Chapter 1

Basic Operations

Let $\mathcal{F}$ be a field, for example the set of real numbers $\mathbf{R}$ or the set of complex numbers $\mathbf{C}$. Let m, n be two integers ≥ 1. An array A of numbers in $\mathcal{F}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} = (a_{ij})$$

is called an $m \times n$ *matrix* with entry a_{ij} in the ith row and jth column. A *row vector* is a $1 \times n$ matrix. A *column vector* is an $n \times 1$ matrix. We have a *zero matrix*, in which $a_{ij} = 0$ for all i, j.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. We define $A + B$ to be the $m \times n$ matrix whose entry in the i-th row and j-th column is $a_{ij} + b_{ij}$. Matrix multiplication is only defined between two matrices if the number of columns of the first matrix is the same as the number of rows of the second matrix. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the matrix product AB is an $m \times p$ matrix defined by

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

for each pair i and j, where $(AB)_{ij}$ denotes the (i, j)th entry in AB. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices with entries in some field. Then their Hadamard product is the entry-wise product of A and B, that is the $m \times n$ matrix $A \bullet B$ whose (i, j)th entry is $a_{ij} b_{ij}$.

Problem 1. Let $\mathbf{x}$ be a column vector in $\mathbf{R}^n$ and $\mathbf{x} \neq \mathbf{0}$. Let

$$A = \frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}$$

where ^{T} denotes the transpose, i.e. $\mathbf{x}^T$ is a row vector. Calculate A^2.

Solution 1. Obviously $\mathbf{x}\mathbf{x}^T$ is a nonzero $n \times n$ matrix and $\mathbf{x}^T\mathbf{x}$ is a nonzero real number. We find

$$\begin{aligned} A^2 &= \left(\frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}\right)^2 \\ &= \frac{(\mathbf{x}\mathbf{x}^T)(\mathbf{x}\mathbf{x}^T)}{(\mathbf{x}^T\mathbf{x})^2} \\ &= \frac{\mathbf{x}(\mathbf{x}^T\mathbf{x})\mathbf{x}^T}{(\mathbf{x}^T\mathbf{x})^2} \\ &= \frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}} \\ &= A \end{aligned}$$

where we used the fact that matrix multiplication is *associative*. Thus $A^2 = A$.

Problem 2. Consider the vector in $\mathbf{R}^8$

$$\mathbf{x}^T = (20.0\ 6.0\ 4.0\ 2.0\ 10.0\ 6.0\ 8.0\ 4.0)$$

where ^{T} denotes transpose. Consider the matrices

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$G_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$H_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad G_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$H_3 = \frac{1}{2}(1 \quad 1), \qquad G_3 = \frac{1}{2}(1 \quad -1)\,.$$

(i) Calculate

$$H_1\mathbf{x}, \qquad G_1\mathbf{x}$$

$$H_2H_1\mathbf{x}, \quad G_2H_1\mathbf{x}, \quad H_2G_1\mathbf{x}, \quad G_2G_1\mathbf{x}$$

$$H_3H_2H_1\mathbf{x}, \quad G_3H_2H_1\mathbf{x}, \quad H_3G_2H_1\mathbf{x}, \quad G_3G_2H_1\mathbf{x},$$

$$H_3H_2G_1\mathbf{x}, \quad G_3H_2G_1\mathbf{x}, \quad H_3G_2G_1\mathbf{x}, \quad G_3G_2G_1\mathbf{x}$$

(ii) Calculate

$$H_jH_j^T, \quad G_jG_j^T, \quad H_jG_j^T$$

for $j = 1, 2, 3$.

(iii) How can we reconstruct the original vector $\mathbf{x}$ from the vector

$$(H_3H_2H_1\mathbf{x}, G_3H_2H_1\mathbf{x}, H_3G_2H_1\mathbf{x}, G_3G_2H_1\mathbf{x}, H_3H_2G_1\mathbf{x}, G_3H_2G_1\mathbf{x}, H_3G_2G_1\mathbf{x}, G_3G_2G_1\mathbf{x})$$

The problem plays a role in *wavelet theory.*

Solution 2. (i) We find

$$H_1\mathbf{x} = \begin{pmatrix} 13.0 \\ 3.0 \\ 8.0 \\ 6.0 \end{pmatrix}, \qquad G_1\mathbf{x} = \begin{pmatrix} 7.0 \\ 1.0 \\ 2.0 \\ 2.0 \end{pmatrix}.$$

Thus we have the vector

$$(13.0 \;\, 3.0 \;\, 8.0 \;\, 6.0 \;\, 7.0 \;\, 1.0 \;\, 2.0 \;\, 2.0)^T.$$

Next we find

$$H_2H_1\mathbf{x} = \begin{pmatrix} 8.0 \\ 7.0 \end{pmatrix}, \quad G_2H_1\mathbf{x} = \begin{pmatrix} 5.0 \\ 1.0 \end{pmatrix},$$

$$H_2G_1\mathbf{x} = \begin{pmatrix} 4.0 \\ 2.0 \end{pmatrix}, \quad G_2G_1\mathbf{x} = \begin{pmatrix} 3.0 \\ 0.0 \end{pmatrix}.$$

Thus we have the vector

$$(8.0 \;\, 7.0 \;\, 5.0 \;\, 1.0 \;\, 4.0 \;\, 2.0 \;\, 3.0 \;\, 0.0)^T.$$

Finally we have

$$H_3H_2H_1\mathbf{x} = 7.5, \quad G_3H_2H_1\mathbf{x} = 0.5, \quad H_3G_2H_1\mathbf{x} = 3.0, \quad G_3G_2H_1\mathbf{x} = 2.0$$

$$H_3H_2G_1\mathbf{x} = 3.0, \quad G_3H_2G_1\mathbf{x} = 1.0, \quad H_3G_2G_1\mathbf{x} = 1.5, \quad G_3G_2G_1\mathbf{x} = 1.5.$$

Thus we obtain the vector

$$(7.5 \;\, 0.5 \;\, 3.0 \;\, 2.0 \;\, 3.0 \;\, 1.0 \;\, 1.5 \;\, 1.5).$$

(ii) We find

$$H_1 H_1^T = \frac{1}{2} I_4, \quad G_1 G_1^T = \frac{1}{2} I_4, \quad H_1 G_1^T = 0_4$$

$$H_2 H_2^T = \frac{1}{2} I_2, \quad G_2 G_2^T = \frac{1}{2} I_2, \quad H_2 G_2^T = 0_2$$

$$H_3 H_3^T = \frac{1}{2}, \quad G_3 G_3^T = \frac{1}{2}, \quad H_3 G_3^T = 0$$

where 0_4 is the 4×4 zero matrix and 0_2 is the 2×2 zero matrix.
(iii) Let

$$\mathbf{w} = (7.5 \ \ 0.5 \ \ 3.0 \ \ 2.0 \ \ 3.0 \ \ 1.0 \ \ 1.5 \ \ 1.5)^T$$

Consider the matrices

$$X_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Then

$$X_1 \mathbf{w} = \begin{pmatrix} 8 \\ 7 \\ 5 \\ 1 \end{pmatrix}, \qquad Y_1 \mathbf{w} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 0 \end{pmatrix}.$$

Now let

$$X_2 = Y_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Then

$$X_2(X_1 \mathbf{w}) = \begin{pmatrix} 13 \\ 3 \\ 8 \\ 6 \end{pmatrix}, \qquad Y_2(Y_1 \mathbf{w}) = \begin{pmatrix} 7 \\ 1 \\ 2 \\ 2 \end{pmatrix}.$$

Thus the original vector $\mathbf{x}$ is reconstructed from

$$\begin{pmatrix} X_2(X_1 \mathbf{w}) \\ Y_2(Y_1 \mathbf{w}) \end{pmatrix}.$$

The odd entries come from $X_2(X_1\mathbf{w}) + Y_2(Y_1\mathbf{w})$ and the even ones from $X_2(X_1\mathbf{w}) - Y_2(Y_1\mathbf{w})$.

