COMPLEX BOUNDARY VALUE PROBLEMS IN A QUARTER PLANE

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Dirichlet and Neumann boundary value problems are considered for the inhomogeneous Cauchy-Riemann equation in a quarter plane. Solvability conditions and solutions are given in explicit form.

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1. Introduction

The Cauchy-Pompeiu representation formula for functions in the upper right quarter plane $\mathbb{Q}_1$ of the complex plane $\mathbb{C}$ follows from the Gauss theorem for regular domains and a limiting process, see [1].

Cauchy-Pompeiu representation Any $w \in C^1(\mathbb{Q}_1;\mathbb{C}) \cap C(\bar{\mathbb{Q}}_1;\mathbb{C})$, where

$$\mathbb{Q}_1 = \{z \in \mathbb{C} : 0 < \text{ Re } z, 0 < \text{Im } z\},$$

for which for some $0 < \delta$ the function $(1+r)^\delta M(r,w)$ with

$$M(r,w) = \max\{|w(z)| : |z| = r, 0 \leq \text{ Re } z, 0 \leq \text{ Im } z\}$$

is bounded in $\mathbb{R}^+$ and $w_{\bar{z}} \in L_1(\mathbb{Q}_1;\mathbb{C})$ is representable as

$$w(z) = \frac{1}{2\pi i}\int\limits_0^{+\infty} w(t)\frac{dt}{t-z} - \frac{1}{2\pi i}\int\limits_0^{+\infty} w(it)\frac{dt}{t+iz} - \frac{1}{\pi}\int\limits_{\mathbb{Q}_1} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}. \quad (1)$$

Introducing the harmonic Green function for $\mathbb{Q}_1$

$$G_1(z,\zeta) = \log\left|\frac{\bar{\zeta}^2 - z^2}{\zeta^2 - z^2}\right|^2 = \log\left|\frac{(\bar{\zeta}-z)(\bar{\zeta}+z)}{(\zeta-z)(\zeta+z)}\right|^2$$

this representation can be altered into

$$w(z) = \frac{1}{\pi}\int\limits_{\mathbb{Q}_1} w(\zeta)\left[\frac{1}{(\bar{\zeta}+z)^2} - \frac{1}{(\bar{\zeta}-z)^2}\right]d\xi d\eta - \frac{1}{\pi}\int\limits_{\mathbb{Q}_1} w_{\bar{\zeta}}(\zeta)\partial_z G_1(z,\zeta)d\xi d\eta, \quad (2)$$

see [1]. For respective representations in the upper half plane see [8], in the unit disc see e.g. [2,3,4,5] and in arbitrary regular domains see [9]. In [1] formula (1) is adjusted to Schwarz boundary data and the solution to the Schwarz boundary value problem is given for the inhomogeneous Cauchy-Riemann and Bitsadze equations.

Here the Dirichlet and Neumann boundary value problems are treated for the Cauchy-Riemann equation. These problems are known to be overdetermined so that solvability conditins have to be determined, see e.g [4,5,6,7].

2. The Dirichlet problem

The complex Gauss theorem in the form

$$\frac{1}{2\pi i}\int\limits_{\partial D} w(z)dz = \frac{1}{\pi}\int\limits_{D} w_{\bar{z}}(z)dxdy$$

for functions $w \in C^1(D;\mathbb{C}) \cap C(\bar{D};\mathbb{C})$ for bounded domains D in the complex plane $\mathbb{C}$ with piecewise smooth boundary ∂D, see e.g. [2], applied for

$$\mathbb{Q}_{1R} = \{z = x + iy : |z| < R, 0 < x, 0 < y\}$$

besides leading to (1) also supplies for $w \in C^1(\mathbb{Q}_1;\mathbb{C}) \cap C(\bar{\mathbb{Q}}_1;\mathbb{C})$ with in $\mathbb{R}^+$ bounded $(1+r)^\delta M(r,w)$ and $w_{\bar{z}} \in L_{p,2}(\mathbb{Q};\mathbb{C}), 2 < p$, and $z \in \mathbb{Q}_1$

$$\frac{1}{2\pi i}\int_0^{+\infty} w(t)\frac{dt}{t-\bar{z}} - \frac{1}{2\pi i}\int_0^{+\infty} w(it)\frac{dt}{t+i\bar{z}} - \frac{1}{\pi}\int_{\mathbb{Q}_1} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}} = 0, \quad (3)$$

$$\frac{1}{2\pi i}\int_0^{+\infty} w(t)\frac{dt}{t+\bar{z}} - \frac{1}{2\pi i}\int_0^{+\infty} w(it)\frac{dt}{t-i\bar{z}} - \frac{1}{\pi}\int_{\mathbb{Q}_1} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta+\bar{z}} = 0, \quad (4)$$

$$\frac{1}{2\pi i}\int_0^{+\infty} w(t)\frac{dt}{t+z} - \frac{1}{2\pi i}\int_0^{+\infty} w(it)\frac{dt}{t-iz} - \frac{1}{\pi}\int_{\mathbb{Q}_1} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta+z} = 0. \quad (5)$$

Properly combined with (1) they lead to solutions to Schwarz problems, see [1]. But they together with (1) also lead to solvability conditions and solutions to the Dirichlet problem for the Cauchy-Riemann equation

Theorem 1 *The Dirichlet problem*

$$w_{\bar{z}} = f \text{ in } \mathbb{Q}_1, w = \gamma_1 \text{ for } 0 \le x, y = 0, w = \gamma_2 \text{ for } 0 \le y, x = 0$$

for $f \in L_{p,2}(\mathbb{Q}_1;\mathbb{C}), 2 < p, \gamma_1, \gamma_2 \in C(\mathbb{R};\mathbb{C})$ *such that* $(1+t)^\delta\gamma_1(t), (1+t)^\delta\gamma_2(t)$ *are bounded for some* $0 < \delta$ *and satisfying the compatibility condition* $\gamma_1(0) = \gamma_2(0)$ *is uniquely weakly solvable in the class* $C^1(\mathbb{Q}_1;\mathbb{C}) \cap C(\bar{\mathbb{Q}}_1;\mathbb{C})$ *if and only if*

$$\frac{1}{2\pi i}\int_0^{+\infty} \gamma_1(t)\frac{dt}{t-\bar{z}} - \frac{1}{2\pi i}\int_0^{+\infty} \gamma_2(t)\frac{dt}{t+i\bar{z}} - \frac{1}{\pi}\int_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}} = 0, \quad (6)$$

$$\frac{1}{2\pi i}\int_0^{+\infty} \gamma_1(t)\frac{dt}{t+\bar{z}} - \frac{1}{2\pi i}\int_0^{+\infty} \gamma_2(t)\frac{dt}{t-i\bar{z}} - \frac{1}{\pi}\int_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta+\bar{z}} = 0, \quad (7)$$

$$\frac{1}{2\pi i}\int_0^{+\infty} \gamma_1(t)\frac{dt}{t+z} - \frac{1}{2\pi i}\int_0^{+\infty} \gamma_2(t)\frac{dt}{t-iz} - \frac{1}{\pi}\int_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta+z} = 0. \quad (8)$$

The solution is

$$w(z) = \frac{1}{2\pi i}\int_0^{+\infty} \gamma_1(t)\frac{dt}{t-z} - \frac{1}{2\pi i}\int_0^{+\infty} \gamma_2(t)\frac{dt}{t+iz} - \frac{1}{\pi}\int_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta-z}. \quad (9)$$

Remark *The boundedness condition on the boundary data* γ_1, γ_2 *at infinity can be weakened to* $\gamma_1, \gamma_2 \in L_2(\mathbb{R}^+; \mathbb{C})$.

