Chapter 1

Generalized Quadrangles

In this chapter we recall basic facts on finite generalized quadrangles, including some important results that will often be used in this monograph.

1.1 Finite Generalized Quadrangles

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (nonempty) sets of objects called *points* and *lines* respectively, and for which I is a symmetric point-line *incidence relation* satisfying the following axioms.

(i) Each point is incident with $1 + t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ points $(s \geq 1)$ and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$.

Generalized quadrangles were introduced by Tits [217] in the appendix of his celebrated work on triality.

If x and L are as in (iii), then we will denote the unique point y on L collinear with x also by $\mathrm{proj}_L x$, and call it the *projection of x onto L*. Dually we define $\mathrm{proj}_x L := M$. We generalize this to points z on L by putting $z = \mathrm{proj}_L z$, if $z \mathrm{I} L$, and to lines M concurrent with L by denoting the intersection point $\mathrm{proj}_L M$. The dual notation is also used.

The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* (s,t); if $s=t$, $\mathcal{S}$ is said to have *order* s. There is a *point-line duality* for GQs (of order (s,t)) for which in any definition or theorem the words "point" and "line" are interchanged and the parameters s and t are interchanged. Hence, we assume without further notice that the dual of a given theorem or definition has also been given.

If the point x is not incident with the line L, in a GQ $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$, we write $p \not\mathrm{I} L$.

Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ be a (finite) GQ of order (s,t). Then $\mathcal{S}$ has $v=|\mathcal{P}|=(1+s)(1+st)$ points and $b=|\mathcal{B}|=(1+t)(1+st)$ lines; see 1.2.1 of Payne and Thas [128]. Also, $s+t$ divides $st(1+s)(1+t)$, and, for $s\neq 1\neq t$, we have $t\leq s^2$ and, dually, $s\leq t^2$ (inequalities of Higman [70]); see 1.2.2 and 1.2.3 of Payne and Thas [128].

Given two (not necessarily distinct) points x,y of $\mathcal{S}$, we write $x\sim y$ and say that x and y are *collinear*, provided that there is some line L for which $x\mathrm{I}L\mathrm{I}y$. And $x\not\sim y$ means that x and y are not collinear. Dually, for $L,M\in\mathcal{B}$, we write $L\sim M$ or $L\not\sim M$ according as L and M are *concurrent* or nonconcurrent, respectively. The line which is incident with distinct collinear points x,y is denoted by xy; the point which is incident with distinct concurrent lines L,M is denoted by either LM or $L\cap M$.

For $x\in\mathcal{P}$, put $x^\perp=\{y\in\mathcal{P}\,\|\,y\sim x\}$, and note that $x\in x^\perp$. If $A\subseteq\mathcal{P}$, A "perp" is defined by $A^\perp=\cap\{x^\perp\,\|\,x\in A\}$. Hence, for $x,y\in\mathcal{P},x\neq y$, we have $\{x,y\}^\perp=x^\perp\cap y^\perp$; we have $|\{x,y\}^\perp|=s+1$ or $t+1$ according as $x\sim y$ or $x\not\sim y$. Further, $\{x,y\}^{\perp\perp}=\{u\in\mathcal{P}\,\|\,u\in z^\perp\ \forall z\in x^\perp\cap y^\perp\}$; we have $|\{x,y\}^{\perp\perp}|=s+1$ or $|\{x,y\}^{\perp\perp}|\leq t+1$ according as $x\sim y$ or $x\not\sim y$. The sets $\{x,y\}^\perp$ and $\{x,y\}^{\perp\perp}$ are respectively called the *trace* and the *span* of the pair $\{x,y\}$. If $x\not\sim y$, then $\{x,y\}^{\perp\perp}$ is also called the *hyperbolic line* defined by x and y.

A *triad* of points is a triple of pairwise noncollinear points. Given a triad T, a *center* of T is just an element of $T^\perp$. If a triad has at least one center, then we say that it is *centric*.

A *subquadrangle*, or also *sub*GQ, $\mathcal{S}'=(\mathcal{P}',\mathcal{B}',\mathrm{I}')$ of a GQ $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ of order (s,t), is a GQ for which $\mathcal{P}'\subseteq\mathcal{P},\mathcal{B}'\subseteq\mathcal{B}$, and where I' is the restriction of I to $(\mathcal{P}'\times\mathcal{B}')\cup(\mathcal{B}'\times\mathcal{P}')$. If the GQ $\mathcal{S}'$ of order (s',t') is a subGQ of the GQ $\mathcal{S}$ of order (s,t), with $\mathcal{S}\neq\mathcal{S}'$, then either $s=s'$ or $s\geq s't'$, and, dually, either $t=t'$ or $t\geq s't'$; see 2.2.1 of Payne and Thas [128].