Problem 3. Consider the 8×8 *Hadamard matrix*

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

(i) Do the 8 column vectors in the matrix H form a basis in $\mathbf{R}^8$? Prove or disprove.
(ii) Calculate HH^T, where ^{T} denotes transpose. Compare the results from (i) and (ii) and discuss.

Solution 3. (i) Calculating the scalar product of the column vectors (and obviously also the row vectors) we find that they are pairwise orthogonal to each other. Since they all nonzero vectors we have a basis in $\mathbf{R}^8$, which is not normalized.
(ii) We find

$$HH^T = 8I_8 .$$

Thus the matrix H is invertible and the column or row vectors must form a basis. Since the right-hand side is $8I_8$ the basis must be orthogonal.

Problem 4. Show that any 2×2 complex matrix has a unique representation of the form

$$a_0 I_2 + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3$$

for some $a_0, a_1, a_2, a_3 \in \mathbf{C}$, where I_2 is the 2×2 identity matrix and σ_1, σ_2, σ_3 are the *Pauli spin matrices*

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solution 4. Since

$$a_0 I_2 + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix}$$

we obtain

$$a_0 I_2 + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta \in \mathbf{C}$. Thus

$$a_0 = \frac{\alpha + \delta}{2}, \quad a_1 = \frac{\beta + \gamma}{2i}, \quad a_2 = \frac{\beta - \gamma}{2}, \quad a_3 = \frac{\alpha - \delta}{2i}.$$

Problem 5. Let A, B be $n \times n$ matrices such that $ABAB = 0_n$. Can we conclude that $BABA = 0_n$?

Solution 5. There are no 1×1 or 2×2 counterexamples. For 2×2 matrices $ABAB = 0_n$ implies $B(ABAB)A = 0_n$. Therefore the matrix BA is *nilpotent*, i.e. $(BA)^3 = 0_n$. If a 2×2 matrix C is nilpotent, its characteristic polynomial is λ^2 and therefore $C^2 = 0_n$ by the *Cayley-Hamilton theorem*. Thus $BABA = 0_n$. For $n = 3$ we find the counterexample

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$BABA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 6. A square matrix A over $\mathbf{C}$ is called *skew-hermitian* if $A = -A^*$. Show that such a matrix is *normal*, i.e., we have $AA^* = A^*A$.

Solution 6. We have

$$AA^* = -A^*A^* = (-A^*)(-A) = A^*A.$$

Problem 7. Let A be an $n \times n$ skew-hermitian matrix over $\mathbf{C}$, i.e. $A^* = -A$. Let U be an $n \times n$ *unitary matrix*, i.e., $U^* = U^{-1}$. Show that $B := U^*AU$ is a skew-hermitian matrix.

Solution 7. We have $A^* = -A$. Thus from $B = U^*AU$ and $U^{**} = U$ it follows that

$$B^* = (U^*AU)^* = U^*A^*U = U^*(-A)U = -U^*AU = -B.$$

Problem 8. Let A, X, Y be $n \times n$ matrices. Assume that

$$XA = I_n, \qquad AY = I_n$$

where I_n is the $n \times n$ unit matrix. Show that $X = Y$.

Solution 8. We have

$$X = XI_n = X(AY) = (XA)Y = I_nY = Y.$$

Problem 9. Let A, B be $n \times n$ matrices. Assume that A is nonsingular, i.e. A^{-1} exists. Show that if $BA = 0_n$, then $B = 0_n$.

Solution 9. We have

$$B = BI_n = B(AA^{-1}) = (BA)A^{-1} = 0_n A^{-1} = 0_n \,.$$

Problem 10. Let A, B be $n \times n$ matrices and

$$A + B = I_n, \qquad AB = 0_n \,.$$

Show that $A^2 = A$ and $B^2 = B$.

Solution 10. Multiplying $A + B = I_n$ with A we obtain $A^2 + AB = A$ and therefore $A^2 = A$. Multiplying $A+B = I_n$ with B yields $AB + B^2 = B$ and therefore $B^2 = B$.

Problem 11. Consider the normalized vectors in $\mathbf{R}^2$

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}.$$

Find the condition on θ_1 and θ_2 such that

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$$

is normalized. A vector $\mathbf{x} \in \mathbf{R}^n$ is called *normalized* if $\|\mathbf{x}\| = 1$, where $\| \; \|$ denotes the Euclidean norm.

Solution 11. From the condition that the vector

$$\begin{pmatrix} \cos\theta_1 + \cos\theta_2 \\ \sin\theta_1 + \sin\theta_2 \end{pmatrix}$$

is normalized it follows that

$$(\sin\theta_1 + \sin\theta_2)^2 + (\cos\theta_1 + \cos\theta_2)^2 = 1 \,.$$

Thus we have

$$\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 = -\frac{1}{2} \,.$$

It follows that

$$\cos(\theta_1 - \theta_2) = -\frac{1}{2} \,.$$

Therefore, $\theta_1 - \theta_2 = 2\pi/3$ or $\theta_1 - \theta_2 = 4\pi/3$.

Problem 12. Let

$$A := \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T \tag{1}$$

where

$$\mathbf{x} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

and $\theta \in \mathbf{R}$. Find the matrix A.

Solution 12. We find

$$A = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The vectors $\mathbf{x}$ and $\mathbf{y}$ form a basis in $\mathbf{R}^2$. Equation (1) is called the *completeness relation.*

Problem 13. Find a 2×2 matrix A over $\mathbf{R}$ such that

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution 13. We find

$$a_{11} = a_{12} = a_{21} = \frac{1}{\sqrt{2}}, \qquad a_{22} = -\frac{1}{\sqrt{2}}.$$

Thus

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This matrix is a Hadamard matrix.

Problem 14. Consider the 2×2 matrix over the complex numbers

$$\Pi(\mathbf{n}) := \frac{1}{2}\left(I_2 + \sum_{j=1}^{3} n_j\sigma_j\right)$$

where $\mathbf{n} := (n_1, n_2, n_3)$ ($n_j \in \mathbf{R}$) is a unit vector, i.e., $n_1^2 + n_2^2 + n_3^2 = 1$. Here σ_1, σ_2, σ_3 are the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and I_2 is the 2×2 unit matrix.

(i) Describe the property of $\Pi(\mathbf{n})$, i.e., find $\Pi^*(\mathbf{n})$, $\mathrm{tr}(\Pi(\mathbf{n}))$ and $\Pi^2(\mathbf{n})$, where tr denotes the trace. The trace is the sum of the diagonal elements of a square matrix.
(ii) Find the vector

$$\Pi(\mathbf{n})\begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}.$$

Discuss.