Proof If this Dirichlet problem has a solution in the class $C^1(\mathbb{Q}_1;\mathbb{C}) \cap C(\bar{\mathbb{Q}}_1;\mathbb{C})$ satisfying $w_{\bar{z}} \in L_{p,2}(\mathbb{Q}_1;\mathbb{C})$ for some $2 < p$ then according to (1) it is of the form (9). On the other hand (9) can be verified to be a solution. From the assumptions on γ_1, γ_2 and f it follows by direct computation and from [10], p. 43, that $(1+r)^\rho M(r,w)$ is bounded on $\mathbb{R}^+$ with $\rho = \min\{\delta, \frac{p-2}{p}\}$. Moreover, w obviously satisfies the differential equation. For checking the boundary conditions the solvability conditions (6) to (8) are needed. Adding (7) to (9) and subtracting (6) and (8) leads to

$$w(z) = \frac{2}{\pi} \int_0^{+\infty} \gamma_1(t) \frac{y}{|t-z|^2} \; \frac{t^2+|z|^2}{|t+z|^2} dt + \frac{2i}{\pi} \int_0^{+\infty} \gamma_2(t) \frac{y}{|t-iz|^2} \; \frac{t^2-|z|^2}{|t-iz|^2} dt$$

$$-\frac{2}{\pi} \int_{\mathbb{Q}_1} f(\zeta) \left(\frac{z}{\zeta^2 - z^2} - \frac{\bar{z}}{\zeta^2 - \bar{z}^2} \right) d\xi d\eta. \tag{10}$$

Adding (8) to (9) and subtracting (6) and (7) shows

$$w(z) = \frac{2}{\pi i} \int_0^{+\infty} \gamma_1(t) \frac{x}{|t-z|^2} \; \frac{t^2-|z|^2}{|t+z|^2} dt + \frac{2}{\pi i} \int_0^{+\infty} \gamma_2(t) \frac{x}{|t+iz|^2} \; \frac{t^2+|z|^2}{|t-iz|^2} dt$$

$$-\frac{2}{\pi} \int_{\mathbb{Q}_1} f(\zeta) \left(\frac{z}{\zeta^2 - z^2} + \frac{\bar{z}}{\zeta^2 - \bar{z}^2} \right) d\xi d\eta. \tag{11}$$

Observing that

$$\frac{1}{2\pi i} \, \frac{z-\bar{z}}{|t-z|^2} = \frac{1}{\pi} \, \frac{y}{|t-z|^2}, \; z = x+iy, \; x,t \in \mathbb{R}, \; 0 < y$$

is the Poisson kernel for the upper half plane, see e.g. [8], from (10) for $t_0 \in \mathbb{R}^+$

$$\lim_{z \to t_0} w(z) = \lim_{z \to t} \frac{2}{\pi} \int_0^{+\infty} \gamma_1(t) \frac{y}{|t-z|^2} \; \frac{t^2+|z|^2}{|t+z|^2} dt = \gamma_1(t_0)$$

and from (11) likewise for $t_0 \in \mathbb{R}^+$

$$\lim_{z \to it_0} w(z) = \lim_{z \to it_0} \frac{2}{\pi i} \int_0^{+\infty} \gamma_2(t) \frac{x}{|t+iz|^2} \; \frac{t^2+|z|^2}{|t-iz|^2} dt = \gamma_2(t_0).$$

The solution to the Dirichlet problem can be used to find the solution and the solvability conditions to the Neumann problem for the inhomogeneous Cauchy-Riemann equation.

3. The Neumann problem

The normal derivative on the real axis is $\partial_y = i(\partial_z - \partial_{\bar{z}})$ while $\partial_x = \partial_z + \partial_{\bar{z}}$ is the normal derivative on the imaginary axis. With respect to the upper right quadrant $\mathbb{Q}_1$ these are derivatives with respect to the inner normals. For simplicity at first the homogeneous Cauchy-Riemann equation is considered.

Theorem 2 *The Neumann problem*

$$\partial_y w = \gamma_1 \text{ for } 0 \leq x, y = 0,\ \partial_x w = \gamma_2 \text{ for } 0 \leq y, x = 0,\ w(0) = c$$

for analytic functions in $\mathbb{Q}_1$ *being continuous in* $\bar{\mathbb{Q}}_1$*, is uniquely solvable for* $\gamma_1, \gamma_2 \in C(\mathbb{R}^+; \mathbb{C}), c \in \mathbb{C}$*, such that for some* $0 < \delta$ *the functions* $(1 + |t|)^\delta \gamma_1(t), (1 + |t|)^\delta \gamma_2(t)$ *are bounded, if and only if*

$$\frac{1}{2\pi} \int_0^{+\infty} \gamma_1(t) \frac{dt}{t - \bar{z}} + \frac{1}{2\pi i} \int_0^{+\infty} \gamma_2(t) \frac{dt}{t + i\bar{z}} = 0, \tag{12}$$

$$\frac{1}{2\pi} \int_0^{+\infty} \gamma_1(t) \frac{dt}{t + \bar{z}} + \frac{1}{2\pi i} \int_0^{+\infty} \gamma_2(t) \frac{dt}{t - i\bar{z}} = 0, \tag{13}$$

$$\frac{1}{2\pi} \int_0^{+\infty} \gamma_1(t) \frac{dt}{t + z} + \frac{1}{2\pi i} \int_0^{+\infty} \gamma_2(t) \frac{dt}{t - iz} = 0. \tag{14}$$

The solution is

$$w(z) = c + \frac{1}{2\pi} \int_0^{+\infty} \gamma_1(t) \log \left| \frac{t^2 - z^2}{t^2} \right|^2 dt + \frac{1}{2\pi} \int_0^{+\infty} \gamma_2(t) \log \left| \frac{t^2 + z^2}{t^2} \right|^2 dt. \tag{15}$$