The following result will appear to be very useful; see 2.2.2 of Payne and Thas [128]. When we refer to it, we will often not make a distinction between the statement and its dual.

Lemma 1.1.1 *If the thick* GQ $\mathcal{S}$ *of order* (s,t) *has a proper subquadrangle* $\mathcal{S}'$ *of order* (s,t'), *then* $t \geq st'$; *in particular* $t' \leq s \leq t$. *Hence if* $s = t$, *then* $t' = 1$. *Also, if* $s = t'$, *then* $t = s^2$.

Also, if in addition $\mathcal{S}''$ *is a proper subquadrangle of* $\mathcal{S}'$ *of order* (s,t''), *then* $t'' = 1$, $s = t'$ *and* $t = s^2$. ■

An *ovoid* of the GQ $\mathcal{S}$ of order (s,t), is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that each line of $\mathcal{S}$ is incident with a unique point of $\mathcal{O}$; a *spread* of $\mathcal{S}$ is a set $\mathcal{T}$ of lines of $\mathcal{S}$ such that each point of $\mathcal{S}$ is incident with a unique line of $\mathcal{T}$. If $\mathcal{O}$ is an ovoid of $\mathcal{S}$, respectively if $\mathcal{T}$ is a spread of $\mathcal{S}$, then $|\mathcal{O}| = |\mathcal{T}| = 1 + st$; see 1.8.1 of Payne and Thas [128].

If the GQ $\mathcal{S}'$ of order (s,t') is a subquadrangle of the GQ $\mathcal{S}$ of order (s,t), with $\mathcal{S} \neq \mathcal{S}'$, then each point of $\mathcal{S}$ not in $\mathcal{S}'$ is collinear with $1 + st'$ points of an ovoid of $\mathcal{S}'$; see 2.2.1 of Payne and Thas [128].

The following result appears to the very useful. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ be a substructure of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s,t) for which the following conditions are satisfied.

(i) If $x, y \in \mathcal{P}', x \neq y$, and $x\mathrm{I}L\mathrm{I}y$, then $L \in \mathcal{B}'$.

(ii) Each element of $\mathcal{B}'$ is incident with $1 + s$ elements of $\mathcal{P}'$.

Then by Thas [157], see also 2.3.1 of Payne and Thas [128], there are four possibilities.

(a) $\mathcal{S}'$ is a dual grid (and then $s = 1$); see §1.3 for the definition of a dual grid.

(b) The elements of $\mathcal{B}'$ are lines which are incident with a distinguished point of $\mathcal{P}$, and $\mathcal{P}'$ consists of those points of $\mathcal{P}$ which are incident with these lines.

(c) $\mathcal{B}' = \emptyset$ and $\mathcal{P}'$ is a set of pairwise noncollinear points of $\mathcal{P}$.

(d) $\mathcal{S}'$ is a subquadrangle of order (s,t').

1.2 Automorphisms

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ. A *collineation* or *automorphism* of $\mathcal{S}$ is a permutation of $\mathcal{P} \cup \mathcal{B}$ that preserves $\mathcal{P}$, $\mathcal{B}$ and incidence. We will denote the set of all collineations of a given GQ $\mathcal{S}$ by $\mathrm{Aut}(\mathcal{S})$ and call it the *full collineation (automorphism) group of* $\mathcal{S}$, as opposed to an ordinary *collineation (automorphism) group of* $\mathcal{S}$, which is just a subgroup of $\mathrm{Aut}(\mathcal{S})$.

1.3 Grids and Dual Grids

A *grid* is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ with $\mathcal{P} = \{x_{ij} \parallel i = 0, 1, \ldots, s_1$ and $j = 0, 1, \ldots, s_2\}$, $s_1 > 0$ and $s_2 > 0$, $\mathcal{B} = \{L_0, L_1, \ldots, L_{s_1}, M_0, M_1, \ldots, M_{s_2}\}$, $x_{ij} \mathrm{I} L_k$ if and only if $i = k$, and $x_{ij} \mathrm{I} M_k$ if and only if $j = k$. We say that such a grid is an $(s_1 + 1) \times (s_2 + 1)$-*grid*. A grid is a GQ if and only if $s_1 = s_2 = s$; in such a case the GQ has order $(s, 1)$. Any GQ of order $(s, 1)$ is isomorphic to an $(s + 1) \times (s + 1)$-grid. A dual grid has parameters t_1, t_2, and it is a GQ if and only if $t_1 = t_2 = t$, in which case it is a GQ of order $(1, t)$. Any GQ of order $(1, t)$ is isomorphic to a dual $(t + 1) \times (t + 1)$-grid. An ordinary quadrangle is a GQ of order (1,1) and is at the same time a grid and a dual grid. This is the motivation for the term "generalized quadrangle".