Solution 14. (i) For the Pauli matrices we have $\sigma_1^* = \sigma_1$, $\sigma_2^* = \sigma_2$, $\sigma_3^* = \sigma_3$. Thus $\Pi(\mathbf{n}) = \Pi^*(\mathbf{n})$. Since

$$\mathrm{tr}\sigma_1 = \mathrm{tr}\sigma_2 = \mathrm{tr}\sigma_3 = 0$$

and the trace operation is linear, we obtain $\mathrm{tr}(\Pi(\mathbf{n})) = 1$. Since

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2$$

and

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0_2, \quad \sigma_2\sigma_3 + \sigma_3\sigma_2 = 0_2, \quad \sigma_3\sigma_1 + \sigma_1\sigma_3 = 0_2$$

the expression

$$\Pi^2(\mathbf{n}) = \frac{1}{4}\left(I_2 + \sum_{j=1}^{3} n_j\sigma_j\right)^2 = \frac{1}{4}I_2 + \frac{1}{2}\sum_{j=1}^{3} n_j\sigma_j + \frac{1}{4}\sum_{j=1}^{3}\sum_{k=1}^{3} n_j n_k \sigma_j\sigma_k$$

simplifies to

$$\Pi^2(\mathbf{n}) = \frac{1}{4}I_2 + \frac{1}{2}\sum_{j=1}^{3} n_j\sigma_j + \frac{1}{4}\sum_{j=1}^{3} n_j^2 I_2\,.$$

Using $n_1^2 + n_2^2 + n_3^2 = 1$ we obtain $\Pi^2(\mathbf{n}) = \Pi(\mathbf{n})$.
(ii) We find

$$\Pi(\mathbf{n})\begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+n_3)e^{i\phi}\cos\theta + (n_1 - in_2)\sin\theta \\ (n_1 + in_2)e^{i\phi}\cos\theta + (1-n_3)\sin\theta \end{pmatrix}.$$

Problem 15. Let

$$\mathbf{x} = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}$$

where $\phi, \theta \in \mathbf{R}$.
(i) Find the matrix $\rho := \mathbf{x}\mathbf{x}^*$.

(ii) Find $\mathrm{tr}\rho$.
(iii) Find ρ^2.

Solution 15. (i) Since $\mathbf{x}^* = (e^{-i\phi}\cos\theta\ ,\ \sin\theta)$ we obtain the 2×2 matrix

$$\rho = \mathbf{x}\mathbf{x}^* = \begin{pmatrix} \cos^2\theta & e^{i\phi}\sin\theta\cos\theta \\ e^{-i\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}.$$

(ii) Since $\cos^2\theta + \sin^2\theta = 1$ we obtain from (i) that $\mathrm{tr}\rho = 1$.
(iii) We have

$$\rho^2 = (\mathbf{x}\mathbf{x}^*)(\mathbf{x}\mathbf{x}^*) = \mathbf{x}(\mathbf{x}^*\mathbf{x})\mathbf{x}^* = \mathbf{x}\mathbf{x}^* = \rho$$

since $\mathbf{x}^*\mathbf{x} = 1$.

Problem 16. Consider the vector space $\mathbf{R}^4$. Find all pairwise orthogonal vectors (column vectors) $\mathbf{x}_1, \ldots, \mathbf{x}_p$, where the entries of the column vectors can only be $+1$ or -1. Calculate the matrix

$$\sum_{j=1}^{p} \mathbf{x}_j \mathbf{x}_j^T$$

and find the eigenvalues and eigenvectors of this matrix.

Solution 16. The number of vectors p cannot exceed 4 since that would imply $\dim(\mathbf{R}^4) > 4$. A solution is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Thus

$$\sum_{j=1}^{4} \mathbf{x}_j \mathbf{x}_j^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$
$$+ \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

The eigenvalue is 4 with multiplicity 4. The eigenvectors are all $\mathbf{x} \in \mathbf{R}^4$ with $\mathbf{x} \neq \mathbf{0}$. Another solution is given by

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 17. Let

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{pmatrix}.$$

(i) Let X be an $m \times n$ matrix. The *column rank* of X is the maximum number of linearly independent columns. The *row rank* is the maximum number of linearly independent rows. The row rank and the column rank of X are equal (called the *rank* of X). Find the rank of A and denote it by k.
(ii) Locate a $k \times k$ submatrix of A having rank k.
(iii) Find 3×3 permutation matrices P and Q such that in the matrix PAQ the submatrix from (ii) is in the upper left portion of A.

Solution 17. (i) The vectors in the first two columns are linearly dependent. Thus the rank of A is 2.
(ii) A 2×2 submatrix having rank 2 is

$$B = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}.$$

(iii) Let

$$P = Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$PAQ = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 6 & -2 \\ 2 & -2 & 2 \end{pmatrix}.$$

Problem 18. Find 2×2 matrices A, B such that $AB = 0_n$ and $BA \neq 0_n$.

Solution 18. An example is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Problem 19. Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. Then the *direct sum* of A and B, denoted by $A \oplus B$, is the $(m+p) \times (n+q)$ matrix defined by

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let A_1, A_2 be $m \times m$ matrices and B_1, B_2 be $n \times n$ matrices. Calculate

$$(A_1 \oplus B_1)(A_2 \oplus B_2).$$

Solution 19. We find

$$(A_1 \oplus B_1)(A_2 \oplus B_2) = (A_1 A_2) \oplus (B_1 B_2).$$

Problem 20. Let A be an $n \times n$ matrix over $\mathbf{R}$. Find all matrices that satisfy the equation

$$A^T A = 0_n.$$

Solution 20. From $A^T A = 0_n$ we find

$$\sum_{i=1}^{n} a_{ij} a_{ij} = 0$$

where $j = 1, 2, \ldots, n$. Thus A must be the zero matrix.

Problem 21. Let π be a permutation on $\{1, 2, \ldots, n\}$. The matrix P_π for which $p_{i*} = e_{\pi(i)*}$ is called the *permutation matrix* associated with π, where p_{i*} is the ith row of P_π and $e_{ij} = 1$ if $i = j$ and 0 otherwise. Let $\pi = (3\ 2\ 4\ 1)$. Find P_π.

Solution 21. P_π is obtained from the $n \times n$ identity matrix I_n by applying π to the rows of I_n. We find

$$P_\pi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} p_{1*} &= e_{3*} = e_{\pi(1)*} \\ p_{2*} &= e_{2*} = e_{\pi(2)*} \\ p_{3*} &= e_{4*} = e_{\pi(3)*} \\ p_{4*} &= e_{1*} = e_{\pi(4)*}. \end{aligned}$$

Problem 22. A matrix A for which $A^p = 0_n$, where p is a positive integer, is called *nilpotent*. If p is the least positive integer for which $A^p = 0_n$ then A is said to be nilpotent of index p. Find all 2×2 matrices over the real numbers which are nilpotent with $p = 2$, i.e. $A^2 = 0_2$.

Solution 22. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then from $A^2 = 0_2$ we obtain the four equations

$$a_{11}^2 + a_{12}a_{21} = 0, \quad a_{12}(a_{11}+a_{22}) = 0, \quad a_{21}(a_{11}+a_{22}) = 0, \quad a_{12}a_{21} + a_{22}^2 = 0.$$

Thus we have to consider the cases $a_{11} + a_{22} \neq 0$ and $a_{11} + a_{22} = 0$. If $a_{11} + a_{22} \neq 0$, then $a_{12} = a_{21} = 0$ and therefore $a_{11} = a_{22} = 0$. Thus we have the 2×2 zero matrix. If $a_{11} + a_{22} = 0$ we have $a_{11} = -a_{22}$ and $a_{11} \neq 0$, otherwise we would find the zero matrix again. Thus $a_{12}a_{21} = -a_{11}^2 = -a_{22}^2$ and for this case we find the solution

$$A = \begin{pmatrix} a_{11} & a_{12} \\ -a_{11}^2/a_{12} & -a_{11} \end{pmatrix}.$$

Problem 23. A square matrix is called *idempotent* if $A^2 = A$. Find all 2×2 matrices over the real numbers which are idempotent and $a_{ij} \neq 0$ for $i, j = 1, 2$.

Solution 23. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then from $A^2 = A$ we obtain

$$a_{11}^2 + a_{12}a_{21} = a_{11}, \quad a_{12}(a_{11} + a_{22}) = a_{12},$$

$$a_{21}(a_{11} + a_{22}) = a_{21}, \quad a_{12}a_{21} + a_{22}^2 = a_{22}.$$

Since $a_{ij} \neq 0$ we obtain $a_{11} + a_{22} = 1$ and

$$a_{21} = \frac{1}{a_{12}}(a_{11} - a_{11}^2).$$

Thus the matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ (a_{11} - a_{11}^2)/a_{12} & 1 - a_{11} \end{pmatrix}$$

with a_{11} and a_{12} arbitrary and nonzero.

