Chapter 5

Present Value

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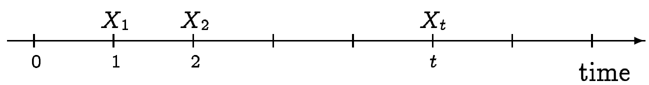
5.1 Definition of Present Value

A basic unit of account in finance is a set of future cash flows. There are two important properties of these cash flows. One, the *amounts*. Two, the *dates* at which the amounts are paid.

For example, a US Government Treasury bill is a promise to pay USD 1000 at a certain future date. This is the typical example of a *risk free* security, one with no uncertainty as to both the amount of and timing of cash flow. As another example, consider an oil company about to start drilling for oil in a new area. The oil company is facing uncertainty at several levels. There is uncertainty about whether the oil company actually locates oil. Even if they locate oil, there is a lot of uncertainty to what price they can sell the oil. In this case it is very hard to find the future cash flows, about the best one can do is to estimate the *expected* future cash flows. Alternatively one can consider *contingent* future cash flows.

We will for the rest of the chapter concentrate on the valuation of a sequence of *certain* future cash flows. The valuation of future *risky* cash flows is the topic of later chapters. We use the symbol X_t for the amount X to be paid at a future date t, and we want to value a set of future cash flows:

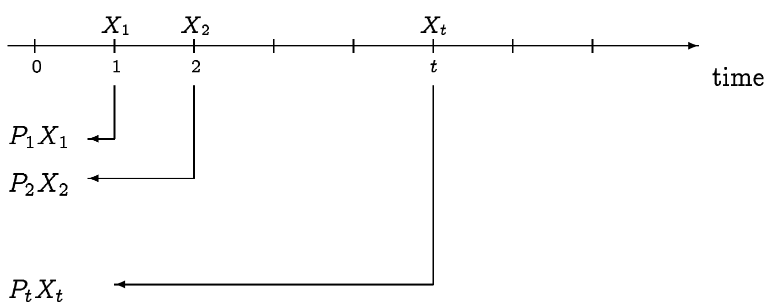


Given the set of dated cash flows, we define its Present Value (PV) as its value today. More precisely, it is the cost to obtain the same stream of cash flows in the market.

Evaluating the PV is simplified by using the axiom of value additivity, since we can then split the problem into summing the values of the individual dated cash flows. The problem is then reduced to finding the value today of a cash flow X_t at some future date t. To do this we use the set of *prices* P_t today of receiving one dollar at time t in the future.

The PV of the entire stream is then:

$$PV = \sum_{t=1}^{T} P_t X_t.$$



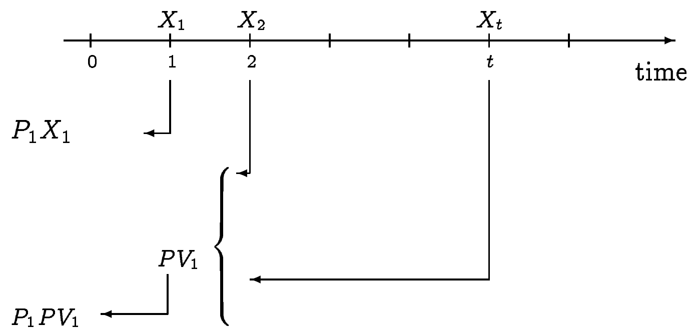
Example

A US Government Treasury bill pays USD 1000 263 days from now. Today's value of one dollar received 263 days from now is USD 0.945. The Present Value of the Treasury Bill is

$$PV = 0.945 \times 1000 = 945.$$

Value additivity buys you other nice things. For example, we can split the evaluation of the present value into several steps. If we let PV_0 be the PV of a stream $X_1, X_2, ...$ and PV_1 be its PV at time $t = 1$ *after* payment of X_1, then

$$PV_0 = \text{Present Value}(X_1, PV_1).$$



5.2 Pricing in Markets for Dated Riskfree Cash Flows

This set of *prices* P_t today of receiving one dollar at time t in the future are important objects in finance. They are typically *estimated* from actual prices in financial markets.