A GQ of order (s, t) is called *thin* if either $s = 1$ or $t = 1$; in the other case it is called *thick*.

1.4 The Classical Generalized Quadrangles

We now give a brief description of three families of examples known as the *classical* GQs, all of which are associated with classical groups and were first recognized as GQs by Tits.

(i) Consider a nonsingular quadric $\mathbf{Q}$ of projective index 1, that is, of Witt index 2, of the projective space $\mathbf{PG}(d, q)$, with $d = 3, 4$ or 5. Then the points of $\mathbf{Q}$ together with the lines of $\mathbf{Q}$ (which are the subspaces of maximal dimension on $\mathbf{Q}$) form a GQ $\mathbf{Q}(d, q)$ with parameters

$$\begin{array}{llll} s = q,\ t = 1, & v = (q+1)^2, & b = 2(q+1), & \text{when } d = 3, \\ s = q,\ t = q, & v = (q+1)(q^2+1), & b = (q+1)(q^2+1) & \text{when } d = 4, \\ s = q,\ t = q^2, & v = (q+1)(q^3+1), & b = (q^2+1)(q^3+1), & \text{when } d = 5. \end{array}$$

Notice that $\mathbf{Q}(3, q)$ is a grid.

(ii) Let **H** be a nonsingular Hermitian variety of the projective space $\mathbf{PG}(d, q^2)$, $d = 3$ or 4. Then the points of **H** together with the lines on **H** form a GQ $\mathbf{H}(d, q^2)$ with parameters

$$s = q^2, t = q, \quad v = (q^2+1)(q^3+1), b = (q+1)(q^3+1), \quad \text{when } d = 3,$$
$$s = q^2, t = q^3, v = (q^2+1)(q^5+1), b = (q^3+1)(q^5+1), \quad \text{when } d = 4.$$

(iii) The points of $\mathbf{PG}(3, q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a GQ $\mathbf{W}(q)$ with parameters

$$s = q, t = q, , v = (q+1)(q^2+1), b = (q+1)(q^2+1).$$

Theorem 1.4.1 (i) *The GQ* $\mathbf{Q}(4, q)$ *is isomorphic to the dual of* $\mathbf{W}(q)$. *Also,* $\mathbf{Q}(4, q)$ *(or* $\mathbf{W}(q)$*) is self-dual if and only if* q *is even.*

(ii) *The GQ* $\mathbf{Q}(5, q)$ *is isomorphic to the dual of* $\mathbf{H}(3, q^2)$.

Proof. See 3.2.1 and 3.2.3 of Payne and Thas [128]. ■

1.5 The Generalized Quadrangles $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$ of Tits

A *k-arc* of $\mathbf{PG}(2, q)$ is a set of k points of $\mathbf{PG}(2, q)$ no three of which are collinear. Then clearly $k \leq q + 2$. By Bose [17], for q odd, $k \leq q + 1$. Further, any nonsingular conic of $\mathbf{PG}(2, q)$ is a $(q+1)$-arc. It can be shown that each $(q+1)$-arc $\mathcal{K}$ of $\mathbf{PG}(2, q)$, q even, extends to a $(q+2)$-arc $\mathcal{K} \cup \{x\}$ (see, e.g., Hirschfeld [73], p. 177); the point x, which is uniquely defined by $\mathcal{K}$, is called the *kernel* or *nucleus* of $\mathcal{K}$. The $(q+1)$-arcs of $\mathbf{PG}(2, q)$ are called *ovals*; the $(q+2)$-arcs of $\mathbf{PG}(2, q)$, q even, are called *hyperovals*. By a celebrated theorem of Segre [137], every oval in $\mathbf{PG}(2, q)$, q odd, is a nonsingular conic. For q even, this is valid if and only if $q \in \{2, 4\}$; see, e.g., Thas [178].