Problem 24. A square matrix A such that $A^2 = I_n$ is called *involutory*. Find all 2×2 matrices over the real numbers which are involutory. Assume that $a_{ij} \neq 0$ for $i, j = 1, 2$.

Solution 24. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then from $A^2 = I_2$ we obtain

$$a_{11}^2 + a_{12}a_{21} = 1, \quad a_{12}(a_{11}+a_{22}) = 0, \quad a_{21}(a_{11}+a_{22}) = 0, \quad a_{12}a_{21} + a_{22}^2 = 1.$$

Since $a_{ij} \neq 0$ we have $a_{11} + a_{22} = 0$ and

$$a_{21} = (1 - a_{11}^2)/a_{12}.$$

Then the matrix is given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ (1 - a_{11}^2)/a_{12} & -a_{11} \end{pmatrix}.$$

Problem 25. Show that an $n \times n$ matrix A is involutary if and only if $(I_n - A)(I_n + A) = 0_n$.

Solution 25. Suppose that $(I_n - A)(I_n + A) = 0_n$. Then

$$(I_n - A)(I_n + A) = I_n - A^2 = 0_n.$$

Thus $A^2 = I_n$ and A is involutary. Suppose that A is involutary. Then $A^2 = I_n$ and

$$0_n = I_n - A^2 = (I_n - A)(I_n + A).$$

Problem 26. Let A be an $n \times n$ symmetric matrix over $\mathbf{R}$. Let P be an arbitrary $n \times n$ matrix over $\mathbf{R}$. Show that $P^T A P$ is symmetric.

Solution 26. Using that $A^T = A$ and $(P^T)^T = P$ we have

$$(P^T A P)^T = P^T A^T (P^T)^T = P^T A P.$$

Thus $P^T A P$ is symmetric.

Problem 27. Let A be an $n \times n$ skew-symmetric matrix over $\mathbf{R}$, i.e. $A^T = -A$. Let P be an arbitrary $n \times n$ matrix over $\mathbf{R}$. Show that $P^T A P$ is skew-symmetric.

Solution 27. Using $A^T = -A$ and $(P^T)^T = P$ we have

$$(P^TAP)^T = P^TA^T(P^T)^T = -P^TAP.$$

Thus P^TAP is skew-symmetric.

Problem 28. Let A be an $m \times n$ matrix. The *column rank* of A is the maximum number of linearly independent columns. The *row rank* is the maximum number of linearly independent rows. The row rank and the column rank of A are equal (called the rank of A). Find the rank of the 4×4 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Solution 28. The *elementary transformations* do not change the rank of a matrix. We subtract the third column from the fourth column, the second column from the third column and the first column from the second column, i.e.

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 5 & 6 & 7 & 1 \\ 9 & 10 & 11 & 1 \\ 13 & 14 & 15 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 5 & 6 & 1 & 1 \\ 9 & 10 & 1 & 1 \\ 13 & 14 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 \\ 9 & 1 & 1 & 1 \\ 13 & 1 & 1 & 1 \end{pmatrix}.$$

From the last matrix we see (three columns are the same) that the rank of A is 2. It follows that two eigenvalues must be 0.

Problem 29. Let A be an invertible $n \times n$ matrix over $\mathbf{C}$ and B be an $n \times n$ matrix over $\mathbf{C}$. We define the $n \times n$ matrix

$$D := A^{-1}BA.$$

Calculate D^n, where $n = 2, 3, \ldots$.

Solution 29. Since $AA^{-1} = I_n$ we obtain

$$D^n = A^{-1}B^nA.$$

Problem 30. A *Cartan matrix* A is a square matrix whose elements a_{ij} satisfy the following conditions:
1. a_{ij} is an integer, one of $\{-3, -2, -1, 0, 2\}$
2. $a_{jj} = 2$ for all diagonal elements of A

3. $a_{ij} \leq 0$ off of the diagonal
4. $a_{ij} = 0$ iff $a_{ji} = 0$
5. There exists an invertible diagonal matrix D such that DAD^{-1} gives a symmetric and positive definite quadratic form.

Give a 2×2 non-diagonal Cartan matrix.

Solution 30. An example is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The first four conditions are obvious for the matrix A. The last condition can be seen from

$$\mathbf{x}^T A\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2(x_1^2 - x_1x_2 + x_2^2) \geq 0$$

for $\mathbf{x} \neq \mathbf{0}$. That the symmetric matrix A is positive definite can also be seen from the eigenvalues of A which are 3 and 1.

Problem 31. Let A, B, C, D be $n \times n$ matrices over $\mathbf{R}$. Assume that AB^T and CD^T are symmetric and $AD^T - BC^T = I_n$, where ^{T} denotes transpose. Show that

$$A^TD - C^TB = I_n.$$

Solution 31. From the assumptions we have

$$AB^T = (AB^T)^T = BA^T$$
$$CD^T = (CD^T)^T = DC^T$$
$$AD^T - BC^T = I_n.$$

Taking the transpose of the third equation we have

$$DA^T - CB^T = I_n.$$

These four equations can be written in the form of block matrices in the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} = \begin{pmatrix} I_n & 0_n \\ 0_n & I_n \end{pmatrix}.$$

Thus the matrices are $(2n) \times (2n)$ marices. If X, Y are $m \times m$ matrices with $XY = I_m$, the identity matrix, then $Y = X^{-1}$ and $YX = I_m$ too. Applying this to our matrix equation with $m = 2n$ we obtain

$$\begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_n & 0_n \\ 0_n & I_n \end{pmatrix}.$$

Equating the lower right blocks shows that $-C^T B + A^T D = I_n$.

Problem 32. Let n be a positive integer. Let A_n be the $(2n+1)\times(2n+1)$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Give A_1 and A_2. Find the rank of A_n.

Solution 32. We have

$$A_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

We use induction on n to prove that $\text{rank}(A_n) = 2n$. The rank of A_1 is 2 since the first vector in the matrix A_1 is linear combination of the second and third vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

and the second and third vectors are linearly independent. Suppose $n \geq 2$ and that $\text{rank}(A_{n-1}) = 2(n-1)$ is known. Adding multiples of the first two rows of A_n to the other rows transforms A_n to a matrix of the form

$$\begin{pmatrix} 0 & -1 & \\ 1 & 0 & * \\ \mathbf{0} & & -A_{n-1} \end{pmatrix}$$

in which $\mathbf{0}$ and $*$ represent blocks of size $(2n-1)\times 2$ and $2\times(2n-1)$, repectively. Thus $\text{rank}(A_n) = 2 + \text{rank}(A_{n-1}) = 2 + 2(n-1) = 2n$.

Problem 33. A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is called a *probability vector* if the components are nonnegative and their sum is 1. Is the vector

$$\mathbf{u} = (1/2,\ 0,\ 1/4,\ 1/4)$$

a probability vector? Can the vector

$$\mathbf{v} = (2,\ 3,\ 5,\ 1,\ 0)$$

be "normalized" so that we obtain a probability vector?

Solution 33. Since all the components are nonnegative and the sum of entries is 1 we find that $\mathbf{u}$ is a probability vector. All the entries in $\mathbf{v}$ are nonnegative but the sum is 11. Thus we can construct the probability vector

$$\widetilde{\mathbf{v}} = \frac{1}{11}(2, 3, 5, 1, 0).$$

Problem 34. An $n \times n$ matrix $P = (p_{ij})$ is called a *stochastic matrix* if each of its rows is a probability vector, i.e., if each entry of P is nonnegative and the sum of the entries in each row is 1. Let A and B be two stochastic $n \times n$ matrices. Is the matrix product AB also a stochastic matrix?