Proof For an analytic function the boundary conditions are

$$\partial_y w = i w' = \gamma_1,\ \partial_x w = w' = \gamma_2.$$

These are Dirichlet conditions for the analytic function w'. Hence, according to Theorem 1, the solution is

$$w'(z) = -\frac{1}{2\pi}\int\limits_0^{+\infty} \gamma_1(t)\frac{dt}{t-z} - \frac{1}{2\pi i}\int\limits_0^{+\infty} \gamma_2(t)\frac{dt}{t+iz} \tag{16}$$

if and only if (12)-(14) are satisfied. Integrating (16) shows (15) or also

$$w(z) = w\left(\frac{1+i}{\sqrt{2}}\right) + \frac{1}{2\pi}\int\limits_0^{+\infty} \gamma_1 \log\frac{|t^2-z^2|^2}{1+t^4} + \frac{1}{2\pi}\int\limits_0^{+\infty} \gamma_2(t)\log\frac{|t^2+z^2|^2}{1+t^4}dt.$$

To verify that (15) in fact is the solution under the solvability conditions (12) to (14) at first w has to be recognized as an analytic function. Differentiation with respect to $\bar{z}$ shows

$$\begin{aligned} w_{\bar{z}}(z) &= -\frac{1}{2\pi}\int\limits_0^{+\infty} \gamma_1(t)\frac{2\bar{z}}{t^2-\bar{z}^2}dt + \frac{1}{2\pi}\int\limits_0^{+\infty} \gamma_2(t)\frac{2\bar{z}}{t^2+\bar{z}^2}dt, \\ &= -\frac{1}{2\pi}\int\limits_0^{+\infty} \gamma_1(t)\left(\frac{1}{t-\bar{z}} - \frac{1}{t-\bar{z}}\right)dt \\ &\quad + \frac{1}{2\pi i}\int\limits_0^{+\infty} \gamma_2(t)\left(\frac{1}{t-i\bar{z}} - \frac{1}{t+i\bar{z}}\right)dt = 0 \end{aligned}$$

according to (12) and (13). Similarly,

$$\begin{aligned} w'(z) &= -\frac{1}{2\pi}\int\limits_0^{+\infty} \gamma_1(t)\left(\frac{1}{t-z} - \frac{1}{t+z}\right)dt \\ &\quad + \frac{1}{2\pi i}\int\limits_0^{+\infty} \gamma_2(t)\left(\frac{1}{t-iz} - \frac{1}{t+iz}\right)dt \\ &= -\frac{1}{2\pi}\int\limits_0^{+\infty} \gamma_1(t)\frac{dt}{t-z} - \frac{1}{2\pi i}\int\limits_0^{+\infty} \gamma_2(t)\frac{dt}{t+iz} \end{aligned}$$

because of (14).

Theorem 3 *The Neumann problem*

$$w_{\bar z} = f \text{ in } \mathbb{Q}_1, \ w(0) = c,$$
$$\partial_y w = \gamma_1 \text{ for } 0 \le x, y = 0, \ \partial_x w = \gamma_2 \text{ for } 0 \le y, x = 0,$$

where $f \in L_{p,2}(\mathbb{Q}_1; \mathbb{C}) \cap C^\alpha(\mathbb{Q}_1; \mathbb{C})$ *for* $2 < p, 0 < \alpha < 1, \gamma_1, \gamma_2 \in C(\mathbb{R}^+; \mathbb{C})$ *such that* $(1+t)^\delta \gamma_1(t), (1+t)^\delta \gamma_2(t), (1+t)^\delta f(t), (1+t)^\delta f(it)$ *are bounded for some* $0 < \delta, c \in \mathbb{C}$ *are uniquely weakly solvable if and only if*

$$\frac{1}{2\pi} \int_0^{+\infty} (\gamma_1(t) + if(t)) \frac{dt}{t - \bar z}$$
$$+ \frac{1}{2\pi i} \int_0^{+\infty} (\gamma_2(t) - f(it)) \frac{dt}{t + i\bar z} + \frac{1}{\pi} \int_{\mathbb{Q}_1} f(\zeta) \frac{d\xi d\eta}{\zeta - \bar z} = 0, \tag{17}$$

$$\frac{1}{2\pi} \int_0^{+\infty} (\gamma_1(t) + if(t)) \frac{dt}{t + \bar z}$$
$$+ \frac{1}{2\pi i} \int_0^{+\infty} (\gamma_2(t) - f(it)) \frac{dt}{t - i\bar z} + \frac{1}{\pi} \int_{\mathbb{Q}_1} f(\zeta) \frac{d\xi d\eta}{\zeta + \bar z} = 0, \tag{18}$$

$$\frac{1}{2\pi} \int_0^{+\infty} (\gamma_1(t) + if(t)) \frac{dt}{t + z}$$
$$+ \frac{1}{2\pi i} \int_0^{+\infty} (\gamma_2(t) - f(it)) \frac{dt}{t - iz} + \frac{1}{\pi} \int_{\mathbb{Q}_1} f(\zeta) \frac{d\xi d\eta}{\zeta + z} = 0. \tag{19}$$

The solution is

$$w(z) = c + \frac{1}{2\pi} \int_0^{+\infty} (\gamma_1(t) + if(t)) \log \left| \frac{t^2 - z^2}{t^2} \right|^2 dt$$
$$+ \frac{1}{2\pi} \int_0^{+\infty} (\gamma_2(t) - f(it)) \log \left| \frac{t^2 + z^2}{t^2} \right|^2 dt$$
$$- \frac{z}{\pi} \int_{\mathbb{Q}_1} \frac{f(\zeta)}{\zeta} \frac{d\xi d\eta}{\zeta - z}. \tag{20}$$

Proof Consider the function

$$\varphi = w - Tf,\ Tf(z) = -\frac{1}{\pi}\int\limits_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta - z}.$$

It is analytic in $\mathbb{Q}_1$ satisfying

$$\begin{aligned}\varphi' &= -i\partial_y(w - Tf) = -i\gamma_1 - \Pi f + f \text{ for } 0 \le x, y = 0,\\ \varphi' &= \partial_x(w - Tf) = \gamma_2 - \Pi f - f \text{ for } 0 \le y, x = 0\end{aligned}$$

with

$$\Pi f(z) = 1/\pi \int_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}.$$

According to [10], p. 63 Πf is Hölder continuous in $\bar{\mathbb{Q}}_1$. Although the asymptotic behaviour of Πf at infinity is not known the following conclusions follow because of the Fubini theorem.