Since most people are impatient, and would put more value today on receiving a dollar tomorrow than one year from now, you would expect the following property to hold:

$$P_1 > P_2 > P_3 > \dots$$

If $P_1 \leq 1$ this can also be shown to be an implication of the no free lunch assumption, which is left as an exercise.

5.3 Interest Rates

Calculating present values is thus very simple as long as one know the prices $P_1, P_2, \dots$ However, it is a long standing convention in finance to use *interest rates* instead of such prices. For each price P_t there is a corresponding interest rate r_t. We will therefore need to spend some time on transformations involving interest rates.

Generally, the *rate of return* on an asset is

$$\text{Rate of return} = \frac{\text{Payments during period} + \text{Value at end of period}}{\text{Value at beginning of period}} - 1$$

Example

The simplest example of interest rates you find when you put money in the bank. The bank offers you a given interest on the balance of your account. If you put \$100 in your bank account, and one year later take out all the money in the account, which by then includes interest of \$8, the interest rate, or rate of return, on your bank account, is

$$\text{rate of return} = \text{interest rate} = \frac{100+8}{100} - 1 = 0.08 = 8\%$$

The relation between the prices P_t and interest rates are found as answers to the following question: *How much would you have to invest now at the (per period) interest rate r_t to get one dollar at time t?*

$$P_t(1+r_t)^t = 1$$

which implies

$$P_t = \frac{1}{(1+r_t)^t} \tag{5.1}$$

and

$$r_t = \sqrt[t]{\frac{1}{P_t}} - 1 = (P_t)^{-\frac{1}{t}} - 1. \tag{5.2}$$

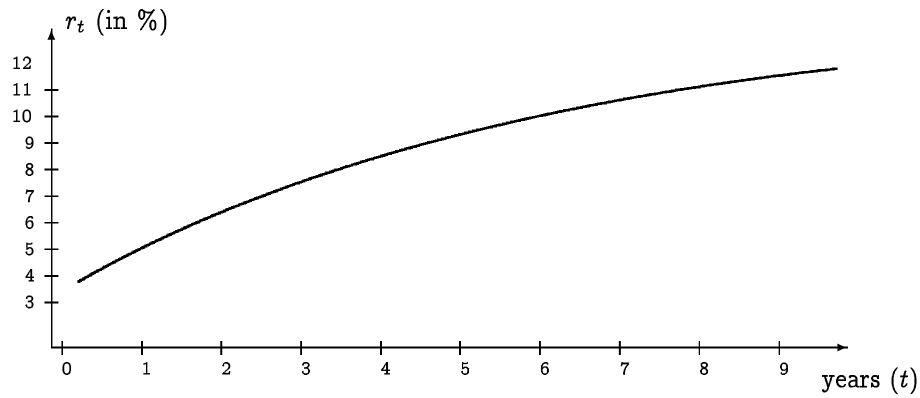
This type of interest is called discretely compounded interest. To complicate matters, there is another facet of interest rates, the frequency of compounding. We will return to this, for now we will use discrete compounding.

One thing to note about the expressions (5.1) and (5.2) for transforming between interest rates r_t and prices P_t is that they are one to one. If you know interest rates you also know prices, and vice versa. For most purposes, such as calculating present values, it is the prices that are of interest, not returns. (Ever seen a grocery store that quotes the prices of its apples as the $-t$'th root of its dollar price?) There are however also cases where interest rates may be more meaningful. The interest rate r_t denotes the percentage return on investing one dollar in a security that promises one dollar at time t. The return is *normalized* to percent per period, so returns on securities with different maturity (different t's) can be compared. Many people will also compare investment opportunities by calculating an implied *return* on the project. Such comparisons are however full of pitfalls, many mistakes continue to be made by people who only use interest rates to compare investment opportunities.

5.4 Term Structure of Interest Rates

The plot of spot interest rates (r_t) against maturity (t) is called the *term structure of interest rates*. The term structure can take a multitude of shapes. Typically, it is rising, but it can also be decreasing, or even "humped." Figure 5.1 shows an example term structure.