Solution 34. From matrix multiplication we have for the (ij) entry of the product

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

It follows that

$$\sum_{j=1}^{n} (AB)_{ij} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} a_{ik} \sum_{j=1}^{n} b_{kj} = 1$$

since

$$\sum_{k=1}^{n} a_{ik} = 1, \qquad \sum_{j=1}^{n} b_{kj} = 1.$$

Problem 35. The *numerical range*, also known as the *field of values*, of an $n \times n$ matrix A over the complex numbers, is defined as

$$F(A) := \{ \mathbf{z}^* A \mathbf{z} \, : \, \|\mathbf{z}\| = 1, \ \mathbf{z} \in \mathbf{C}^n \}.$$

Find the numerical range for the 2×2 matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Find the numerical range for the 2×2 matrix

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

The *Toeplitz-Hausdorff convexity theorem* tells us that the numerical range of a square matrix is a convex compact subset of the complex plane.

Solution 35. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and

$$\mathbf{z} = \begin{pmatrix} e^{i\phi}\cos\theta \\ e^{i\chi}\sin\theta \end{pmatrix}, \qquad \phi, \chi, \theta \in \mathbf{R}.$$

Therefore $\mathbf{z}$ is an arbitrary complex number of length 1, i.e., $\|\mathbf{z}\| = 1$. Then

$$\begin{aligned} \mathbf{z}^* A \mathbf{z} &= (\, e^{-i\phi}\cos\theta \quad e^{-i\chi}\sin\theta\,) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e^{i\phi}\cos\theta \\ e^{i\chi}\sin\theta \end{pmatrix} \\ &= a_{11}\cos^2\theta + (a_{12}e^{i(\chi-\phi)} + a_{21}e^{i(\phi-\chi)})\sin\theta\cos\theta + a_{22}\sin^2\theta\,. \end{aligned}$$

Thus for the matrix B we have $\mathbf{z}^T B\mathbf{z} = \cos^2\theta$, where $\cos^2\theta \in [0,1]$ for all $\theta \in \mathbf{R}$. For the matrix C we have

$$\mathbf{z}^* C\mathbf{z} = e^{i(\phi-\chi)}\sin\theta\cos\theta + \sin^2\theta\,.$$

Thus the numerical range $F(C)$ is the closed elliptical disc in the complex plane with foci at $(0,0)$ and $(1,0)$, minor axis 1, and major axis $\sqrt{2}$.

Problem 36. Let A be an $n\times n$ matrix over $\mathbf{C}$. The *field of values* of A is defined as the set

$$F(A) := \{\, \mathbf{z}^* A\mathbf{z} \,:\, \mathbf{z} \in \mathbf{C}^n\,,\ \mathbf{z}^*\mathbf{z} = 1\,\}\,.$$

Let $\alpha \in \mathbf{R}$ and

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha \end{pmatrix}$$

(i) Show that the set $F(A)$ lies on the real axis.
(ii) Show that

$$|\mathbf{z}^* A\mathbf{z}| \le \alpha + 16\,.$$

Solution 36. (i) Since

$$\mathbf{z}^* = (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}, \overline{z_5}, \overline{z_6}, \overline{z_7}, \overline{z_8}, \overline{z_9})$$

with $\mathbf{z}^*\mathbf{z} = 1$ and applying matrix multiplication we obtain

$$\mathbf{z}^* A\mathbf{z} = \alpha + \sum_{j=1}^{8} (\overline{z_j} z_{j+1} + \overline{z_{j+1}} z_j) \, .$$

Let $z_j = r_j e^{i\theta_j}$. Then $\overline{z_j} = r_j e^{-i\theta_j}$. Let $j \neq k$. Then

$$\overline{z_j} z_k + z_j \overline{z_k} = 2 r_j r_k \cos(\theta_j - \theta_k)$$

with $0 \leq r_j \leq 1$. Thus we have

$$\mathbf{z}^* A\mathbf{z} = \alpha + 2 \sum_{j=1}^{8} r_j r_{j+1} \cos(\theta_{j+1} - \theta_j) \, .$$

Thus the set $F(A)$ lies on the real axis.
(ii) Since $0 \leq r_j \leq 1$ and $|\cos\theta| \leq 1$ we obtain

$$|\mathbf{z}^* A\mathbf{z}| \leq \alpha + 16 \, .$$

Problem 37. Let A be an $n \times n$ matrix over $\mathbf{C}$ and $F(A)$ the field of values. Let U be an $n \times n$ unitary matrix.
(i) Show that

$$F(U^* A U) = F(A) \, .$$

(ii) Apply the theorem to the two matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which are unitarily equivalent.

Solution 37. (i) Since a unitary matrix leaves invariant the surface of the Euclidean unit ball, the complex numbers that comprise the sets $F(U^*AU)$ and $F(A)$ are the same. If $\mathbf{z} \in \mathbf{C}^n$ and $\mathbf{z}^*\mathbf{z} = 1$, we have

$$\mathbf{z}^*(U^* A U)\mathbf{z} = \mathbf{w}^* A \mathbf{w} \in F(A)$$

where $\mathbf{w} = U\mathbf{z}$, so that $\mathbf{w}^*\mathbf{w} = \mathbf{z}^* U^* U \mathbf{z} = \mathbf{z}^*\mathbf{z} = 1$. Thus $F(U^*AU) \subset F(A)$. The reverse containment is obtained similarly.
(ii) For A_1 we have

$$\mathbf{z}^* A_1 \mathbf{z} = \overline{z_1} z_2 + \overline{z_2} z_1 = 2 r_1 r_2 \cos(\theta_2 - \theta_1)$$

with the constraints $0 \leq r_1, r_2 \leq 1$ and $r_1^2 + r_2^2 = 1$. For A_2 we find

$$\mathbf{z}^* A_2 \mathbf{z} = \overline{z_1} z_1 - \overline{z_2} z_2 = r_1^2 - r_2^2$$

with the constraints $0 \le r_1, r_2 \le 1$ and $r_1^2 + r_2^2 = 1$. Both define the same set, namely the interval $[-1, 1]$.

Problem 38. Can one find a unitary matrix U such that

$$U^* \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} U = \begin{pmatrix} 0 & ce^{i\theta} \\ de^{-i\theta} & 0 \end{pmatrix}$$

where $c, d \in \mathbf{C}$ and $\theta \in \mathbf{R}$?

Solution 38. We find

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} .$$

Problem 39. An $n^2 \times n$ matrix J is called a *selection matrix* such that J^T is the $n \times n^2$ matrix

$$[E_{11} \; E_{22} \; \ldots \; E_{nn}]$$

where E_{ii} is the $n \times n$ matrix of zeros except for a 1 in the (i, i)th position.
(i) Find J for $n = 2$ and calculate $J^T J$.
(ii) Calculate $J^T J$ for arbitrary n.

Solution 39. (i) We have

$$J^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Thus

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

Therefore $J^T J = I_2$.
(ii) For the general case we find $J^T J = I_n$.

