FIXED POINT ACTIONS, SYMMETRIES AND SYMMETRY TRANSFORMATIONS ON THE LATTICE *

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Unwanted symmetry breaking by the lattice regularization will produce cut-off effects and distort the results. Symmetries are realized on the lattice frequently in an unusual way. Fixed point actions preserve all the classical symmetries of the theory and help to abstract not only the symmetry conditions, but the form of the lattice symmetry transformations also.

1. Introduction

Defining the action of a field theory on a hypercubic lattice is the first step towards a non-perturbative treatment of the corresponding quantum field theory. It seems to be obvious that certain symmetries of the continuum action (like space-time independent, 'internal' symmetries) will be trivially respected, while certain space time symmetries (like infinitesimal translations, rotations) unavoidably will be broken by the lattice action. This is, however not true. The internal chiral symmetry in its well known continuum formulation can not be kept on the lattice without violating some basic principles [1]. On the other hand, simple renormalization group considerations suggest that there exist lattice *actions* which inherit all the features of translation and rotation symmetries of the continuum. Even more, they inherit all the features of all the continuum symmetries. Since the action defines the classical field theory, these lattice formulations are classically perfect [2]. Refering to their role in renormalization group theory we shall call these actions here 'fixed point (FP) actions'.

These theoretical considerations are directly connected to earth bound

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numerical experiments. Unwanted symmetry breaking by the lattice regularization will produce cut-off effects and distort the results.

On the basis of the accumulated experience on stochastic calculations since the first primitive computers entered the scene more than fifty years ago, it is not very probable that we shall see a great breakthrough in full QCD calculations in the near future. It is hard to expect miracles: good scaling, good chiral properties, theoretical safety and expenses will remain in balance. Probably, we shall see a plethora of full QCD simulations adapted to the physical problem investigated. In some interesting, but difficult situations like the ϵ-, or δ-regime the approximated fixed point action might be a competitive choice.

2. Cut-off effects: a numerical experiment on the running coupling

We shall start with an example. Fig 1. shows the stochastic values of a physical quantity as the function of the lattice resolution a^3. The results are plotted against a^2, since the quantity is expected to approach the continuum limit with a^2 corrections if a^2 is small. Actually, the figure refers to a $d = 2$ quantum field theory, the two dimensional $O(3)$ non-linear σ-model. This model has many analogies with a $d = 4$ Yang-Mills theory: it is asymptotically free, has dimensional transmutation (i.e. it has massive excitations although the classical theory is scale invariant) and has exact classical solutions with topology (instantons).

The continuum extrapolation does not seem to be easy: the onset of the a^2 behaviour is delayed. One might perform the continuum extrapolation with an Ansatz including a^2 powers and logs and obtain an estimate for the continuum limit. One might also perform further simulations on larger lattices with smaller a and make the extrapolation more reliable.

An alternative procedure is to simulate a better action in order to make the extrapolation easier. For example, one might consider a theoretical construction for a local action on the lattice which has no cut-off effects at all in the classical field theory. Such local actions, called fixed point (FP) actions, exist and, as we shall discuss, are defined by classical equations. These equations can be solved approximately and a local action can be constructed which is approximately classically perfect. Simulating this action produces the full triangles in Fig.2. An excellent global behaviour is observed. Of course, these results should be weighted with the expenses of the numerical procedure. Constructing a good approximation to the FP action requests analytic and numerical work. The simulation is more expensive than that