Figure 5.1 Example Term Structure of Interest Rates



The prices P_t (and, hence r_t) are usually estimated from prices of government fixed income securities, such as US Treasury bills and US Treasury bonds.

Example

A two-year Treasury bond with a face value of 1000 and an annual coupon payment of 8% sells for 982.50. A one–year T bill, with a face value of 100, and no coupons, sells for 90. Given these market prices, we can find P_1 and P_2 that gives the securities the correct prices:

$$\begin{bmatrix} 982.50 = P_1 80 + P_2 1080 \\ 90 = P_1 100 \end{bmatrix}$$

Solving these we find prices

$$\begin{bmatrix} P_1 = 0.9 \\ P_2 = 0.84 \end{bmatrix}$$

and interest rates

$$\begin{bmatrix} r_1 = 11\% \\ r_2 = 9\% \end{bmatrix}$$

5.5 Net Present Value

The Present Value (PV) discussed above is the cost to obtain a set of future cash flows in the market. The Net Present Value (NPV) of an investment project is the difference between the Present Value and how much it costs *you* to generate the same cash flows (with your project). If the Net Present Value of a project is positive, it is obviously a valuable project: you are creating value; you generate a stream of cash flows at a cost lower than the market.

Example

An investment project promises the following future cashflows

t	=	1	2	3
X_t	=	1100	1100	1100

Starting the project costs 3000 today. If the interest rates are $r_1 = r_2 = r_3 = 15\%$, is this a valuable project?

We calculate the NPV of the project by first finding the Present Value of the future cash flows as

$$PV = \frac{1100}{(1+0.15)^1} + \frac{1100}{(1+0.15)^2} + \frac{1100}{(1+0.15)^3} = 2511.6$$

Subtracting the cost today gives the NPV of the project:

$$NPV = 2511.6 - 3000 = -488.4$$

Since the $NPV < 0$, this is clearly a undesirable project.

5.6 Capital Budgeting

We use the term *Capital Budgeting* for the valuation and management of investment projects. This goes for any kind of investment project. The basic decision rule is to

Invest in any project with a positive Net Present Value

Positive NPV occurs when your cost to generate a stream of cash flows is less than the price that the market charges.

Example

Suppose you know how to generate a particular risky stream of cash flows by building new computer chips and selling them. The investment is cheap. You would have to pay a lot more to get the a stream of cash flows with the same properties (distribution over time and type of randomness) by combining equity, futures, bonds, etc.

Positive NPV reflects the presence of *economic rents*. (You own or know something that nobody else does.)

5.7 Perpetuities

Interest rates are useful for another reason, they can be used to simplify certain calculations. One example is a the calculation of a *perpetuity*. A perpetuity is a sequence of payments each period into indefinite future. Its value is calculated as

$$PV = \sum_{t=1}^{\infty} P_t X_t = \sum_{t=1}^{\infty} \frac{X_t}{(1+r_t)^t}$$

In the case where the future cash flows are the same each year ($X_t = X$) and the interest rate r_t is constant, the above formula simplifies to[1]

$$PV = \frac{X}{r}$$

Example

What is the present value of an annual payment of \$10 if the interest rate is 10%?

$$PV = \frac{10}{0.1} = \$100.$$

A *growing perpetuity* is a perpetuity that grows at a constant rate g. The cash flow in year t is

$$X_t = X_1(1+g)^{t-1}$$

Using this in the present value formula with constant interest rate r,

$$PV = \sum_{t=1}^{\infty} \frac{X_1(1+g)^{t-1}}{(1+r)^t}$$

we can show that this simplifies to

$$PV = \frac{X_1}{r-g}$$

[1] This is shown relatively simply:

$$PV = \sum_{t=1}^{\infty} \frac{X}{(1+r)^t} = X \sum_{t=1}^{\infty} \left(\frac{1}{1+r}\right)^t$$

$PV - PV\left(\frac{1}{1+r}\right)$ is the price of receiving X at $t = 1$ only, i.e.