Problem 40. Let A and B be $m \times n$ matrices. The *Hadamard product* $A \circ B$ is defined as the $m \times n$ matrix

$$A \bullet B := (a_{ij} b_{ij}) .$$

(i) Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ 7 & 1 \end{pmatrix} .$$

Calculate $A \bullet B$.
(ii) Let C, D be $m \times n$ matrices. Show that

$$\mathrm{rank}(A \bullet B) \leq (\mathrm{rank}A)(\mathrm{rank}B)\,.$$

Solution 40. (i) Entrywise multiplication of the two matrices yields

$$A \bullet B = \begin{pmatrix} 0 & 4 \\ 7 & 0 \end{pmatrix}.$$

(ii) Any matrix of rank r can be written as a sum of r rank one matrices, each of which is an outer product of two vectors (column vector times row vector). Thus, if $\mathrm{rank}A = r_1$ and $\mathrm{rank}B = r_2$, we have

$$A = \sum_{i=1}^{r_1} \mathbf{x}_i \mathbf{y}_i^*, \qquad B = \sum_{j=1}^{r_2} \mathbf{u}_j \mathbf{v}_j^*$$

where $\mathbf{x}_i, \mathbf{u}_j \in \mathbf{C}^m$, $\mathbf{y}_i, \mathbf{v}_j \in \mathbf{C}^n$, $i = 1, \ldots, r_1$ and $j = 1, \ldots, r_2$. Then

$$A \bullet B = \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} (\mathbf{x}_i \bullet \mathbf{u}_j)(\mathbf{y}_i \bullet \mathbf{v}_j)^*\,.$$

This shows that $A \bullet B$ is a sum of at most $r_1 r_2$ rank one matrices. Thus $\mathrm{rank}(A \bullet B) \leq r_1 r_2 = (\mathrm{rank}A)(\mathrm{rank}B)$.

Problem 41. Consider a symmetric matrix A over $\mathbf{R}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}$$

and the orthonormal basis (so-called *Bell basis*)

$$\mathbf{x}^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{x}^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{y}^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{y}^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

The Bell basis forms an orthonormal basis in $\mathbf{R}^4$. Let $\widetilde{A}$ denote the matrix A in the Bell basis. What is the condition on the entries a_{ij} such that the matrix A is diagonal in the Bell basis?

Solution 41. Obviously we have

$$\widetilde{a}_{ij} = \widetilde{a}_{ji},$$

i.e., the matrix $\widetilde{A}$ is also symmetric. Straightforward calculation yields

$$\begin{aligned}
\widetilde{a}_{11} &= (\mathbf{x}^+)^T A \mathbf{x}^+ = \frac{1}{2}(a_{11} + 2a_{14} + a_{44}) \\
\widetilde{a}_{12} &= (\mathbf{x}^+)^T A \mathbf{x}^- = \frac{1}{2}(a_{11} - a_{44}) \\
\widetilde{a}_{13} &= (\mathbf{x}^+)^T A \mathbf{y}^+ = \frac{1}{2}(a_{12} + a_{13} + a_{24} + a_{34}) \\
\widetilde{a}_{14} &= (\mathbf{x}^+)^T A \mathbf{y}^- = \frac{1}{2}(a_{12} - a_{13} + a_{24} - a_{34}) \\
\widetilde{a}_{22} &= (\mathbf{x}^-)^T A \mathbf{x}^- = \frac{1}{2}(a_{11} - 2a_{14} + a_{44}) \\
\widetilde{a}_{23} &= (\mathbf{x}^-)^T A \mathbf{y}^+ = \frac{1}{2}(a_{12} + a_{13} - a_{24} - a_{34}) \\
\widetilde{a}_{24} &= (\mathbf{x}^-)^T A \mathbf{y}^- = \frac{1}{2}(a_{12} - a_{13} - a_{24} + a_{34}) \\
\widetilde{a}_{33} &= (\mathbf{y}^+)^T A \mathbf{y}^+ = \frac{1}{2}(a_{22} + 2a_{23} + a_{33}) \\
\widetilde{a}_{34} &= (\mathbf{y}^+)^T A \mathbf{y}^- = \frac{1}{2}(a_{22} - a_{33}) \\
\widetilde{a}_{44} &= (\mathbf{y}^-)^T A \mathbf{y}^- = \frac{1}{2}(a_{22} - 2a_{23} + a_{33}) .
\end{aligned}$$

The condition that the matrix $\widetilde{A}$ should be diagonal leads to

$$a_{11} - a_{44} = 0, \qquad a_{22} - a_{33} = 0$$

and

$$a_{12} = a_{13} = a_{24} = a_{34} = 0$$

with the entries a_{14} and a_{23} arbitrary. Thus the matrix A has the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{23} & a_{22} & 0 \\ a_{14} & 0 & 0 & a_{11} \end{pmatrix} .$$

Problem 42. Let A be an $m \times n$ matrix over $\mathbf{C}$. The *Moore-Penrose pseudoinverse matrix* A^+ is the unique $n \times m$ matrix which satisfies

$$\begin{aligned}
AA^+A &= A \\
A^+AA^+ &= A^+ \\
(AA^+)^* &= AA^+ \\
(A^+A)^* &= A^+A .
\end{aligned}$$

We also have that

$$\mathbf{x} = A^+\mathbf{b} \tag{1}$$

is the shortest length least square solution to the problem

$$A\mathbf{x} = \mathbf{b}\,. \tag{2}$$

(i) Show that if $(A^*A)^{-1}$ exists, then $A^+ = (A^*A)^{-1}A^*$.
(ii) Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{pmatrix}.$$

Find the Moore-Penrose matrix inverse A^+ of A.

Solution 42. (i) Suppose that $(A^*A)^{-1}$ exists we have

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^*A\mathbf{x} &= A^*\mathbf{b} \\ \mathbf{x} &= (A^*A)^{-1}A^*\mathbf{b}\,. \end{aligned}$$

Using (1) we obtain

$$A^+ = (A^*A)^{-1}A^*\,.$$

(ii) We have

$$A^*A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 26 \\ 26 & 50 \end{pmatrix}.$$

Since $\det(A^*A) \neq 0$ the inverse of A^*A exists and is given by

$$(A^*A)^{-1} = \frac{1}{12}\begin{pmatrix} 25 & -13 \\ -13 & 7 \end{pmatrix}.$$

Thus

$$\begin{aligned} A^+ = (A^*A)^{-1}A^* &= \frac{1}{12}\begin{pmatrix} 25 & -13 \\ -13 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} \\ &= \frac{1}{12}\begin{pmatrix} -14 & -2 & 10 \\ 8 & 2 & -4 \end{pmatrix}. \end{aligned}$$

Problem 43. Given a signal as the column vector

$$\mathbf{x} = (3.0\ \ 0.5\ \ 2.0\ \ 7.0)^T\,.$$

The *pyramid algorithm* (for *Haar wavelets*) is as follows: The first two entries $(3.0\ 0.5)^T$ in the signal give an average of $(3.0 + 0.5)/2 = 1.75$ and a difference average of $(3.0 - 0.5)/2 = 1.25$. The second two entries $(2.0\ 7.0)$ give an average of $(2.0 + 7.0)/2 = 4.5$ and a difference average of $(2.0 - 7.0)/2 = -2.5$. Thus we end up with a vector

$$(1.75\ 1.25\ 4.5\ \ -2.5)^T .$$

Now we take the average of 1.75 and 4.5 providing $(1.75 + 4.5)/2 = 3.125$ and the difference average $(1.75 - 4.5)/2 = -1.375$. Thus we end up with the vector

$$\mathbf{y} = (3.125\ \ -1.375\ 1.25\ \ -2.5)^T .$$

(i) Find a 4×4 matrix A such that

$$\mathbf{x} \equiv \begin{pmatrix} 3.0 \\ 0.5 \\ 2.0 \\ 7.0 \end{pmatrix} = A\mathbf{y} \equiv A \begin{pmatrix} 3.125 \\ -1.375 \\ 1.25 \\ -2.5 \end{pmatrix} .$$

(ii) Show that the inverse of A exists. Then find the inverse matrix of A.