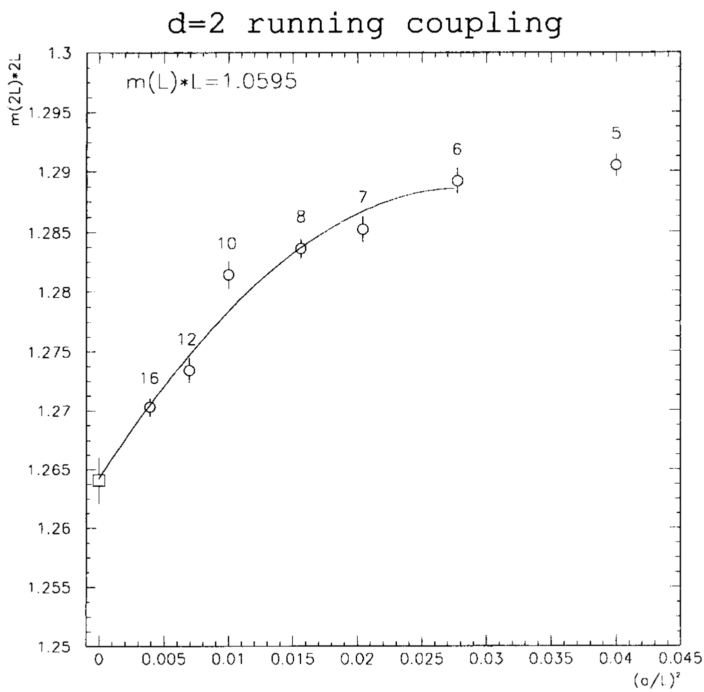

Fig. 1. The cutoff effects and the continuum extrapolation of $m(2L)2L$ if $m(L)L = 1.0595$ [3], where L is the spatial size of the lattice and $m(L)$ is the mass gap in this box.

of the simplest discretizations. The gain might also depend on the physical quantity measured. In this example the FP action seems to be a very useful idea.

The physical quantity considered above is a specially defined running coupling in this model. Since an analogous quantity plays an important role in QCD, let us discuss it in this simpler model.

The continuum action of the $d = 2, O(3)$ non-linear σ-model in Euclidean space reads

$$\mathcal{A} = \frac{1}{2g}\int d^2x \partial_\mu \mathbf{S} \partial_\mu \mathbf{S}, \qquad \mu = 0, 1\,, \qquad \mathbf{S} = (S^1, S^2, S^3), \qquad \mathbf{S}(x)^2 = 1\,, \tag{1}$$

where the bare coupling g is dimensionless. Consider this system in a finite, periodic box of size L. Take a small box $L \ll 1/m$, where m is the infinite volume mass-gap. From the point of view of low energy excitations the sector with non-zero momenta, and so the x^1-dependence of the field

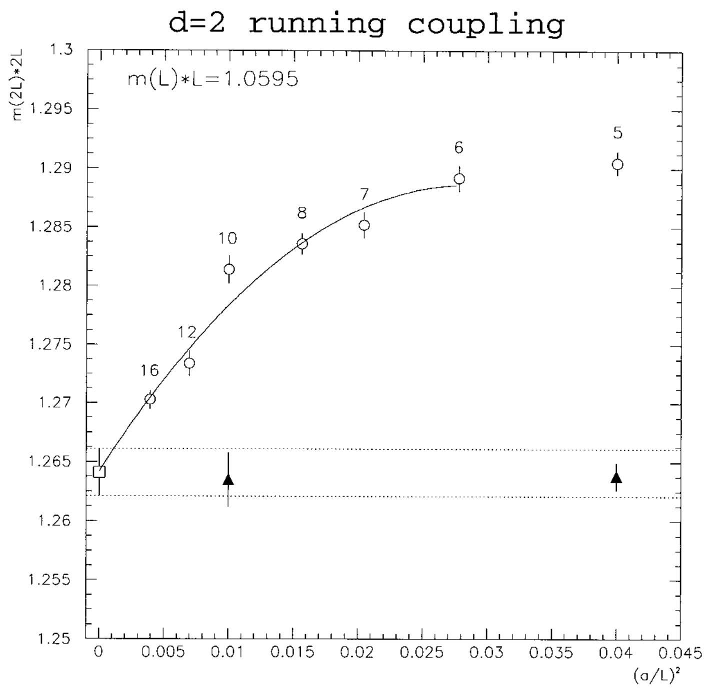

Fig. 2. The same as Fig.1 but adding data obtained with an approximation of the fixed point action[2].

$\mathbf{S}(x^0, x^1)$, can be suppressed in this case

$$\mathbf{S}(x^0, x^1) \sim \mathbf{S}(x^0), \qquad \mathcal{A} \sim \frac{L}{2g} \int dx^0 \partial_0 \mathbf{S} \partial_0 \mathbf{S}\,. \tag{2}$$

Here $\mathbf{S}$ is a unit vector with three components which depend on x^0, so the action in Eq. (2) describes the quantum mechanics of a free particle moving on the surface of a unit sphere: we have a quantum rotator. The energy spectrum is $E_l = l(l+1)/2\Theta$ with $\Theta = g/L$. The lowest excitation in this box (the mass-gap $m(L)$) is

$$m(L) = E_1 - E_0 = g/L\,. \tag{3}$$

Since the bare coupling $g \to 0$ in the continuum limit, the rotator excitations are indeed much lighter than those with non-zero momenta. The dimensionless quantity $Lm(L)$ is a valid definition of the scale dependent ('running') coupling constant $g(L)$. In a small box $L \ll \Lambda_{O(3)}^{-1}$ ($\Lambda_{O(3)}$ is the Lambda-parameter of the theory) i.e. in the ultraviolet region, it is equal to the bare coupling g.