Thus on the basis of Theorem 1 this Dirichlet problem is solvable if and only if

$$\frac{1}{2\pi i}\int\limits_0^{+\infty}[i\gamma_1(t) + \Pi f(t) - f(t)]\frac{dt}{t - \bar{z}}$$

$$+\frac{1}{2\pi i}\int\limits_0^{+\infty}[\gamma_2(t) - \Pi f(it) - f(it)]\frac{dt}{t + i\bar{z}} = 0, \tag{21}$$

$$\frac{1}{2\pi i}\int\limits_0^{+\infty}[\gamma_1(t) + \Pi f(t) - f(t)]\frac{dt}{t + \bar{z}}$$

$$+\frac{1}{2\pi i}\int\limits_0^{+\infty}[\gamma_2(t) - \Pi f(it) - f(it)]\frac{dt}{t - i\bar{z}} = 0, \tag{22}$$

$$\frac{1}{2\pi i}\int\limits_0^{+\infty}[\gamma_1(t) + \Pi f(t) - f(t)]\frac{dt}{t + z}$$

$$+\frac{1}{2\pi i}\int\limits_0^{+\infty}[\gamma_2(t) - \Pi f(it) - f(it)]\frac{dt}{t - iz} = 0. \tag{23}$$

The solution is

$$\varphi'(z) = -\frac{1}{2\pi i}\int\limits_{0}^{+\infty}[i\gamma_1(t) + \Pi f(t) - f(t)]\frac{dt}{t-z}$$

$$-\frac{1}{2\pi i}\int\limits_{0}^{+\infty}[\gamma_2(t) - \Pi f(it) - f(it)]\frac{dt}{t+iz}. \tag{24}$$

For $z \in \mathbb{Q}_1$

$$\frac{1}{2\pi i}\int\limits_{0}^{+\infty}\Pi f(t)\frac{dt}{t-\bar{z}} - \frac{1}{2\pi i}\int\limits_{0}^{+\infty}\Pi f(it)\frac{dt}{t+i\bar{z}}$$

$$= \frac{1}{\pi}\int\limits_{\mathbb{Q}_1} f(\zeta)\left[\frac{1}{2\pi i}\int\limits_{0}^{+\infty}\frac{dt}{(t-\zeta)(t-\bar{z})} - \frac{1}{2\pi i}\int\limits_{0}^{+\infty}\frac{idt}{(it-\zeta)(it-\bar{z})}\right]d\xi d\eta$$

$$= \frac{1}{\pi}\int\limits_{\mathbb{Q}_1} f(\zeta)\frac{1}{2\pi i}\int\limits_{\partial\mathbb{Q}_1}\frac{d\tilde{\zeta}}{(\tilde{\zeta}-\zeta)(\tilde{\zeta}-\bar{z})}d\xi d\eta = \frac{1}{\pi}\int\limits_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}} \tag{25}$$

and similarly

$$\frac{1}{2\pi i}\int\limits_{0}^{+\infty}\Pi f(t)\frac{dt}{t+\bar{z}} - \frac{1}{2\pi i}\int\limits_{0}^{+\infty}\Pi f(it)\frac{dt}{t-iz}$$

$$= \frac{1}{\pi}\int\limits_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta+\bar{z}}, \tag{26}$$

$$\frac{1}{2\pi i}\int\limits_{0}^{+\infty}\Pi f(t)\frac{dt}{t+z} - \frac{1}{2\pi i}\int\limits_{0}^{+\infty}\Pi f(it)\frac{dt}{t-iz}$$

$$= \frac{1}{\pi}\int\limits_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{t+z} \tag{27}$$

and

$$\frac{1}{2\pi i}\int\limits_{0}^{+\infty}\Pi f(t)\frac{dt}{t-z} - \frac{1}{2\pi i}\int\limits_{0}^{+\infty}\Pi f(it)\frac{dt}{t+iz} = 0. \tag{28}$$

This shows (17), (18), (19) and

$$\varphi'(z) = -\frac{1}{2\pi i}\int_0^{+\infty}[i\gamma_1(t) - f(t)]\frac{dt}{t-z} - \frac{1}{2\pi i}\int_0^{+\infty}[\gamma_2(t) - f(it)]\frac{dt}{t+iz}.$$

Integrating this last equation shows

$$\begin{aligned}\varphi(z) = \varphi(0) &+ \frac{1}{2\pi i}\int_0^{+\infty}[i\gamma_1(t) - f(t)]\log\left|\frac{t^2-z^2}{t^2}\right|^2 dt\\ &+ \frac{1}{2\pi}\int_0^{+\infty}[\gamma_2(t) - f(it)]\log\left|\frac{t^2+z^2}{t^2}\right|^2 dt\end{aligned}$$

so that

$$w(z) = \varphi(t) + Tf(z) = w(0) + Tf(z) - Tf(0)$$

which is (20).

Dirichtlet and Neumann problems are usually solved for the Poisson equation. They can also be considered for the Bitsadze equation, see [4], part II, in the case of the unit disc. For the quater plane $\mathbb{Q}_1$ these problems can be treated in a similar manner as in this paper.

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SOME EQUATIONS OVER 4-DIMENSIONAL CLIFFORD ALGEBRA

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Some real matrix representations of 4-dimensional Clifford numbers are presented and some properties of 4-dimensional Clifford numbers are obtained. As an application, an explicit expression of the general solution of equation $axb = d$ and $ax = xb$ are obtained in terms of a and b in 4-dimensional Clifford algebra $C\ell_{0,3}$. As a by-product, we obtain the necessary and sufficient conditions for two Clifford numbers to be similar.

Keywords: Quaternion, Clifford number, matrix representation, equation.

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1. Introduction

From the general definition of Clifford algebra [1], the basis of 4-dimensional Clifford algebra $C\ell_{0,3}$ is $e_i, i = 0, \cdots, 7$ and the multiplication rules for the basis of $C\ell_{0,3}$ are as follows.

For $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7 \in C\ell_{0,3}$,we define $|a| = \sqrt{\sum_{i=0}^{7} a_i^2}$ will be the Euclidean norm of a.We point out that when $n > 3$, $|a^2| = |a|^2$ will fail to hold and there exist zero-divisors. For

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1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	$-e_3$	-1	e_1	e_6	$-e_7$	$-e_4$	e_5
e_3	e_2	$-e_1$	-1	e_7	e_6	$-e_5$	$-e_4$
e_4	$-e_5$	$-e_6$	e_7	-1	e_1	e_2	$-e_3$
e_5	e_4	$-e_7$	$-e_6$	$-e_1$	-1	e_3	e_2
e_6	e_7	e_4	e_5	$-e_2$	$-e_3$	-1	$-e_1$
e_7	$-e_6$	e_5	$-e_4$	$-e_3$	e_2	$-e_1$	1

instance:

$$|(1+e_7)^2| = 2\sqrt{2}, \text{while } |1+e_7|^2 = 2, \tag{1}$$

$$(1+e_7)(1-e_7) = 0. \tag{2}$$

In this paper we will concentrate on the properties of Clifford numbers and some equations in $C\ell_{0,3}$. Our first task is to investigate the properties of Clifford numbers by using the real quaternions and their matrix representations in Section 2. Based on the matrix representation of Clifford numbers, we will give the concepts of Moore-Penrose inverse of Clifford numbers and obtain the explicit solutions to equation $axb = d$ in Section 3. The nontrivial task of solving equation $ax = xb$ is treated in Section 4.