We now introduce the notion of "ovoid" as defined by Tits in [221]. An *ovoid* $\mathcal{O}$ of $\mathbf{PG}(3, q)$ is a set of points of $\mathbf{PG}(3, q)$ no three of which are collinear and such that for any point of $\mathcal{O}$ the union of the lines which meet $\mathcal{O}$ only in that point, that is, the *tangent lines* at that point, is a $\mathbf{PG}(2, q)$. If $\mathcal{O}$ is an ovoid, its number of points is $q^2 + 1$. A *k-cap* $\mathcal{K}$ of $\mathbf{PG}(3, q)$ is a set of k points no three of which are collinear. For any k-cap $\mathcal{K}$ of $\mathbf{PG}(3, q)$, with $q \neq 2, k \leq q^2 + 1$; for any k-cap $\mathcal{K}$ of $\mathbf{PG}(3, 2), k \leq 8$ holds and the

8-caps of $\mathbf{PG}(3,2)$ are the complements of planes. For q odd this result is due to Bose [17], for q even to Qvist [132]. A (q^2+1)-cap of $\mathbf{PG}(3,q)$, $q \neq 2$, is precisely an *ovoid* (cf. Barlotti [8] or Hirschfeld [72]); the *ovoids* of $\mathbf{PG}(3,2)$ are the sets of 5 points no 4 of which are coplanar. It is easy to show that each nonsingular elliptic quadric of $\mathbf{PG}(3,q)$ is an ovoid. By a celebrated theorem, due independently to Barlotti [8] and Panella [106], every ovoid in $\mathbf{PG}(3,q), q$ odd or $q = 4$, is an elliptic quadric.

Theorem 1.5.1 (Barlotti [8]; Panella [106]) *Each ovoid of* $\mathbf{PG}(3,q)$, q *odd, is an elliptic quadric.* ■

To the contrary, in the even case, Tits [221] showed that for any $q = 2^{2e+1}$, with $e \geq 1$, there exists an ovoid which is not an elliptic quadric; these ovoids are called *Tits ovoids*, or also *Suzuki-Tits ovoids*, and are related to the simple Suzuki groups $\mathbf{Sz}(q)$. In fact, for $q = 8$ Segre [138] discovered an ovoid which is not an elliptic quadric, and which was shown to be a Tits ovoid by Fellegara [57]. For even q no other ovoids than the elliptic quadrics and the Tits ovoids are known.

If $\mathcal{O}$ is an ovoid in $\mathbf{PG}(3,q)$, then any plane π of $\mathbf{PG}(3,q)$ intersects $\mathcal{O}$ in either one point or in an oval. If $|\pi \cap \mathcal{O}| = 1$, then we say that π is a *tangent plane* of $\mathcal{O}$. At each of its points $\mathcal{O}$ has exactly one tangent plane. For more details, see Hirschfeld [72]. Finally, a beautiful result due to Brown [23] tells us that any ovoid $\mathcal{O}$ of $\mathbf{PG}(3,q)$ containing at least one conic section, is an elliptic quadric.

Let $d = 2$ (respectively, $d = 3$) and let $\mathcal{O}$ be an oval (respectively, an ovoid) of $\mathbf{PG}(d,q)$. Further, let $\mathbf{PG}(d,q) = H$ be embedded as a hyperplane in $\mathbf{PG}(d+1,q) = P$. Define

- POINTS as
  - (i) the points of $P \backslash H$,
  - (ii) the hyperplanes X of P for which $|X \cap \mathcal{O}| = 1$, and
  - (iii) one new symbol (∞).
- LINES are defined as
  - (a) the lines of P which are not contained in H and meet $\mathcal{O}$ (necessarily in a unique point), and
  - (b) the points of $\mathcal{O}$.

- INCIDENCE is defined as follows. A point of Type (i) is incident only with lines of Type (a); here the incidence is that of P. A point of Type (ii) is incident with all lines of Type (a) contained in it and with the unique element of $\mathcal{O}$ in it. The point (∞) is incident with no line of Type (a) and all lines of Type (b).

It is an easy exercise to show that the incidence structure $\mathbf{T}_2(\mathcal{O})$ (respectively, $\mathbf{T}_3(\mathcal{O})$) so defined is a GQ. The parameters are

$$s = q,\ t = q,\ \ v = (q+1)(q^2+1),\ b = (q+1)(q^2+1),\quad \text{when } d = 2,$$
$$s = q,\ t = q^2,\ v = (q+1)(q^3+1),\ b = (q^2+1)(q^3+1),\ \text{when } d = 3.$$

Theorem 1.5.2 (i) *The* GQ $\mathbf{T}_2(\mathcal{O})$ *is isomorphic to the* GQ $\mathbf{Q}(4,q)$ *if and only if* $\mathcal{O}$ *is a nonsingular conic. The* GQ $\mathbf{T}_2(\mathcal{O})$ *is isomorphic to the* GQ $\mathbf{W}(q)$ *if and only if* q *is even and* $\mathcal{O}$ *is a conic.*