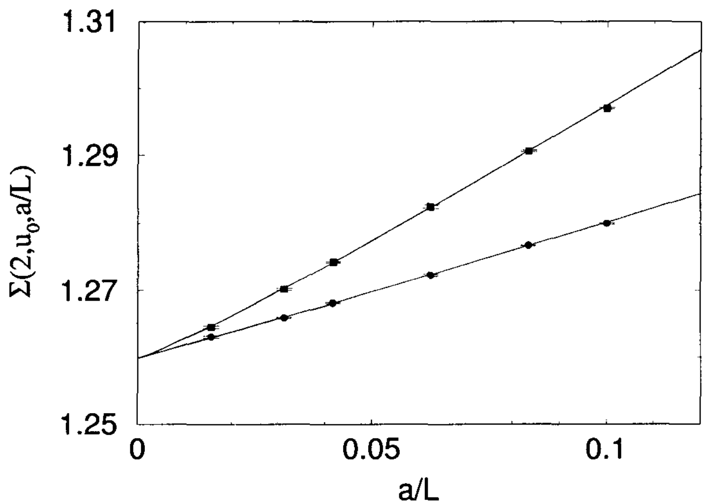

Fig. 3. The cutoff dependence of high precision data obtained with two different actions[4] are shown in this figure. The starting coupling is $g(L) = Lm(L) = 1.0595$ as previously. The cutoff dependent running coupling in a box of $2L$ is denoted here by Σ. In the continuum limit $\Sigma \to g(2L)$.

Assume that at some value of L the running coupling takes the value $g(L) = r$, where r is some positive real number. The dynamics of the $O(3)$ σ-model defines then the value of $g(2L)$ in the continuum limit. Since the theory is asymptotically free, we expect that $g(2L) > g(L)$. At finite lattice constant a this number depends on the resolution. This cutoff dependence is shown in Fig. 1 for the arbitrarily taken case with $r = 1.0595$.

The data in Fig. 1 are more than ten years old. They have large statistical errors and do not go deep in the continuum limit. Fig. 3 shows some more recent results with two simple actions [4†]. Universality predicts that the two sets of data hit the same point in the continuum limit. This makes the extrapolation easier and leads to the prediction $g(2L) = 1.2614(3)$. A few years later the exact continuum number became known also [5]: 1.261208.... It is satisfying to see that using special cluster algorithms and estimating the errors carefully high precision results can be obtained in this $d = 2$ model. For much more difficult problems, like QCD, one would like to see more theoretical inputs, however.

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†Note that the data are plotted against a/L in this figure.

3. Renormalization group and the fixed point action

A QFT is defined over a large span of scales from low physical scales up to the cut-off which goes to infinity in the continuum limit. Although field variables associated with very high scales do influence the physical predictions through a complicated cascade process, no physical question involves them directly. Their presence and indirect influence makes it difficult to establish an intuitive connection between the form of the interaction and the final predictions. The presence of a large number of degrees of freedom makes the problem technically difficult also. It is, therefore a natural idea to partially integrate them out in the path integral. This process, which reduces the number of degrees of freedom, taking into account their effect on the remaining variables exactly, is called a renormalization group transformation [6,7] (RGT). A technically simple introduction to lattice regularization and Wilson's RG can be found in Sect.2-3 of Ref. 8.

Consider some lattice regularization of the $SU(N)$ Yang-Mills gauge action

$$\mathcal{A} = \frac{\beta}{8N}\int F^a_{\mu\nu}(x)F^a_{\mu\nu}(x), \qquad a = 1, ..., N^2-1\,, \tag{4}$$

where $\beta = 2N/g^2$ and g is very small. Imagine performing repeatedly RG transformations by some gauge invariant averaging. Tree level perturbation theory predicts that all but one of the possible gauge invariant operators die out rapidly under this repeated averaging. The surviving operator is a special discretization of Eq. (4) whose detailed form depends on the averaging procedure. This special operator (action) is called the fixed point of the RG transformation since it is reproduced by the transformation. The form of the fixed point action is determined by classical saddle-point equations[10]. The theoretical properties of the fixed point action and the construction of approximate solutions to be used in simulations are based on these saddle-point equations which will be our starting point in the next section. For further details, which will not be needed here, we refer to the literature[10].