2. The real matrix representation of elements in $C\ell_{0,3}$

We will represent 4-dimensional Clifford numbers with two quaternions. At first, we recall some properties of division ring of real quaternions $\mathbb{H}$ and their matrix representations.

It is well known that the real quaternion algebra $\mathbb{H}$ is algebraically isomorphic to the matrix algebra corresponding to the bijective map

$$L : q = q_0 + q_1e_1 + q_2e_2 + q_3e_3 \in \mathbb{H} \mapsto L(q) := \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix}.$$

Denote $\overrightarrow{x} = (x_0, x_1, x_2, x_3)^T$ for $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H}$, where A^T stands for the transposed matrix of A. Then the following equations hold.

$$\overrightarrow{qx} = L(q)\overrightarrow{x}, \ \overrightarrow{xq} = R(q)\overrightarrow{x}, \tag{3}$$

where $R(q) = E_4 L(q)^T E_4$ and $E_4 = diag(1, -1, -1, -1)$. For more details of the properties of quaternions and related topics, see [2,3,4,5] and the references therein.

We can write

$$a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 = a_h + a_H e_4, \quad (4)$$

where $a_h = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, a_H = a_4 + a_5 e_1 + a_6 e_2 + a_7 e_3 \in \mathbb{H}$. We define

$$\bar{a} = a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3 - a_4 e_4 - a_5 e_5 - a_6 e_6 + a_7 e_7 \quad (5)$$

and

$$\mathcal{C}\mathtt{re}(a) = \frac{1}{2}(a + \bar{a}) = a_0 + a_7 e_7, \; \mathcal{C}\mathtt{im}(a) = a - \mathcal{C}\mathtt{re}(a).$$

Note that $\mathbb{R} + \mathbb{R}e_7$ is the center field of $C\ell_{0,3}$. For $a \in C\ell_{0,3}$, we define $\mathcal{T} : C\ell_{0,3} \mapsto \mathbb{R}$ by

$$\mathcal{T}(a) = a_0 a_7 + a_2 a_5 - a_1 a_6 - a_3 a_4. \quad (6)$$

Then

$$a\bar{a} = \bar{a}a = |a|^2 + 2\mathcal{T}(a) e_7 \quad (7)$$

and

$$|ab|^2 = |a|^2 |b|^2 + 4\mathcal{T}(a)\mathcal{T}(b), \; \mathcal{T}(ab) = |a|^2 \mathcal{T}(b) + |b|^2 \mathcal{T}(a), \; \mathcal{T}(a) = \mathcal{T}(\bar{a}).$$

Based on the real matrix representations of quaternions, we can introduce the real matrix representations of numbers in $C\ell_{0,3}$ as follows.

Definition 2.1. Let $a = a_h + a_H e_4 \in C\ell_{0,3}$. Then the real 8-by-8 real matrix

$$\omega(a) := \begin{pmatrix} L(a_h) & -L(a_H)P_4 \\ L(a_H)P_4 & L(a_h) \end{pmatrix} \quad (8)$$

is called the left matrix representation of a over $\mathbb{R}$, where $P_4 = diag(1, -1, -1, 1)$. The real 8-by-8 real matrix

$$\nu(a) := K_8 \omega(a)^T K_8 \quad (9)$$

is called the right matrix representation of a, where $K_8 = diag(-1, 1, 1, 1, 1, 1, 1, -1)$.

We list some properties of the representation matrix as following propositions.

Proposition 2.1. *Denote* $\overrightarrow{x} = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T$ *for* $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$. *Then*

$$\overrightarrow{ax} = \omega(a)\overrightarrow{x} \quad \text{and} \quad \overrightarrow{xa} = \nu(a)\overrightarrow{x}. \tag{10}$$

Proposition 2.2. *Let* $a, b \in C\ell_{0,3}$ *and* $\lambda \in \mathbb{R}$. *Then*

(1) $a = b \Longleftrightarrow \omega(a) = \omega(b) \Longleftrightarrow \nu(a) = \nu(b)$;
(2) $\omega(a+b) = \omega(a)+\omega(b), \omega(\lambda a) = \lambda\omega(a), \omega(ab) = \omega(a)\omega(b), \omega(1) = I_8$;
(3) $\nu(a + b) = \nu(a) + \nu(b), \nu(\lambda a) = \lambda\nu(a), \nu(ab) = \nu(b)\nu(a), \nu(1) = I_8$;
(4) $\omega(\bar{a}) = \omega(a)^T, \nu(\bar{a}) = \nu(a)^T, \omega(a)\nu(b) = \nu(b)\omega(a)$;

where I_8 *denotes the* 8×8 *unit matrix.*

Proposition 2.3. *The eigenvalues of right matrix representation of a,* $\omega(a)$, *are given by*

$$\lambda_{1,2} = a_0 + a_7 \pm i\sqrt{|a|^2 + 2\mathcal{T}(a) - (a_0 + a_7)^2}$$

and

$$\lambda_{3,4} = a_0 - a_7 \pm i\sqrt{|a|^2 - 2\mathcal{T}(a) - (a_0 - a_7)^2},$$

where $i = \sqrt{-1}$ *and each eigenvalue occurs with algebraic multiplicity 2.*

In order to find the solution of $ax = xb$ in Section 4, we need the following propositions which proof can be verify directly.

Proposition 2.4. *The eigenvalues of* $\omega(a) - \nu(b)$ *can be expressed by*

$$\begin{aligned}\mu_{1,2} = {} & (a_0 + a_7) - (b_0 + b_7) + i(\sqrt{(a_1 - a_6)^2 + (a_3 - a_4)^2 + (a_2 + a_5)^2} \\ & \pm \sqrt{(b_1 - b_6)^2 + (b_3 - b_4)^2 + (b_2 + b_5)^2}\,),\end{aligned}$$

$$\begin{aligned}\mu_{3,4} = {} & (a_0 + a_7) - (b_0 + b_7) - i(\sqrt{(a_1 - a_6)^2 + (a_3 - a_4)^2 + (a_2 + a_5)^2} \\ & \pm \sqrt{(b_1 - b_6)^2 + (b_3 - b_4)^2 + (b_2 + b_5)^2}\,),\end{aligned}$$

$$\begin{aligned}\mu_{5,6} = {} & (a_0 - a_7) - (b_0 - b_7) + i(\sqrt{(a_1 + a_6)^2 + (a_3 + a_4)^2 + (a_2 - a_5)^2} \\ & \pm \sqrt{(b_1 + b_6)^2 + (b_3 + b_4)^2 + (b_2 - b_5)^2}\,),\end{aligned}$$

$$\begin{aligned}\mu_{7,8} = {} & (a_0 - a_7) - (b_0 - b_7) - i(\sqrt{(a_1 + a_6)^2 + (a_3 + a_4)^2 + (a_2 - a_5)^2} \\ & \pm \sqrt{(b_1 + b_6)^2 + (b_3 + b_4)^2 + (b_2 - b_5)^2}\,).\end{aligned}$$