$$PV - PV\left(\frac{1}{1+r}\right) = PV\left(1 - \frac{1}{1+r}\right) = X\left(\frac{1}{1+r}\right)$$

$$PV = \frac{X}{r}$$

Example

Your bank offers you the following set of future cash flows: You receive 10 next year. Each subsequent year your payment will be 5% larger than the previous year. The interest rate r is 10%. How much are you willing to pay the bank today for this set of cash flows?

$$PV = \sum_{t=1}^{\infty} \frac{10(1+0.05)^{(t-1)}}{(1+0.1)^t} = \frac{X_1}{r-g} = \frac{10}{0.1-0.05} = 20.$$

5.8 Annuities

An *annuity* is an asset that pays a fixed amount each year for a specified finite number of years. The present value of an annuity that last T periods is found as:[2]

$$PV = \sum_{t=1}^{T} \frac{X}{(1+r)^t} = X \left[\frac{1}{r} - \frac{1}{r} \frac{1}{(1+r)^T} \right]$$

The term in brackets is called an *annuity factor*. Such annuity factors are tabulated in a lot of places. Most financial calculators will also provide them.

Example

You have just won the Lotto and can choose between \$12 million immediately or \$1 million per year for the next 20 years, payments starting one year from now. You can currently earn 5% by investing in treasury securities.

The choice should be based on the alternative with the highest present value. The present value of the latter alternative is calculated as

$$\begin{aligned} PV &= 1 \text{ mill} \left[\frac{1}{r} - \frac{1}{r(1+r)^T} \right] \\ &= 1 \text{ mill} \left[\frac{1}{0.05} - \frac{1}{0.05(1+0.05)^{20}} \right] \\ &= 12.46 \text{ mill.} \end{aligned}$$

The annual payment of 1 million is preferred to getting the 12 million immediately.

[2] This is easily found as the difference between two perpetuities: one that starts one period from now, worth $\frac{X}{r}$, less the present value of one that starts $T+1$ periods from now, worth $\frac{X}{r}\frac{1}{(1+r)^T}$

5.9 Compound Interest

Compounding refers to the frequency with which interest is added to the principal. To calculate the future value at time t of compounding n times per period at a constant interest rate r we use the formula

$$FV_t = PV\left(1 + \frac{r}{n}\right)^{nt}$$

Example

If you invest $100 at a 10% annual interest rate, this will grow to

$$100(1 + 0.1)^{10} = 259.4$$

after 10 year with annual compounding, but it will grow to

$$100\left(1 + \frac{0.1}{360}\right)^{360\cdot 10} = 271.8$$

with daily compounding.

In the case of *continuous compounding* the above formula collapses to

$$FV_t = PV(e^{rt})$$

In continuous compounding, the return is computed as if you continuously received a dividend which you immediately reinvested. In discrete compounding, you receive a dividend only once (or a number of times n, depending on the case) every period. The continuous-time case is really the limit of discrete compounding, whereby the rate r is paid and reinvested faster and faster:

$$\lim_{n\to\infty}\left(\left(1 + \frac{r}{n}\right)^{n}\right)^{t} = e^{rt}.$$

Example

If you invest $100 at a 10% annual interest rate, with continuous compounding this will grow to $100e^{0.1\cdot 10} = 271.8$ after 10 years.

We can also find the present values for respectively discrete and continuous compounding as:

$$PV = FV_t\left(1 + \frac{r}{n}\right)^{-nt}$$

$$PV = FV_t(e^{-rt}),$$

If we are given *prices* P_t, we can find the corresponding interest rate with continuous compounding as follows

$$r_t = -\frac{\ln(P_t)}{t}$$

Example

If $P_1 = 0,9$, the corresponding interest rate with continuous compounding is

$$r_1 = -\frac{\ln(P_t)}{t} = -\ln 0.9 = 0.1053 = 10.53\%$$

With annual compounding the interest rate would have been

$$r_1 = \frac{1}{P_t} = \frac{1}{0.9} = 0.1111 = 11.11\%$$

5.10 Valuing Fixed Income Securities

A fixed income security is a security that offers a predetermined sequence of future payments. The typical fixed income security is a bond.

Example

A US Government Bond (T Bond) with maturity 10 years and stated interest 7% is a promise to pay interest of 3.5% of the principal twice a year for 10 years, and repay the principal after 10 years.