4. Saddle point equation for the fixed point action in QCD

The saddle point equation and its solution, the fixed point action, depend on the averaging procedure. Let $U(n)_\mu, \psi_n, \bar{\psi}_n$ denote the gauge and fermion fields on a $d = 4$ Euclidean ('fine') lattice whose points are indexed by n. The averaged ('blocked') gauge and fermion fields will be denoted by $V(n_B)_\mu, \chi_{n_B}, \bar{\chi}_{n_B}$. This lattice ('coarse') is indexed by n_B. The gauge fields are $SU(N)$ matrices, the fermion fields are Grassmann variables.

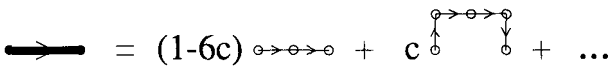

Fig. 4. A simple example for a gauge invariant blocking

The averaging procedure for the gauge variables is coded by an $N \times N$ complex matrix $Q_\mu(n_B)$. It is a gauge covariant average of the fine link variables U in the neighbourhood of the coarse link (n_B, μ), i.e. it is a weighted sum of paths (each path being a product of U matrices) between the points n_B and $n_B + \hat{\mu}$ of the coarse lattice. A simple example for a scale 2 blocking is (see Fig. 4):

$$\begin{aligned} Q_\mu(n_B) = (1-6c)U_\mu(n)U_\mu(n+\hat{\mu}) + \\ c\sum_{\nu\neq\mu}\left[U_\nu(n)U_\mu(n+\hat{\nu})U_\mu(n+\hat{\mu}+\hat{\nu})U_\nu^\dagger(n+2\hat{\mu}) + \right. \\ \left. U_\nu^\dagger(n-\hat{\nu})U_\mu(n-\hat{\nu})U_\mu(n+\hat{\mu}-\hat{\nu})U_\nu(n+2\hat{\mu}-\hat{\nu})\right] , \end{aligned} \tag{5}$$

where c, the relative weight of the staples versus the central link, is a parameter.

The fixed point action has the standard structure

$$A^{FP}(\overline{\chi}, \chi, V) = \mathcal{A}_g^{FP}(V) + \sum_{n_B, n'_B} \overline{\chi}_{n_B} D^{FP}(n_B, n'_B)\chi_{n'_B} , \tag{6}$$

where $\mathcal{A}_g^{FP}(V)$ is the gauge action and D^{FP} is the fixed point Dirac operator. The fixed point gauge action is determined by the equation[10,11]

$$\mathcal{A}_g^{FP}(V) = \min_{\{U\}}\left(\mathcal{A}_g^{FP}(U) - \frac{\kappa_g}{N}\sum_{n_B,\mu}\left[ReTr(V_\mu(n_B)Q_\mu^\dagger(n_B)) - f(Q_\mu(n_B))\right]\right) , \tag{7}$$

where

$$f(Q) = \max_W\left[ReTr(WQ^\dagger)\right], \qquad W \in SU(N) . \tag{8}$$

Given an arbitrary gauge field configuration $\{V\}$ on the coarse lattice one is looking for the value of the FP action (a real number) on this configuration. This number is given by the minimum on the r.h.s. of Eq. (7) with respect to the gauge field configuration U on the fine lattice. The FP action should be local, only such solutions are acceptable. The parameters κ_g and c in Eq. (7) and Eq. (6) can be used to optimise the action being not only local but as compact as possible.

The FP Dirac operator satisfies the equation

$$D^{FP}(V)^{-1}_{n_B,n'_B} = \frac{1}{\kappa_f}\delta_{n_B,n'_B} + b_f^2 \sum_{n,n'} \omega(U_{\min})_{n_B,n} D^{FP}(U_{\min})^{-1}_{n,n'} \omega(U_{\min})^{\dagger}_{n',n'_B} \tag{9}$$

where $U_{\min}$ is the minimising field in Eq. (7), $\omega(U)$ defines the gauge covariant averaging of fermions, while κ_f and b_f are parameters‡. Eqs. (7,9) determine the FP action of QCD.