Proposition 2.5. *The eigenvalues of $\omega(a)+\nu(b)$ can be expressed by*

$$\begin{aligned}\mu_{1,2} = (a_0+a_7)+(b_0+b_7)+i(&\sqrt{(a_1-a_6)^2+(a_3-a_4)^2+(a_2+a_5)^2}\\ &\pm\sqrt{(b_1-b_6)^2+(b_3-b_4)^2+(b_2+b_5)^2}\,),\end{aligned}$$

$$\begin{aligned}\mu_{3,4} = (a_0+a_7)+(b_0+b_7)-i(&\sqrt{(a_1-a_6)^2+(a_3-a_4)^2+(a_2+a_5)^2}\\ &\pm\sqrt{(b_1-b_6)^2+(b_3-b_4)^2+(b_2+b_5)^2}\,),\end{aligned}$$

$$\begin{aligned}\mu_{5,6} = (a_0-a_7)+(b_0-b_7)+i(&\sqrt{(a_1+a_6)^2+(a_3+a_4)^2+(a_2-a_5)^2}\\ &\pm\sqrt{(b_1+b_6)^2+(b_3+b_4)^2+(b_2-b_5)^2}\,),\end{aligned}$$

$$\begin{aligned}\mu_{7,8} = (a_0-a_7)+(b_0-b_7)-i(&\sqrt{(a_1+a_6)^2+(a_3+a_4)^2+(a_2-a_5)^2}\\ &\pm\sqrt{(b_1+b_6)^2+(b_3+b_4)^2+(b_2-b_5)^2}\,).\end{aligned}$$

3. The Moore-Penrose inverse of elements in $C\ell_{0,3}$

Definition 3.1. Let $a \in C\ell_{0,3}$. If there exists an $x \in C\ell_{0,3}$ such that $ax = 1$, then x is called the inverse of a, which is denoted by a^{-1}.

For $a \in C\ell_{0,3}$, we define $p : C\ell_{0,3} \mapsto \mathbb{R}$ by

$$p(a) = |a|^4 - 4\mathcal{T}(a)^2. \tag{11}$$

Proposition 3.1. *If $p(a) = |a|^4 - 4\mathcal{T}(a)^2 \neq 0$, then a is invertible and*

$$a^{-1} = \frac{(|a|^2 - 2\mathcal{T}(a)e_7)}{p(a)}\bar{a}.$$

Lemma 3.1. *The following equations*

$$ax = 0,\ \bar{a}x = 0,\ \bar{a}ax = 0 \tag{12}$$

have the same solutions. Moreover, if a is not invertible then the general solution is

$$x = (|a|^2 - 2\mathcal{T}(a)e_7)p,\ \forall p \in C\ell_{0,3}. \tag{13}$$

Definition 3.2. The Moore-Penrose inverse of $a \in C\ell_{0,3}$, denoted by a^+, is a Clifford number x such that

$$axa = a,\ xax = x,\ \overline{ax} = ax. \tag{14}$$

Note that if x is the Moore-Penrose inverse of a then $\overline{xa} = xa$ and thus $\omega(x)$ is the unique Moore-Penrose inverse of $\omega(a)$.

If $a = 0$ then $a^+ = 0$ and if a is invertible then $a^+ = a^{-1} = \frac{(\bar{a}a)^*}{p(a)}\bar{a}$. For non-zero element a, by $\bar{a}a = a\bar{a} = |a|^2 + 2\mathcal{T}(a)e_7$, we have

$$a\frac{\bar{a}}{2|a|^2}a = a, \quad \frac{\bar{a}}{2|a|^2}a\frac{\bar{a}}{2|a|^2} = \frac{\bar{a}}{2|a|^2}, \quad \overline{a\frac{\bar{a}}{2|a|^2}} = a\frac{\bar{a}}{2|a|^2}, \tag{15}$$

which implies that the Moore-Penrose inverse of a (which is not invertible) is

$$a^+ = \frac{\bar{a}}{2|a|^2}. \tag{16}$$

It is easy to verify that $\omega(a^+) = \omega(a)^+$ and the following proposition.

Proposition 3.2. *If $a \in C\ell_{0,3}$ is not invertible, then a satisfies one of the following two cases:*
(1) $a\bar{a} = |a|^2(1 + e_7)$ and $a = ae_7$; (2) $a\bar{a} = |a|^2(1 - e_7)$ and $a = -ae_7$.

Examples: $a = e_2 + e_5$, $b = e_2 + e_3 + e_5$, $ab = -2 + e_1 - e_6 - 2e_7$, $ba = -2 - e_1 + e_6 - 2e_7$, $a^+ = -\frac{1}{4}(e_2 + e_5)$, $b^{-1} = -\frac{1}{5}(e_2 + 3e_3 + 2e_4 + e_5)$, $ae_7 = a$.

By the theory of linear equation, we have the following two theorems.

Theorem 3.1. *Let $a \in C\ell_{0,3}$ be a nonzero element and not invertible. Then the linear equation $ax = d$ has a solution if and only if $\frac{a\bar{a}}{2|a|^2}d = d$. In that case, the general solution can be expressed by*

$$x = \frac{\bar{a}d}{2|a|^2} + (2|a|^2 - a\bar{a})p, \ \forall\, p \in C\ell_{0,3}.$$

Theorem 3.2. *Let $a, b \in C\ell_{0,3}$ be nonzero elements and not invertible. Then the linear equation $axb = d$ has a solution if and only if $\frac{ab\overline{ab}}{4|a|^2|b|^2}d = d$. In that case, the general solution can be expressed by*

$$x = \frac{\bar{a}d\bar{b}}{4|a|^2|b|^2} + (4|a|^2|b|^2 - ab\overline{ab})p, \ \forall\, p \in C\ell_{0,3}.$$

4. Linear equation $ax = xb$ over $C\ell_{0,3}$

Due to the non-commutativity and existence of non-division elements of $C\ell_{0,3}$, it is non-trivial to find the solutions of equation in $C\ell_{0,3}$.

