} |\mathbf{g}^{i,j,n}|$ is less than some specified value.

Finally, it is noted that the convexity of the modified strain energy ensures that computations based on the foregoing scheme are not *mesh-dependent*. This property is fundamental to the convergence of the method with respect to mesh refinement.

For pure position boundary-value problems with assigned pressure, the foregoing claims regarding the convexity of the variational problem and the conclusions which follow therefrom are not valid, although the problem continues to satisfy important existence criteria such as the quasiconvexity condition [Dacorogna (1989)]⁶. The stronger statement pertaining to (partial) uniqueness also does not apply, and this fact is reflected in some of the simulations to be discussed.

1.4. Examples

The examples discussed here are described in detail in Eliza's thesis [Haseganu (1994)]⁵ and pertain to the problem of assigned boundary position and simultaneous inflation by a uniform pressure of fixed intensity. Eliza had intended to collect these together into a paper for publication. Her previous work [Haseganu and Steigmann (1994)]¹ was concerned exclusively with mixed position/traction data in the absence of pressure.

In the first example, a circular membrane of unit radius is subjected to a lateral dimensionless pressure $\tilde{p} = p/G = 2.0$ while the radius of the boundary is fixed at one-half its reference value. The meshed reference configuration is shown in Figure 1.1. Contraction of the boundary tends to promote membrane wrinkling whereas pressure tends to suppress it. The competition between these two influences is evident in Figure 1.2, in which dashed lines indicate trajectories of the active principal stress in wrinkled regions; elsewhere the membrane is under biaxial tension. From the figure it is evident that the large circumferential contraction of the boundary and the attendant wrinkling combine to yield a non-axisymmetric shape despite the axisymmetry of the data.

Figure 1.3 depicts a representative meridian corresponding to the same boundary data under a succession of pressures. A snap bifurcation from a low-volume solution to a shape enclosing a much larger volume was found to occur between $\tilde{p} = 2.675$ and 2.6775 , the latter figure representing a 0.93% increase over the former. This kind of behaviour is a well-known feature of the response of spherical membranes inflated by a controlled pressure undergoing spherically symmetric deformations. In the latter case analytical solutions are available which exhibit an unstable branch of equilibria con-

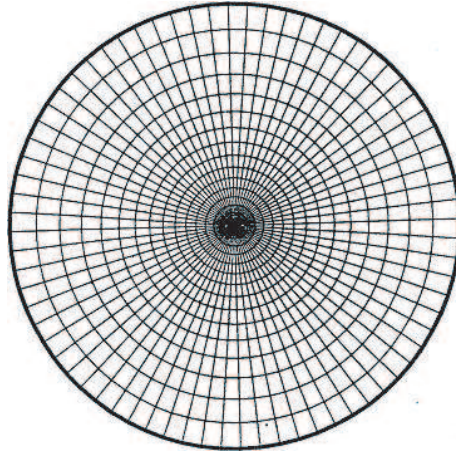


Fig. 1.1. Circular membrane; meshed reference configuration.

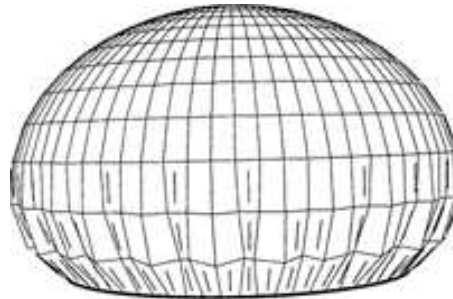


Fig. 1.2. Circular Ogden membrane subjected to a non dimensional pressure of 2.0 combined with a 50% reduction in the length of boundary circumference.

necting the stable branches, the latter requiring an increase in pressure to effect an increase in volume. It may be argued on physical grounds that the unstable equilibrium branch does not represent a meaningful solution, since the membrane is likely to undergo dynamical behaviour instead if subjected to even the smallest disturbance. The dynamic relaxation method mirrors this expectation as it never generates the unstable branch of equilibria. It is also noteworthy that computations using the neo-Hookean strain energy in place of (15)–(16) did not yield equilibria for pressures above $\tilde{p} = 2.612$. This is due to the insufficient growth of the latter strain-energy function at large stretches.