5. Perfect classical lattice theories

The FP Yang-Mills action and the FP Dirac operator have amazing properties: they are perfect on any physical question in the classical limit even on coarse lattices. So, for example, the classical (Euler-Lagrange) equation obtained from $A_g^{FP}(V)$ has exact, scale invariant instanton solutions on the lattice. The value of the instanton action is the same as in the continuum. This is true even if the instanton size is a few lattice unit only. Another example is the free FP Dirac operator: $D^{FP}(V=1)$. The energy-impulse dispersion relation $E = E(\mathbf{p}) = |\mathbf{p}|$ is exact and $E \in (0,\infty)$ like in the continuum.

For illustration let us prove the statement on instantons. We show first that if the lattice gauge configuration V satisfies the FP classical (Euler-Lagrange) equations $\delta\mathcal{A}_g^{FP}(V)/\delta V = 0$ and V is a local minimum of $\mathcal{A}_g^{FP}(V)$ then the configuration $U_{\min}$ which minimises the r.h.s. of the FP equation Eq. (7) satisfies the FP classical equations as well. In addition, the value of the action is the same: $\mathcal{A}^{FP}(U_{\min}(V)) = \mathcal{A}^{FP}(V)$.

The argument is as follows. In the equation $\delta\mathcal{A}_g^{FP}(V)/\delta V = 0$ the FP action can be replaced by the r.h.s. of the FP equation Eq. (7). The r.h.s. of Eq. (7) depends on V explicitly and also implicitly through the V dependence of $U_{\min}$. This last contribution is zero since $U_{\min}$ is the minimum. The explicit dependence comes from the middle term in the r.h.s. of Eq. (7), therefore this term takes its maximum and so the last terms cancel each other. Therefore $\mathcal{A}_g^{FP}(U)$ takes its minimum at $U_{\min}$, i.e. $U_{\min}$ is a solution of the Euler-Lagrange equation of the action $\mathcal{A}_g^{FP}(U)$ on the fine lattice and

$$\mathcal{A}_g^{FP}(V) = \mathcal{A}_g^{FP}(U_{\min}) \,. \tag{10}$$

‡For simplicity we quoted the FP equation for the inverse of the Dirac operator, which can be used only if D^{FP} has no zero modes.

This result implies that to any solution V with a characteristic size ρ there exists another solution $U_{\min}(V)$ of size 2ρ [§] with the same value of the action, i.e. these are scale invariant solutions. Repeating the argument there exist solutions with scale $2^2\rho, \ldots, 2^k\rho, \ldots$ The very large solutions become arbitrarily smooth and the value of the action is equal to the continuum value. Since this value is independent of ρ, all the solutions have the continuum value.

6. The FP Dirac operator satisfies the Ginsparg-Wilson relation

Nielsen and Ninomiya[1] demonstrated that if the lattice Dirac operator satisfies some basic conditions (locality and massless fermions without doublers), than the standard chiral symmetry relation $D\gamma_5 + \gamma_5 D = 0$ is unavoidably violated. In a following paper Ginsparg and Wilson[12] argued that the correct chiral symmetry relation on the lattice is $D\gamma_5 + \gamma_5 D = D\gamma_5 R D$, where R is an arbitrary local operator, trivial in Dirac space. No Dirac operators were around that time which satisfied this non-linear relation and so this work and the message remained largely unnoticed. More than fifteen years later the FP [13] and soon after the overlap [14] Dirac operators were identified as solutions of the Ginsparg-Wilson relation.

Using the FP equation Eq. (7) it is easy to demonstrate that D^{FP} satisfies the GW relation. Eq. (7) refers to a fine and a coarse lattice whose lattice units differ by a factor of 2^k. Start from a very fine lattice, take k very large and consider the anticommutator of γ_5 with the inverse of FP Dirac operator on the coarse lattice:

$$\{\gamma_5, D^{FP}_{n_B,n'_B}(V)^{-1}\}\,. \tag{11}$$

On the r.h.s. of Eq. (7) the first term gives $2/\kappa_f \delta_{n_B,n'_B}$. In the second term D^{FP} lives on the very fine (for $k \to \infty$, infinitely fine, i.e. continuum) gauge configuration $U_{\min}(V)$. Any legitim Dirac operator goes over the continuum (massless) Dirac operator in this limit. We get then

$$\{\gamma_5, D^{FP}_{n_B,n'_B}(V)^{-1}\} = \frac{2}{\kappa_f}\,, \tag{12}$$

or equivalently

$$\{\gamma_5, D^{FP}(V)\} = \frac{2}{\kappa_f} D^{FP}(V)\gamma_5 D^{FP}(V)\,. \tag{13}$$