Valuing bonds should by now be straightforward. We need to find the present value of the promised sequence of payments, using either prices P_t or interest rates r_t.

Example

A bond promises the following sequence of payments:

t	=	1	2	3	4
Cashflow X_t	=	10	10	10	110

The interest rates r_t and prices P_t of future risk free cash flows are as follows

t	=	1	2	3	4
r_t	=	5.3%	5.4%	5.6%	5.7%
P_t	=	0.95	0.9	0.85	0.80

$$\text{Bond Price} = \sum_{t=1}^{4} P_t X_t = 0.95 \cdot 10 + 0.9 \cdot 10 + 0.85 \cdot 10 + 0.8 \cdot 110 = 115$$

5.11 Valuing Equities

An owner of one common stock owns a small part of the listed company the stock is issued by. Collectively, the equity owners are the residual claimants on the value of the firm, net of all the firm's liabilities. For one stock, however, this is not the relevant starting point for valuation. What counts is the cash flows accruing to the stock. For the company the cash flows are the dividends paid. The individual owner of a stock has an alternative source of cash flow, though: he can sell the stock to somebody else.

Example

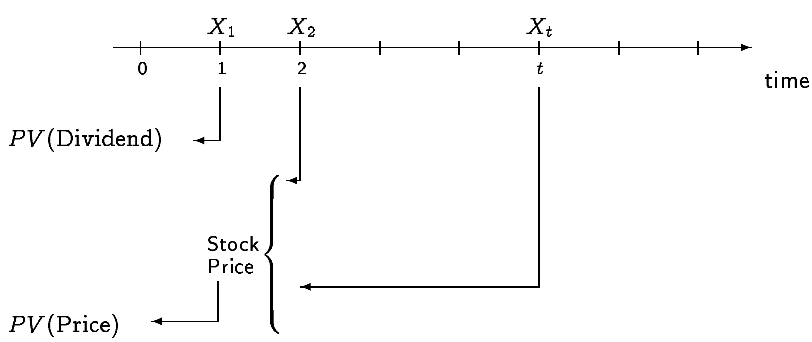
You currently own one stock in the XYZ company. XYZ will pay dividend one year from now of one dollar. You also know that the price of one XYZ share one year from now, just after the dividend payments, will be 100 for sure. The current one year interest rate is 10%. What is the current value of the stock?

This is just the present value:

$$PV = \frac{1}{1+r}(1+100) = 91.81$$

But why is it possible to sell the share one year from now? Clearly the buyer of the share must believe that the value of one XYZ share *at that time* is 100. The source of the value must be cash flow from the XYZ share at some point in the future.

We are in other words in the following situation:



The price at time 1 is a present value of all the future dividends.

To value a stock it is not necessary to estimate some future stock price. One can also concentrate on the cash flows from the company, namely the dividends. The price of a stock is the present value of all future dividends.

$$\text{Stock Price} = \sum_{t=1}^{\infty} P_t \text{Dividend}_t$$

But this just replaces one estimation problem with another. How to estimate all these future dividends? In general this is clearly impossible. However, if we assume that dividends will grow by a fixed percentage g each year, this is an example of a growing annuity, which we have just seen, and if we further use a fixed interest rate r we can calculate the price today as

$$\text{Stock Price} = \frac{\text{Dividend}_1}{r-g}$$

Example

The XYZ corporation will pay dividends of 1 next year. Dividends are expected to grow by 2% per year. The current interest rate is 10%. You estimate todays price of XYZ stock as

$$\text{Stock Price} = \frac{1}{0.1-0.02} = 12.5$$

5.12 Risky Cash flows

It's not obvious how you would "count" the number of units of dated cash flow if it is *risky*. Usually one unit equals one dollar of *expected* cash flow. That is, if X_t now denotes the cash flow itself (a random variable), and $E[X_t]$ its expected value, then one measures the risky cash flow in terms of number of expected dollars, and writes:

$$PV = \sum_{t=1}^{T} P_t E[X_t].$$

But it is far from clear that you would want to use the same prices P_t as when you were calculating the present value of future riskless cash flows. In fact, you will want to use different prices depending on the level of risk. We will be returning to this risky case.

