Chapter Eight

EIGENVECTORS AND EIGENVALUES

An eigenvector of an $n \times n$ matrix A is a non-zero n-column vector X such that $AX = cX$ for some scalar c, which is called an eigenvalue of A. Thus the effect of left multiplication of an eigenvector by A is merely to multiply it by a scalar, and when $n \leq 3$, a parallel vector is obtained. Similarly, if T is a linear operator on a vector space V, an eigenvector of T is a non-zero vector $\mathbf{v}$ of V such that $T(\mathbf{v}) = c\mathbf{v}$ for some scalar c called an *eigenvalue.* For example, if T is a rotation in $\mathbf{R}^3$, the eigenvectors of T are the non-zero vectors parallel to the axis of rotation and the eigenvalues are all equal to 1.

A large amount of information about a matrix or linear operator is carried by its eigenvectors and eigenvalues. In addition, the theory of eigenvectors and eigenvalues has important applications to systems of linear recurrence relations, Markov processes and systems of linear differential equations. We shall describe the basic theory in the first section and then we give applications in the following two sections of the chapter.

8.1 Basic Theory of Eigenvectors and Eigenvalues

We begin with the fundamental definition. Let A be an $n \times n$ matrix over a field of scalars F. An *eigenvector* of A is a non-zero n-column vector X over F such that

$$AX = cX$$

for some scalar c in F; the scalar c is then referred to as the *eigenvalue* of A associated with the eigenvector X.

In order to clarify the definition and illustrate the technique for finding eigenvectors and eigenvalues, an example will be worked out in detail.

Example 8.1.1

Consider the real 2×2 matrix

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}.$$

The condition for the vector

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

to be an eigenvector of A is that $AX = cX$ for some scalar c. This is equivalent to $(A - cI_2)X = 0$, which simply asserts that X is a solution of the linear system

$$\begin{pmatrix} 2-c & -1 \\ 2 & 4-c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now by 3.3.2 this linear system will have a non-trivial solution x_1, x_2 if and only if the determinant of the coefficient matrix vanishes,

$$\begin{vmatrix} 2-c & -1 \\ 2 & 4-c \end{vmatrix} = 0,$$

that is, $c^2 - 6c + 10 = 0$. The roots of this quadratic equation are $c_1 = 3 + \sqrt{-1}$ and $c_2 = 3 - \sqrt{-1}$, so these are the eigenvalues of A.

The eigenvectors for each eigenvalue are found by solving the linear systems $(A - c_1 I_2)X = 0$ and $(A - c_2 I_2)X = 0$. For example, in the case of c_1 we have to solve

$$\begin{cases} (-1-\sqrt{-1})x_1 - \phantom{(1-\sqrt{-1})}x_2 = 0 \\ \phantom{(-1-\sqrt{-1})}2x_1 + (1-\sqrt{-1})x_2 = 0 \end{cases}$$

The general solution of this system is $x_1 = \frac{d}{2}(-1 + \sqrt{-1})$ and $x_2 = d$, where d is an arbitrary scalar. Thus the eigenvectors of A associated with the eigenvalue c_1 are the non-zero vectors of the form

$$d \begin{pmatrix} (-1 + \sqrt{-1})/2 \\ 1 \end{pmatrix}.$$

Notice that these, together with the zero vector, form a 1-dimensional subspace of $\mathbf{C}^2$. In a similar manner the eigenvectors for the eigenvalue $3 - \sqrt{-1}$ are found to be the vectors of the form

$$d \begin{pmatrix} -(1 + \sqrt{-1})/2 \\ 1 \end{pmatrix}$$

where $d \neq 0$. Again these form with the zero vector a subspace of $\mathbf{C}^2$.

It should be clear to the reader that the method used in this example is in fact a general procedure for finding eigenvectors and eigenvalues. This will now be described in detail.

The characteristic equation of a matrix

Let A be an $n \times n$ matrix over a field of scalars F, and let X be a non-zero n-column vector over F. The condition for X to be an eigenvector of A is $AX = cX$, or

$$(A - cI_n)X = 0,$$

where c is the corresponding eigenvalue. Hence the eigenvectors associated with c, together with the zero vector, form the null space of the matrix $A - cI_n$. This subspace is often referred to as the *eigenspace* of the eigenvalue c.

Now $(A - cI_n)X = 0$ is a linear system of n equations in n unknowns. By 3.3.2 the condition for there to be a non-trivial solution of the system is that the coefficient matrix have zero determinant,

$$\det(A - cI_n) = 0.$$

Conversely, if the scalar c satisfies this equation, there will be a non-zero solution of the system and c will be an eigenvalue. These considerations already make it clear that the determinant

$$\det(A - xI_n) = \begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x \end{vmatrix}$$

must play an important role. This is a polynomial of degree n in x which is called the *characteristic polynomial* of A. The equation obtained by setting the characteristic polynomial equal to zero is the *characteristic equation.* Thus the eigenvalues of A are the roots of the characteristic equation (or characteristic polynomial) which lie in the field F.

At this point it is necessary to point out that A may well have no eigenvalues in F. For example, the characteristic polynomial of the real matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is $x^2 + 1$, which has no real roots, so the matrix has no eigenvalues in $\mathbf{R}$.