[§] We consider a scale=2 RG transformation

If the RG transformation is a factor of 2 coarsing, then the equation for D^{FP} should be iterated to push the fine lattice towards the continuum. In this case one obtains

$$\{\gamma_5, D^{FP}(V)\} = \frac{2}{\kappa_f} D^{FP}(V) 2R\gamma_5 D^{FP}(V)\,, \tag{14}$$

where $2R$ is a local operator and is trivial in Dirac space.

7. Lattice regularization and symmetry transformations

The GW relation implies the existence of an exact chiral symmetry transformation on the lattice[15]. Assume, the lattice Dirac operator satisfies the GW relation

$$\{\gamma_5, D\} = 2D\gamma_5 D\,. \tag{15}$$

Then the lattice fermion action $\bar{\psi}_n D_{n,n'} \psi_{n'}$ is invariant under the modified chiral transformation

$$\begin{aligned} \delta\psi &= \mathrm{i}\epsilon\gamma_5(1 - a\mathcal{D})\psi\,, \\ \delta\overline{\psi} &= \mathrm{i}\epsilon\overline{\psi}(1 - a\mathcal{D})\gamma_5\,, \end{aligned} \tag{16}$$

where, exceptionally, the dependence on the lattice unit a is also indicated. The leading term in the transformation has the standard continuum form, while the $O(a)$ correction depends on the Dirac operator and so on the gauge field also.

Although many of the consequences of the GW relation can be obtained by other methods[16], knowing the symmetry transformation opens the way towards standard powerful techniques like Ward identities derived from the path integral. Although the action is invariant under the transformation in Eq. (16), due to the gauge field dependence there is a non-trivial measure in the case of a $U(1)$ chiral transformation which produces the correct chiral anomaly.

There is a systematic way to derive the GW relation and the related symmetry transformation. The method gives a better intuitive understanding and allows also the generalisation for other symmetries[17]. We shall consider a free fermionic theory but the procedure can be applied to interacting theories also.

Consider free massless fermions in the continuum with the action $\overline{\psi}(x)D(xx')\psi(x')$, where $D_{xx'} = (\gamma^\mu \partial_\mu)_{xx'}$. Put a lattice over the continuum Euclidean space and perform a RG transformation averaging the continuum variables into lattice variables: 'blocking out of continuum'[18].

Since a RG transformation does not change the physical content of the theory, the lattice action (which is actually the fixed point action) inherits all the symmetries of the starting continuum action. We give now a simple procedure to find the lattice symmetry transformations.

The path integral describing this block transformation is Gaussian which is equivalent to a formal minimization problem:

$$\overline{\chi}\mathcal{D}\chi = \min_{\overline{\psi},\psi} \left\{ \overline{\psi}D\psi + \left(\overline{\chi} - \overline{\psi}\omega^{\dagger}\right)\left(\chi - \omega\psi\right) \right\} , \tag{17}$$

where the fermion fields ψ_x and χ_n live in the continuum and on the lattice respectively, ω_{nx} is the blocking matrix, $D_{xx'} = (\gamma^{\mu}\partial_{\mu})_{xx'}$ and $\mathcal{D}_{nn'}$ are the continuum and lattice Dirac operators. For the blocking we take a flat, non-overlapping averaging

$$\omega_{nx} = \begin{cases} 1 & \text{if} \quad x \in \text{block}\, n \,, \\ 0 & \text{otherwise} . \end{cases} \tag{18}$$

With this choice one has $\sum_x \omega_{nx}\omega^{\dagger}_{xn'} = \delta_{nn'}$, i.e. $\omega\omega^{\dagger} = 1$.