References

Textbook References

Any basic text book on corporate finance, such as Brealey and Myers (2002) or Ross, Westerfield, and Jaffe (2005) covers this material in much more detail.

Problems

5.1 *Present Value* [3]

You are given the following prices P_t today for receiving risk free payments t periods from now.

t	=	1	2	3
P_t	=	0.95	0.9	0.85

1. Calculate the implied interest rates and graph the term structure of interest rates.

2. Calculate the present value of the following cash flows:

t	=	1	2	3
X_t	=	100	100	100

5.2 *Borrowing* [2]

BankTwo is offering personal loans at 10%, compounded quarterly. BankThree is offering personal loans at 10.5%, compounded annually. Which is the better offer?

5.3 *Arbitrage* [4]

You are given the following prices P_t today for receiving risk free payments t periods from now.

t	=	1	2	3
P_t	=	0.95	0.9	0.95

There are traded securities that offer $1 at any future date, available at these prices. How would you make a lot of money?

5.4 *Bank Loans* [2]

Your company is in need of financing of environmental investments. Three banks have offered loans. The first bank offers 4.5% interest, with biannual compounding. The second bank offers 4.3% interest, with monthly compounding. The third bank offers 4.25% with annual compounding.

Determine which is is the best offer.

5.5 *Stock* [4]

A stock has just paid a dividend of 10. Dividends are expected to grow with 10% a year for the next 2 years. After that the company is expecting a constant growth of 2% a year. The required return on the stock is 10%. Determine todays stock price.

5.6 *Bonds* [6]

You observe the following three bonds:

		Cashflow in period		
Bond	Price	1	2	3
A	95	100	0	0
B	90	10	110	0
C	85	10	10	110

1. What is the current value of receiving one dollar at time 3?

Consider now the bond D, with the following characteristics

	Cashflow in period		
Bond	1	2	3
D	20	20	520

2. What is the current price of bond D?

Consider next bond E, which last for four periods. Bond E has the following characteristics:

	Cashflow in period			
Bond	1	2	3	4
E	10	10	10	110

3. If the market does not allow any free lunches (arbitrage), what is the *maximal* price that bond E can have?

5.7 *Growing Perpetuity* [8]

The present value of a perpetuity that pays X_1 the first year and then grows at a rate g each year is:

$$PV = \sum_{t=1}^{\infty} \frac{X_1(1+g)^{t-1}}{(1+r)^t}$$

Show that this simplifies to

$$PV = \frac{X_1}{r-g}$$

5.8 *Annuity* [6]

Show that the present value of an annuity paying X per period for T years when the interest rate is r can be simplified as

$$PV = \sum_{t=1}^{T} \frac{X}{(1+r)^t} = X\left[\frac{1}{r} - \frac{1}{r}\frac{1}{(1+r)^T}\right]$$

5.9 *Stock* [2]

The current price for a stock is 50. The company is paying a dividend of 5 next period. Dividend is expected to grow by 5% annually. The relevant interest rate is 14%. In an efficient market, can these numbers be sustained?

5.10 *Growing Annuity* [6]

Consider an T-period annuity that pays X next period. After that, the payments grows at a rate of g per year for the next T years.

The present value of the annuity is

$$PV = \sum_{t=1}^{T} \frac{X(1+g)^{(t-1)}}{(1+r)^t}$$

Can you find a simplified expression for this present value?

5.11 *Jane* [3]

Jane, a freshman in college, needs 55000 in 4 years to start studying for an MBA. Her investments earn 5% interest per year.

1. How much must she invest today to have that amount at graduation?
2. If she invested once a year for four years beginning today until the end of the 4 years how much must she invest?

5.12 *Bonds* [3]

The current interest rate is 7%. Given the opportunity to invest in one of the three bonds listed below, which would you buy? Sell short?

Bond	Face value	Annual coupon rate	Maturity	Price
A	1000	4%	1 year	990
B	1000	7.5%	17 years	990
C	1000	8.5%	25 years	990