Solution 43. (i) Since we can write

$$\begin{pmatrix} 3.0 \\ 0.5 \\ 2.0 \\ 7.0 \end{pmatrix} = 3.125 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1.375 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 1.25 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - 2.5 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

we obtain the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} .$$

(ii) All the column vectors in the matrix A are nonzero and all the pairwise scalar products are equal to 0. Thus the column vectors form a basis (not normalized) in $\mathbf{R}^n$. Thus the matrix is invertible. The inverse matrix is given by

$$A^{-1} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix} .$$

Problem 44. A *Hadamard matrix* is an $n \times n$ matrix H with entries in $\{-1, +1\}$ such that any two distinct rows or columns of H have inner product 0. Construct a 4×4 Hadamard matrix starting from the column vector

$$\mathbf{x}_1 = (1\ 1\ 1\ 1)^T .$$

Solution 44. The vector

$$\mathbf{x}_2 = (1\ 1\ -1\ -1)^T$$

is perpendicular to the vector $\mathbf{x}_1$. Next the vector

$$\mathbf{x}_3 = (-1\ 1\ 1\ -1)^T$$

is perpendicular to $\mathbf{x}_1$ and $\mathbf{x}_2$. Finally the vector

$$\mathbf{x}_4 = (1\ -1\ 1\ -1)$$

is perpendicular to $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. Thus we obtain the 4×4 Hadamard matrix

$$H = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix} .$$

Problem 45. A *binary Hadamard matrix* is an $n \times n$ matrix M (where n is even) with entries in $\{0, 1\}$ such that any two distinct rows or columns of M have *Hamming distance* $n/2$. The Hamming distance between two vectors is the number of entries at which they differ. Find a 4×4 binary Hadamard matrix.

Solution 45. Hadamard matrices (see previous problem) are in *bijection* with binary Hadamard matrices with the mapping $1 \to 1$ and $-1 \to 0$. Thus using the result from the previous problem we obtain

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

Problem 46. Let $\mathbf{x}$ be a normalized column vector in $\mathbf{R}^n$, i.e. $\mathbf{x}^T\mathbf{x} = 1$. A matrix T is called a *Householder matrix* if

$$T := I_n - 2\mathbf{x}\mathbf{x}^T .$$

Calculate T^2.

Solution 46. Since the matrix product is associative we have

$$\begin{aligned} T^2 &= (I_n - 2\mathbf{x}\mathbf{x}^T)(I_n - 2\mathbf{x}\mathbf{x}^T) \\ &= I_n - 2\mathbf{x}\mathbf{x}^T - 2\mathbf{x}\mathbf{x}^T + 4\mathbf{x}(\mathbf{x}^T\mathbf{x})\mathbf{x}^T \\ &= I_n - 4\mathbf{x}\mathbf{x}^T + 4\mathbf{x}\mathbf{x}^T \\ &= I_n . \end{aligned}$$

Problem 47. An $n \times n$ matrix P is a *projection matrix* if

$$P^* = P, \qquad P^2 = P .$$

(i) Let P_1 and P_2 be projection matrices. Is $P_1 + P_2$ a projection matrix?
(ii) Let P_1 and P_2 be projection matrices. Is $P_1 P_2$ a projection matrix?
(iii) Let P be a projection matrix. Is $I_n - P$ a projection matrix? Calculate $P(I_n - P)$.
(iv) Is

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

a projection matrix?

Solution 47. (i) Obviously $(P_1 + P_2)^* = P_1^* + P_2^* = P_1 + P_2$. We have

$$(P_1 + P_2)^2 = P_1^2 + P_1 P_2 + P_2 P_1 + P_2^2 = P_1 + P_1 P_2 + P_2 P_1 + P_2 .$$

Thus $P_1 + P_2$ is a projection matrix only if $P_1 P_2 = 0_n$, where we used that from $P_1 P_2 = 0_n$ we can conclude that $P_2 P_1 = 0_n$. From $P_1 P_2 = 0_n$ it follows that

$$(P_1 P_2)^* = P_2^* P_1^* = P_2 P_1 = 0_n .$$

(ii) We have $(P_1 P_2)^* = P_2^* P_1^* = P_2 P_1$ and

$$(P_1 P_2)^2 = P_1 P_2 P_1 P_2 .$$

Thus we see that $P_1 P_2$ is a projection matrix if and only if $P_1 P_2 = P_2 P_1$.
(iii) We have $(I_n - P)^* = I_n^* - P^* = I_n - P$ and

$$(I_n - P)^2 = I_n - P - P + P^2 = I_n - P .$$

Thus $I_n - P$ is a projection matrix. We have

$$P(I_n - P) = P - P^2 = P - P = 0_n .$$

(iv) We find $P^* = P$ and $P^2 = P$. Thus P is a projection matrix.

Problem 48. Let

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

be vectors in $\mathbf{R}^3$. Let $\times$ denote the vector product.
(i) Show that we can find a 3×3 matrix $S(\mathbf{a})$ such that

$$\mathbf{a} \times \mathbf{b} = S(\mathbf{a})\mathbf{b}\,.$$

(ii) Express the *Jacobi identity*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{0}$$

using the matrices $S(\mathbf{a})$, $S(\mathbf{b})$ and $S(\mathbf{c})$.

Solution 48. (i) The *vector product* is defined as

$$\mathbf{a} \times \mathbf{b} := \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Thus $S(\mathbf{a})$ is the skew-symmetric matrix

$$S(\mathbf{a}) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

(ii) Using the result from (i) we obtain

$$\begin{aligned}
\mathbf{a} \times (S(\mathbf{b})\mathbf{c}) + \mathbf{c} \times (S(\mathbf{a})\mathbf{b}) + \mathbf{b} \times (S(\mathbf{c})\mathbf{a}) &= \mathbf{0} \\
S(\mathbf{a})(S(\mathbf{b})\mathbf{c}) + S(\mathbf{c})(S(\mathbf{a})\mathbf{b}) + S(\mathbf{b})(S(\mathbf{c})\mathbf{a}) &= \mathbf{0} \\
(S(\mathbf{a})S(\mathbf{b}))\mathbf{c} + (S(\mathbf{c})S(\mathbf{a}))\mathbf{b} + (S(\mathbf{b})S(\mathbf{c}))\mathbf{a} &= \mathbf{0}
\end{aligned}$$

where we used that matrix multiplication is associative.

Problem 49. Let s (*spin quantum number*)

$$s \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots \right\}.$$

Given a fixed s. The indices j, k run over $s, s-1, s-2, \dots, -s+1, -s$. Consider the $(2s+1)$ unit vectors (standard basis)

$$\mathbf{e}_{s,s} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_{s,s-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \mathbf{e}_{s,-s} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Obviously the vectors have $(2s+1)$ components. The $(2s+1)\times(2s+1)$ matrices s_+ and s_- are defined as

$$\begin{aligned} s_+\mathbf{e}_{s,m} &:= \hbar\sqrt{(s-m)(s+m+1)}\mathbf{e}_{s,m+1}, \quad m = s-1, s-2, \dots, -s \\ s_-\mathbf{e}_{s,m} &:= \hbar\sqrt{(s+m)(s-m+1)}\mathbf{e}_{s,m-1}, \quad m = s, s-1, \dots, -s+1 \end{aligned}$$

and $s_+\mathbf{e}_{ss} = \mathbf{0}$, $s_-\mathbf{e}_{s-s} = \mathbf{0}$, where $\hbar$ is the Planck constant.
(i) Find the matrix representation of s_+ and s_-.
(ii) The $(2s+1)\times(2s+1)$ matrix s_z is defined as

$$s_z\mathbf{e}_{s,m} := m\hbar\mathbf{e}_{s,m}, \quad m = s, s-1, \dots, -s\,.$$