We say $a, b \in C\ell_{0,3}$ satisfy *condition 1* if

$$a_0 + a_7 = b_0 + b_7 \text{ and } |a|^2 + 2\mathcal{T}(a) = |b|^2 + 2\mathcal{T}(b) \tag{17}$$

and *condition 2* if

$$a_0 - a_7 = b_0 - b_7 \text{ and } |a|^2 - 2\mathcal{T}(a) = |b|^2 - 2\mathcal{T}(b). \tag{18}$$

By the equivalently of $(\omega(a) - \nu(b))\overrightarrow{x} = 0$ and the linear equation

$$ax = xb, \tag{19}$$

Proposition 2.4 implies that

Proposition 4.1. *Equation (19) has a nonzero solution if and only if a, b satisfy condition 1 or condition 2.*

In order to find the general solution of Eq.(19), we list some properties of its solutions.

Proposition 4.2. *The solution x of Eq.(19) with $|a| \neq |b|$ satisfies*
(1) $x = xe_7$ provided condition 1 holds; (2) $x = -xe_7$ provided condition 2 holds.

Proposition 4.3. *Equation (19) with $|a| = |b|$ has a nonzero solution if and only if one of the following three cases holds: (1) $\mathcal{T}(a) = \mathcal{T}(b), a_0 + a_7 = b_0 + b_7, a_0 \neq b_0$; (2) $\mathcal{T}(a) = \mathcal{T}(b), a_0 - a_7 = b_0 - b_7, a_0 \neq b_0$; (3) $\mathcal{T}(a) = \mathcal{T}(b), a_0 = b_0, a_7 = b_7$. Moreover, the solution x of Eq.(19) satisfies (4) $x = xe_7$ provided (1) holds ; (5) $x = -xe_7$ provided (2) holds.*

Proposition 4.4. *Equation (19) has a solution which is invertible if and only if*

$$\mathcal{C}\mathtt{re}(a) = \mathcal{C}\mathtt{re}(b), |a| = |b|, \textit{ and } \mathcal{T}(a) = \mathcal{T}(b).$$

Remark 4.1. We say two Clifford numbers a and b are *similar* if there exists invertible Clifford number $q \in C\ell_{0,3}$ such that $a = qbq^{-1}$. The above proposition provides the necessary and sufficient conditions for two Clifford numbers to be similar, which generalizes the corresponding concept of quaternions.

We are now ready to consider the general solution of Eq.(19). This will be divided into three cases.

Case 1: $\mathcal{C}\mathtt{im}(a) = 0$ *or* $\mathcal{C}\mathtt{im}(b) = 0$

The case 1, say $\mathcal{C}\mathtt{im}(b) = 0$, implies that Eq.(19) is equivalent to $(a-b)x = 0$. By Lemma 3.1, we can summarize these results as the following theorem.

Theorem 4.1. *If case 1 and condition 1 or condition 2 hold, then the general solution of Eq.(19) is*

$$x = (|a-b|^2 - 2\mathcal{T}(a-b)e_7)p, \ \forall\, p \in C\ell_{0,3}.$$

Case 2: $|a| \neq |b|$, $\mathcal{C}\mathtt{im}(a) \neq 0$ *and* $\mathcal{C}\mathtt{im}(b) \neq 0$

Theorem 4.2. *If case 2 and condition 1 hold, then*

(1) if $|\mathcal{C}\mathtt{im}(a)|^2 = -2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$ *and* $|\mathcal{C}\mathtt{im}(b)|^2 = -2\mathcal{T}(\mathcal{C}\mathtt{im}(b))$ *then the general solution can be expressed by*

$$x = (1+e_7)p, \quad \forall p \in C\ell_{0,3}; \tag{20}$$

(2) otherwise the general solution can be expressed by

$$x = (1+e_7)(\mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b)), \quad \forall\, p \in C\ell_{0,3}. \tag{21}$$

If case 2 and condition 2 hold, then

(3) if $|\mathcal{C}\mathtt{im}(a)|^2 = 2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$ *and* $|\mathcal{C}\mathtt{im}(b)|^2 = 2\mathcal{T}(\mathcal{C}\mathtt{im}(b))$ *then the general solution can be expressed by*

$$x = (1-e_7)p, \quad \forall\, p \in C\ell_{0,3}; \tag{22}$$

(4) otherwise the general solution can be expressed by

$$x = (1-e_7)(\mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b)), \quad \forall\, p \in C\ell_{0,3}. \tag{23}$$

Proof. At first, we assume that condition 1 holds. By Proposition 4.2, if x_0 is a nonzero solution of $ax = xb$ then $(1-e_7)x_0 = 0$. It follows from $ax_0 - x_0 b = (\mathcal{C}\mathtt{re}(a) - \mathcal{C}\mathtt{re}(b))x_0 + \mathcal{C}\mathtt{im}(a)x_0 - x_0\mathcal{C}\mathtt{im}(b)$ that

$$\mathcal{C}\mathtt{im}(a)x_0 = x_0\mathcal{C}\mathtt{im}(b).$$

If $|\mathcal{C}\mathtt{im}(a)|^2 = -2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$ and $|\mathcal{C}\mathtt{im}(b)|^2 = -2\mathcal{T}(\mathcal{C}\mathtt{im}(b))$ hold, then each solution of Eq.(19) satisfies $\mathcal{C}\mathtt{im}(a)x = x\mathcal{C}\mathtt{im}(b) = 0$. By Lemma 3.1, the general solution can be expressed as

$$x = (1+e_7)p, \quad \forall\, p \in C\ell_{0,3}.$$

This completes the proof of Theorem 4.2 (1).

For the remaining cases, since

$$\mathcal{C}\mathtt{im}(a)^2 - \mathcal{C}\mathtt{im}(b)^2 = (\mathcal{C}\mathtt{re}(a)^2 - \mathcal{C}\mathtt{re}(b)^2) + (|b|^2 - |a|^2 + 2(\mathcal{T}(b) - \mathcal{T}(a))e_7),$$

we have $(1+e_7)(\mathcal{C}\mathtt{im}(a)^2 - \mathcal{C}\mathtt{im}(b)^2) = 0$. Thus, $x = (1+e_7)(\mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b))$, which satisfies $ax - xb = 0$, is a solution of Eq.(19) for each p in $\mathcal{C}\ell_{0,3}$.

Suppose that x_0 is any solution of $ax = xb$ of the cases in Theorem 4.2 (2). Let $p = \mathcal{C}\mathtt{im}(a)^+ x_0/4$ provided $\mathcal{C}\mathtt{im}(a)$ is invertible or $|\mathcal{C}\mathtt{im}(a)|^2 = 2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$, and let $p = x_0\mathcal{C}\mathtt{im}(b)^+/4$ provided $\mathcal{C}\mathtt{im}(b)$ is invertible or $|\mathcal{C}\mathtt{im}(b)|^2 = 2\mathcal{T}(\mathcal{C}\mathtt{im}(b))$. Then $x_0 = (1+e_7)(\mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b))$ for such p.