In the next class of examples the membrane is subjected to a relatively

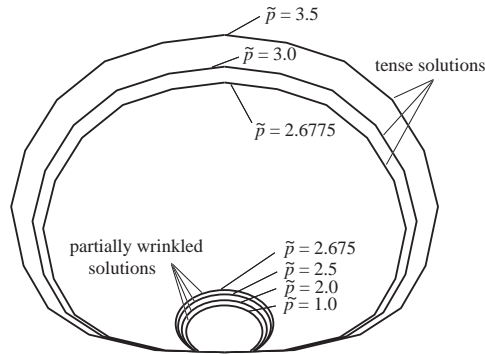


Fig. 1.3. Cross-sections through meridian of deformed configurations of circular Ogden membrane for different numerical values of pressure

modest pressure of intensity $\tilde{p} = 1.0$ while the central node of the mesh at the center of the membrane is displaced vertically downward by a fixed amount. The radius of the outer boundary of the membrane is decreased to 0.8. The resulting deformation is axisymmetric and exhibits wrinkling in a region adjoining the boundary, as depicted in Figure 1.4.

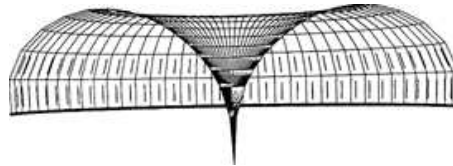


Fig. 1.4. Cross section through pressurized circular membrane subjected to a central vertical point load directed downwards, combined with a 20% reduction in the length of boundary circumference

A further reduction of the boundary radius to 0.5 holding the pressure and central displacement fixed generates a remarkable deformation in which the membrane exhibits pleats or *hinges* separating three distinct lobes. Oblique and overhead views are shown in Figures 1.5 and 1.6.

The computed solution is not to be regarded as definitive since the predicted deformation entails self penetration of the material. A refined solution would penalize self penetration through the introduction of a reactive contact pressure distribution. Eliza did not have the opportunity to incorporate this refinement into the computations. Nevertheless the results are remarkable to the extent that they capture the typical three-lobe pat-

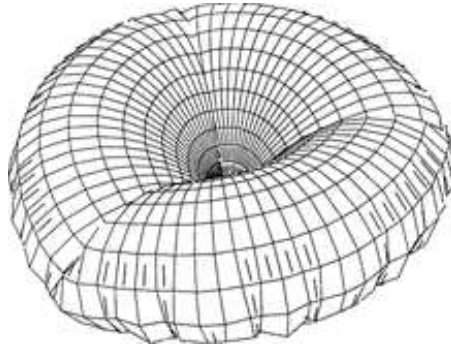


Fig. 1.5. Pressurized circular membrane subjected to a central vertical point load directed downwards, combined with a 50% reduction in the length of boundary circumference

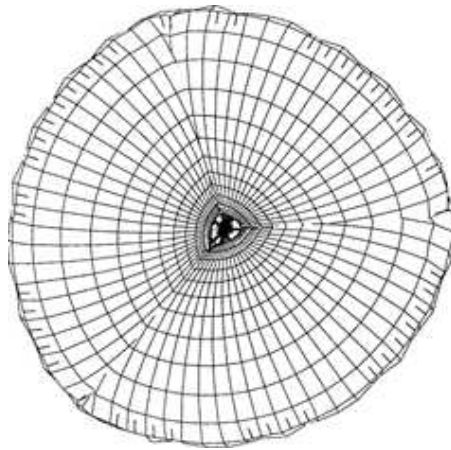


Fig. 1.6. Pressurized circular membrane subjected to a central vertical point load directed downwards, combined with a 50% reduction in the length of boundary circumference; top view

tern observed when a weight is suspended from the apex of a pressurized high-altitude balloon [Baginski (2002)]⁷.

Solutions possessing the unusual and unexpected features exhibited here were made possible by Eliza's pioneering efforts in this rich branch of computational elasticity. It is my hope that they will stimulate further advances in the study of the intricate and difficult problem of nonlinear membrane behaviour.

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