Let

$$\mathbf{s} := (s_x, s_y, s_z)$$

where $s_+ = \frac{1}{2}(s_x + is_y)$ and $s_- = \frac{1}{2}(s_x - is_y)$. Find the $(2s+1)\times(2s+1)$ matrix

$$\mathbf{s}^2 := s_x^2 + s_y^2 + s_z^2\,.$$

(iii) Calculate the expectation values

$$\mathbf{e}^*_{s,s}s_+\mathbf{e}_{s,s}, \quad \mathbf{e}^*_{s,s}s_-\mathbf{e}_{s,s}, \quad \mathbf{e}^*_{s,s}s_z\mathbf{e}_{s,s}\,.$$

Solution 49. (i) We have

$$\begin{aligned} (s_+)_{jk} = (s_-)_{kj} &= \hbar\sqrt{(s-k)(s+k+1)}\delta_{j,k+1} \\ &= \hbar\sqrt{(s+j)(s-j+1)}\delta_{j,k+1} \end{aligned}$$

$$\begin{aligned} (s_-)_{jk} = (s_+)_{kj} &= \hbar\sqrt{(s+k)(s-k+1)}\delta_{j,k-1} \\ &= \hbar\sqrt{(s-j)(s+j+1)}\delta_{j,k-1}\,. \end{aligned}$$

Therefore

$$s_+ = \hbar\begin{pmatrix} 0 & \sqrt{2s} & 0 & 0 & \dots & 0 \\ 0 & 0 & \sqrt{2(2s-1)} & 0 & \dots & 0 \\ 0 & 0 & 0 & \sqrt{3(2s-2)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \sqrt{2s} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Thus $s_- = (s_+)^*$.
(ii) We have

$$\mathbf{s}^2 = \frac{1}{2}(s_+s_- + s_-s_+) + s_z^2\,.$$

Thus

$$(\mathbf{s})^2_{jk} = s(s+1)\hbar^2\delta_{jk}\,.$$

Therefore $\mathbf{s}^2$ is a diagonal matrix.
(iii) We find

$$\begin{aligned}\mathbf{e}^*_{ss}s_+\mathbf{e}_{ss} &= 0\\ \mathbf{e}^*_{ss}s_-\mathbf{e}_{ss} &= 0\\ \mathbf{e}^*_{ss}s_z\mathbf{e}_{ss} &= \hbar s\,.\end{aligned}$$

Problem 50. The *Fibonacci numbers* are defined by the recurrence relation (linear difference equation of second order with constant coefficients)

$$s_{n+2} = s_{n+1} + s_n$$

where $n = 0, 1, \ldots$ and $s_0 = 0$, $s_1 = 1$. Write this recurrence relation in matrix form. Find s_6, s_5, and s_4.

Solution 50. We have $s_2 = 1$. We can write

$$\begin{pmatrix} s_{n+1} & s_n \\ s_n & s_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n, \qquad n = 1, 2, \ldots\quad .$$

Thus

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^5 = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}.$$

It follows that $s_6 = 8$, $s_5 = 5$ and $s_4 = 3$.

Problem 51. (i) Find four unit (column) vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ in $\mathbf{R}^3$ such that

$$\mathbf{x}_j^T\mathbf{x}_k = \frac{4}{3}\delta_{jk} - \frac{1}{3} = \begin{cases} 1 & \text{for } j = k \\ -1/3 & \text{for } j \neq k\,. \end{cases}$$

Give a geometric interpretation.
(ii) Calculate

$$\sum_{j=1}^{4} \mathbf{x}_j\,.$$

(iii) Calculate

$$\sum_{j=1}^{4} \mathbf{x}_j\mathbf{x}_j^T\,.$$

Solution 51. (i) The four vectors consist of the vectors pointing from the center of a cube to nonadjacent corners. Alternatively, one may picture these vectors as the normal vectors for the faces of the tetrahedron that is defined by the other four corners of the cube. Thus an example is

$$\mathbf{x}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix},$$

$$\mathbf{x}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{x}_4 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

(ii) We obviously find

$$\sum_{j=1}^{4} \mathbf{x}_j = \mathbf{0}$$

which states that the four vectors are linearly dependent.
(iii) We obtain

$$\sum_{j=1}^{4} \mathbf{x}_j \mathbf{x}_j^T = \frac{4}{3} I_3$$

where I_3 is the 3×3 identity matrix.

Problem 52. Assume that

$$A = A_1 + iA_2$$

is a nonsingular $n \times n$ matrix, where A_1 and A_2 are real $n \times n$ matrices. Assume that A_1 is also nonsingular. Find the inverse of A using the inverse of A_1.

Solution 52. We have the identity

$$(A_1 + iA_2)(I_n - iA_1^{-1}A_2) \equiv A_1 + A_2 A_1^{-1} A_2 .$$

Thus we find the inverse

$$A^{-1} = (A_1 + A_2 A_1^{-1} A_2)^{-1} - iA_1^{-1} A_2 (A_1 + A_2 A_1^{-1} A_2)^{-1} .$$

Problem 53. The 8×8 matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

plays a role in the *discrete wavelet transform.*
(i) Show that the matrix is invertible without calculating the determinant.
(ii) Find the inverse.

Solution 53. (i) All the column vectors in the matrix H are nonzero. All the pairwise scalar products of the column vectors are 0. Thus the matrix has maximum rank, i.e. rank$H$ = 8 and the column vectors form a basis (not normalized) in $\mathbf{R}^n$.
(ii) The inverse matrix is given by

$$H^{-1} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 \\ 4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4 \end{pmatrix}.$$

Problem 54. Let A and B be $n \times n$ matrices over $\mathbf{R}$. Assume that $A \neq B$, $A^3 = B^3$ and $A^2B = B^2A$. Is $A^2 + B^2$ invertible?

Solution 54. We have

$$(A^2 + B^2)(A - B) = A^3 - B^3 - A^2B + B^2A = 0_n .$$

Since $A \neq B$, we can conclude that $A^2 + B^2$ is not invertible.

Problem 55. Let A be a positive definite $n \times n$ matrix over $\mathbf{R}$. Let $\mathbf{x} \in \mathbf{R}$. Show that $A + \mathbf{x}\mathbf{x}^T$ is also positive definite.

Solution 55. For all vector $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{y} \neq \mathbf{0}$, we have $\mathbf{y}^T A\mathbf{y} > 0$. We have

$$\mathbf{y}^T(\mathbf{x}\mathbf{x}^T)\mathbf{y} = (\mathbf{y}^T\mathbf{x})(\mathbf{x}^T\mathbf{y}) = \left(\sum_{j=1}^n y_j x_j\right)\left(\sum_{j=1}^n x_j y_j\right) = \left(\sum_{j=1}^n x_j y_j\right)^2 \geq 0$$

and therefore we have $\mathbf{y}^T(A + \mathbf{x}\mathbf{x}^T)\mathbf{y} > 0$ for all $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{y} \neq \mathbf{0}$.

Problem 56. Let A, B be $n \times n$ matrices over $\mathbf{C}$. The matrix A is called *similar* to the matrix B if there is an $n \times n$ invertible matrix S such that

$$A = S^{-1}BS .$$

If A is similar to B, then B is also similar to A, since $B = SAS^{-1}$.
(i) Consider the two matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Are the matrices similar?
(ii) Consider the two matrices

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Are the matrices similar?

Solution 56. (i) From $A = S^{-1}BS$ we obtain $SA = BS$. Let

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

Then from $SA = BS$ we obtain

$$\begin{pmatrix} s_{11} + 2s_{12} & s_{12} \\ s_{21} + 2s_{22} & s_{22} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

It follows that $s_{12} = 0$ and $s_{22} = 0$. Thus S is not invertible and therefore A and B are not similar.
(ii) The matrices C and D are similar. We find

$$s_{11} = s_{21}, \qquad s_{12} = -s_{22}.$$

Since S must be invertible, all four matrix elements are nonzero. For example, we can select

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$