The above statements show that, for cases of Theorem 4.2 (2), any solution of $ax = xb$ can be expressed by (21). This completes the proof of Theorem 4.2 (2). By similar arguments, we can obtain the proof of Theorem 4.2 (3),(4). ☐

Case 3: $|a| = |b|$, $\mathcal{C}\mathtt{im}(a) \neq 0$ *and* $\mathcal{C}\mathtt{im}(b) \neq 0$

Theorem 4.3. *If case 3 and the hypotheses in Proposition 4.3 (1) hold, then*

(1) if $|\mathcal{C}\mathtt{im}(a)|^2 = -2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$ *and* $|\mathcal{C}\mathtt{im}(b)|^2 = -2\mathcal{T}(\mathcal{C}\mathtt{im}(b))$ *then the general solution can be expressed by*

$$x = (1+e_7)p, \quad \forall p \in \mathcal{C}\ell_{0,3}; \tag{24}$$

(2) if $|\mathcal{C}\mathtt{im}(a)|^2 \neq -2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$ *or* $|\mathcal{C}\mathtt{im}(b)|^2 \neq -2\mathcal{T}(\mathcal{C}\mathtt{im}(b))$ *then the general solution can be expressed by*

$$x = (1+e_7)(\mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b)), \quad \forall\, p \in \mathcal{C}\ell_{0,3}. \tag{25}$$

If case 3 and the hypotheses in Proposition 4.3 (2) hold, then

(3) if $|\mathcal{C}\mathtt{im}(a)|^2 = 2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$ *and* $|\mathcal{C}\mathtt{im}(b)|^2 = 2\mathcal{T}(\mathcal{C}\mathtt{im}(b))$ *then the general solution can be expressed by*

$$x = (1-e_7)p, \quad \forall\, p \in \mathcal{C}\ell_{0,3}; \tag{26}$$

(4) if $|\mathcal{C}\mathtt{im}(a)|^2 \neq 2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$ *or* $|\mathcal{C}\mathtt{im}(b)|^2 \neq 2\mathcal{T}(\mathcal{C}\mathtt{im}(b))$ *then the general solution can be expressed by*

$$x = (1-e_7)(\mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b)), \quad \forall\, p \in \mathcal{C}\ell_{0,3}. \tag{27}$$

If case 3 and the hypotheses in Proposition 4.3 (3) hold, then

(5) if $\mathcal{C}\mathtt{im}(a)$ *is invertible then the general solution can be expressed by*

$$x = \mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b), \quad \forall\, p \in \mathcal{C}\ell_{0,3}; \tag{28}$$

(6) if $\mathcal{C}\mathtt{im}(a)$ *is not invertible and*

(a) *if* $|\mathcal{C}\mathtt{im}(a)|^2 = -2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$, *then the general belongs to the direct sum*

$$Q_1 \oplus Q_2 = \{x = x_1 + x_2 : x_1 \in Q_1, x_2 \in Q_2\},$$

where $Q_1 = \{x : x = \mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b), \forall p \in C4\}$ *and* $Q_2 = \{x : x = (1 + e_7)p, \forall\ p \in C\ell_{0,3}\}$.

(b) *if* $|\mathcal{C}\mathtt{im}(a)|^2 = 2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$, *then the general belongs to the direct sum*

$$Q_1 \oplus Q_2 = \{x = x_1 + x_2 : x_1 \in Q_1, x_2 \in Q_2\},$$

where $Q_1 = \{x : x = \mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b), \forall p \in C4\}$ *and* $Q_2 = \{x : x = (1 - e_7)p, \forall\ p \in C\ell_{0,3}\}$.

Proof. The proofs of the Theorem 4.3 (1)-(4) follow from the similar reasoning as that in the proof of Theorem 4.2.

If the hypotheses in Proposition 4.3 (3) hold then $\mathcal{C}\mathtt{im}(a)^2 - \mathcal{C}\mathtt{im}(b)^2 = \mathcal{C}\mathtt{re}(a)^2 - \mathcal{C}\mathtt{re}(b)^2 = 0$. Thus $x = \mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b)$, which satisfies $ax - xb = (\mathcal{C}\mathtt{im}(a)^2 - \mathcal{C}\mathtt{im}(b)^2)p = 0$, is a solution of Eq.(19) for each p in $C\ell_{0,3}$. Under the hypotheses in Proposition 4.3 (3), x is a solution of Eq.(19) if and only if $\mathcal{C}\mathtt{im}(a)x = x\mathcal{C}\mathtt{im}(b)$.

If $\mathcal{C}\mathtt{im}(a)$ is invertible, then any solution x_0 can be expressed by $\mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b)$ for $p = \mathcal{C}\mathtt{im}(a)^{-1}x_0/2$. This completes the proof of Theorem 4.3 (5).

If $\mathcal{C}\mathtt{im}(a)$ is not invertible, say $|\mathcal{C}\mathtt{im}(a)|^2 = -2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$, then $|\mathcal{C}\mathtt{im}(b)|^2 = -2\mathcal{T}(\mathcal{C}\mathtt{im}(b))$. Thus $x = (1 + e_7)p$, which satisfies $\mathcal{C}\mathtt{im}(a)x = x\mathcal{C}\mathtt{im}(b)$, is a solutions of Eq.(19) for each p in $C\ell_{0,3}$.

Let $Q_1 = \{x : x = (1 + e_7)p, \forall\ p \in C\ell_{0,3}\}$ and $Q_2 = \{x : x = \mathcal{C}\mathtt{im}(a)p + p\mathcal{C}\mathtt{im}(b), \forall\ p \in C\ell_{0,3}\}$. We can prove that $Q_1 \cap Q_2 = \{0\}$.

By Propositions 2.4 and 2.5, we know that $rank[\omega(a) - \nu(b)] = 2$ and $rank[\omega(\mathcal{C}\mathtt{im}(a)) + \nu(\mathcal{C}\mathtt{im}(b))] = 2$. Thus all the solutions of Eq.(19) belong to direct sum $Q_1 \oplus Q_2$ of Q_1 and Q_2.

Similarly, we can obtain the proof of the case $|\mathcal{C}\mathtt{im}(a)|^2 = 2\mathcal{T}(\mathcal{C}\mathtt{im}(a))$. This completes the proofs of Theorem 4.3 (6). ☐

Remark 4.2. By proposition 3.2, if $a \in C\ell_{0,3}$ is not invertible, then $a = \frac{1}{2}(1 + e_7)a$ or $a = \frac{1}{2}(1 - e_7)a$. In both cases, we have $(1 - e_7)a = 0$ or $(1 + e_7)a = 0$. Thus, in the case (6) of Theorem 4.3, if $y = y_1 + y_2 \in Q_1 \oplus Q_2$ with $0 \neq y_1 \in Q_1$ and $0 \neq y_2 \in Q_2$ then y is invertible.

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