(ii) *The* GQ $\mathbf{T}_3(\mathcal{O})$ *is isomorphic to the* GQ $\mathbf{Q}(5,q)$ *if and only if* $\mathcal{O}$ *is a nonsingular elliptic quadric.*

Proof. See 3.2.2 and 3.2.4 of Payne and Thas [128]. ■

Remark 1.5.3 (i) For q odd any oval is a nonsingular conic, hence for q odd we always have $\mathbf{T}_2(\mathcal{O}) \cong \mathbf{Q}(4,q)$.

(ii) For q odd any ovoid is an elliptic quadric, hence for q odd we always have $\mathbf{T}_3(\mathcal{O}) \cong \mathbf{Q}(5,q)$.

1.6 The Generalized Quadrangles $\mathbf{T}_2^*(\mathcal{O})$

Let $\mathcal{O}$ be a hyperoval of $\mathbf{PG}(2,q)$, so q is even. Embed $\mathbf{PG}(2,q) = H$ as a hyperplane in $\mathbf{PG}(3,q) = P$.

- The POINTS of the GQ $\mathbf{T}_2^*(\mathcal{O})$ are the points of $P\backslash H$,
- LINES of the GQ are the lines of P not in H which meet $\mathcal{O}$, and
- the INCIDENCE is inherited from P.

Then $\mathbf{T}_2^*(\mathcal{O})$ is a GQ with parameters

$$s = q-1, t = q+1, v = q^3, b = (q+2)q^2.$$

1.7 Orders of the Known Generalized Quadrangles

The orders (s,t) of the known GQs are

$$\begin{array}{ll} (s,1), & s \in \mathbb{N}\backslash\{0\}, \\ (1,t), & t \in \mathbb{N}\backslash\{0\}, \\ (q,q), & q \text{ any prime power}, \\ (q,q^2), & q \text{ any prime power}, \\ (q^2,q), & q \text{ any prime power}, \\ (q^2,q^3), & q \text{ any prime power}, \\ (q^3,q^2), & q \text{ any prime power}, \\ (q-1,q+1), & q \text{ any prime power}, \\ (q+1,q-1), & q \text{ any prime power}. \end{array}$$

1.8 Generalized Quadrangles with Small Parameters

The proofs of all the results in this section are contained in Chapter 6 of Payne and Thas [128].

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a finite GQ of order $(s,t), 1 < s \leq t$.

(a) $s = 2$

By Section 1.1, $s+t$ divides $st(s+1)(t+1)$ and $t \leq s^2$. Hence $t \in \{2,4\}$. Up to isomorphism there is only one GQ of order 2 and only one GQ of order (2,4). It follows that the GQs $\mathbf{W}(2)$ and $\mathbf{Q}(4,2)$ are self-dual and mutually isomorphic. It is easy to show that the GQ of order 2 is unique.

The uniqueness of the GQ of order (2,4) was proved independently at least five times, by Seidel [141], Shult [142], Thas [164], Freudenthal [61] and Dixmier and Zara [53].

(b) $s = 3$

Again by 1.1 we have $t \in \{3,5,6,9\}$. Any GQ of order $(3,5)$ must be isomorphic to the GQ $\mathbf{T}_2^*(\mathcal{O})$ arising from the unique hyperoval in $\mathbf{PG}(2,4)$, any GQ of order $(3,9)$ must be isomorphic to $\mathbf{Q}(5,3)$, and a GQ of order 3 is isomorphic to either $\mathbf{W}(3)$ or to its dual $\mathbf{Q}(4,3)$. Finally, there is no GQ of order (3,6).

The uniqueness of the GQ of order (3,5) was proved by Dixmier and Zara [53], the uniqueness of the GQ of order (3,9) was proved independently by Dixmier and Zara [53] and Cameron (see Payne and Thas [126]), the determination of all GQs of order 3 is due independently to Payne [108] and to Dixmier and Zara [53]. Dixmier and Zara [53] proved that there is no GQ of order (3,6).

(c) $s = 4$

Using 1.1 it is easy to check that $t \in \{4, 6, 8, 11, 12, 16\}$. Nothing is known about $t = 11$ or $t = 12$. In the other cases unique examples are known, but the uniqueness question is settled only in the case $t = 4$. The proof of this uniqueness that appears in Payne and Thas [128] is that of Payne [110, 111], with a gap filled in by Tits.