The minimising fields $\psi_0 = \psi_0(\chi)$ and $\overline{\psi}_0 = \overline{\psi}_0(\overline{\chi})$ from Eq. (17) are given by

$$\begin{aligned} \psi_0(\chi) &= A^{-1}\omega^{\dagger}\chi, \\ \overline{\psi}_0(\overline{\chi}) &= \overline{\chi}\omega A^{-1} , \end{aligned} \tag{19}$$

where

$$A = D + \omega^{\dagger}\omega. \tag{20}$$

Inserting Eq. (19) into Eq. (17) gives the lattice Dirac operator

$$\mathcal{D} = 1 - \omega A^{-1}\omega^{\dagger} . \tag{21}$$

From the equations above it is easy to derive the following useful relations

$$\begin{aligned} \omega\psi_0(\chi) &= (1-\mathcal{D})\chi , & \overline{\psi}_0(\overline{\chi})\omega^{\dagger} &= \overline{\chi}(1-\mathcal{D}) , \\ D\psi_0(\chi) &= \omega^{\dagger}\mathcal{D}\chi , & \overline{\psi}_0(\overline{\chi})D &= \overline{\chi}\mathcal{D}\omega . \end{aligned} \tag{22}$$

The Ginsparg-Wilson relation can be obtained then from Eq. (21) by using $\{D,\gamma_5\} = 0$ and the relations above¶:

$$\{\mathcal{D},\gamma_5\} = 2\mathcal{D}\gamma_5\mathcal{D} . \tag{23}$$

¶Note that with our choice of the coefficients in Eq. (17) the factor 2 appears in the GW relation. This is, of course, just a convention.

We formulate now a general procedure to find the lattice symmetry *transformation* to any infinitesimal symmetry transformation in the continuum[17].

Let $\delta\psi$ and $\delta\overline{\psi}$ be the change of the corresponding continuum fields under an infinitesimal symmetry transformation which leaves the continuum action $\overline{\psi}D\psi$ invariant.

Define the infinitesimal change of the lattice fields by

$$\delta\chi = \omega\delta\psi_0(\chi)\,, \quad \delta\overline{\chi} = \delta\overline{\psi}_0(\overline{\chi})\omega^\dagger\,. \tag{24}$$

Then the lattice action $\overline{\chi}\mathcal{D}\chi$ is invariant under this infinitesimal transformations.

Rather than detailing the proof[17], let us consider a few examples.

U(1) axial transformation

The standard infinitesimal axial rotation in the continuum reads

$$\delta\psi_0(\chi) = \mathrm{i}\epsilon\gamma_5\psi_0(\chi), \quad \delta\overline{\psi}_0(\overline{\chi}) = \mathrm{i}\epsilon\overline{\psi}_0(\overline{\chi})\gamma_5\,. \tag{25}$$

The corresponding lattice transformation has the form

$$\begin{aligned} \delta\chi &= \mathrm{i}\epsilon\gamma_5\omega\psi_0(\chi) = \mathrm{i}\epsilon\gamma_5(1-\mathcal{D})\chi\,, \\ \delta\overline{\chi} &= \mathrm{i}\epsilon\overline{\psi}_0(\overline{\chi})\gamma_5\omega^\dagger = \mathrm{i}\epsilon\overline{\chi}(1-\mathcal{D})\gamma_5\,, \end{aligned} \tag{26}$$

where we used Eq. (24) and Eq. (22). These transformations have the well known form found by Lüscher [15]. Notice, however that the axial transformation in the continuum is not unique and so the lattice transformation is not unique either.

Infinitesimal translation

In the continuum we have $\delta\psi_0(\chi) = \epsilon\hat{\partial}_\mu\psi_0(\chi)$, $\delta\overline{\psi}_0(\overline{\chi}) = \epsilon\overline{\psi}_0(\overline{\chi})\hat{\partial}_\mu^\dagger$, where $\left(\hat{\partial}_\mu\right)_{xy} = \partial_\mu^x\delta(x-y)$. Our general procedure leads to the lattice transformations

$$\begin{aligned} \delta\chi &= \epsilon\omega\hat{\partial}_\mu\psi_0(\chi)\,, \\ \delta\overline{\chi} &= \epsilon\overline{\psi}_0(\overline{\chi})\hat{\partial}_\mu^\dagger\omega^\dagger\,. \end{aligned} \tag{27}$$

Using $[D, \hat{\partial}_\mu] = 0$ it is a simple exercise to show explicitly that the lattice action is invariant under this infinitesimal translation.

Further examples, also in the presence of interactions, can be found in Ref. 17.

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