However, if A is a complex $n \times n$ matrix, its characteristic equation will have n complex roots, some of which may be equal. The reason for this is a well-known result known as *The Fundamental Theorem of Algebra*; it asserts that every polynomial f of positive degree n with complex coefficients can be expressed as a product of n linear factors; thus the equation $f(x) = 0$ has exactly n roots in $\mathbf{C}$. Because of this we can be sure that complex matrices always have all their eigenvalues and eigenvectors in $\mathbf{C}$. It is this case that principally concerns us here.

Let us sum up our conclusions about the eigenvalues of complex matrices so far.

Theorem 8.1.1
Let A be an $n \times n$ complex matrix.

(i) *The eigenvalues of A are precisely the n roots of the characteristic polynomial* $\det(A - xI_n)$;

(ii) *the eigenvectors of A associated with an eigenvalue c are the non-zero vectors in the null space of the matrix $A - cI_n$.*

Thus in Example 8.1.1 the characteristic polynomial of the matrix is

$$\begin{vmatrix} 2-x & -1 \\ 2 & 4-x \end{vmatrix} = x^2 - 6x + 10.$$

The eigenvalues are the roots of the characteristic equation $x^2 - 6x + 10 = 0$, that is, $c_1 = 3 + \sqrt{-1}$ and $c_2 = 3 - \sqrt{-1}$; the eigenspaces of c_1 and c_2 are generated by the vectors

$$\begin{pmatrix} (-1+\sqrt{-1})/2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -(1+\sqrt{-1})/2 \\ 1 \end{pmatrix}$$

respectively.

Example 8.1.2
Find the eigenvalues of the upper triangular matrix

$$\begin{pmatrix} a_{11}-x & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-x & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_{nn}-x \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{vmatrix} a_{11}-x & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-x & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_{nn}-x \end{vmatrix},$$

which, by 3.1.5, equals $(a_{11} - x)(a_{22} - x)\ \ldots\ (a_{nn} - x)$. The eigenvalues of the matrix are therefore just the diagonal entries $a_{11},\ a_{22}, \ldots, a_{nn}$.

Example 8.1.3
Consider the 3×3 matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{vmatrix} 2-x & -1 & -1 \\ -1 & 2-x & -1 \\ -1 & -1 & -x \end{vmatrix} = -x^3 + 4x^2 - x - 6.$$

Fortunately one can guess a root of this cubic polynomial, namely $x = -1$. Dividing the polynomial by $x + 1$ using long division, we obtain the quotient $-x^2 + 5x - 6 = -(x-2)(x-3)$. Hence the characteristic polynomial can be factorized completely as $-(x+1)(x-2)(x-3)$, and the eigenvalues of A are $-1, 2$ and 3.

To find the corresponding eigenvectors, we have to solve the three linear systems $(A + I_3)X = 0$, $(A - 2I_3)X = 0$ and $(A - 3I_3)X = 0$. On solving these, we find that the respective eigenvectors are the non-zero scalar multiples of the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The eigenspaces are generated by these three vectors and so each has dimension 1.

Properties of the characteristic polynomial

Now let us see what can be said in general about the characteristic polynomial of an $n \times n$ matrix A. Let $p(x)$ denote this polynomial; thus

$$p(x) = \begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x \end{vmatrix}.$$

At this point we need to recall the definition of a determinant as an alternating sum of terms, each term being a product of entries, one from each row and column. The term of $p(x)$ with highest degree in x arises from the product

$$(a_{11} - x) \cdots (a_{nn} - x)$$

and is clearly $(-x)^n$. The terms of degree $n-1$ are also easy to locate since they arise from the same product. Thus the coefficient of x^{n-1} is

$$(-1)^{n-1}(a_{11} + \cdots + a_{nn})$$

and the sum of the diagonal entries of A is seen to have significance; it is given a special name, the *trace* of A,

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

The term in $p(x)$ of degree $n-1$ is therefore $\mathrm{tr}(A)\ (-x)^{n-1}$.

The constant term in $p(x)$ may be found by simply putting $x = 0$ in $p(x) = \det(A - xI_n)$, thereby leaving $\det(A)$. Our knowledge of $p(x)$ so far is summarized in the formula

$$p(x) = (-x)^n + \mathrm{tr}(A)(-x)^{n-1} + \cdots + \det(A).$$

The other coefficients in the characteristic polynomial are not so easy to describe, but they are in fact expressible as subdeterminants of $\det(A)$. For example, take the case of x^{n-2}. Now terms in x^{n-2} arise in two ways: from the product $(a_{11} - x) \cdots (a_{nn} - x)$ or from products like

$$-a_{12}a_{21}(a_{33} - x) \cdots (a_{nn} - x).$$

So a typical contribution to the coefficient of x^{n-2} is

$$(-1)^{n-2}(a_{11}a_{22} - a_{12}a_{21}) = (-1)^{n-2} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

From this it is clear that the term of degree $n-2$ in $p(x)$ is just $(-x)^{n-2}$ times the sum of all the 2×2 determinants of the form

$$\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

where $i < j$.

In general one can prove by similar considerations that the following is true.

Theorem 8.1.2
The characteristic polynomial of the $n \times n$ matrix A equals

$$\sum_{i=0}^{n} d_i(-x)^{n-i}$$

where d_i is the sum of all the $i \times i$ subdeterminants of det(A) whose principal diagonals are part of the principal diagonal of A.

Now assume that the matrix A has complex entries. Let $c_1, c_2, \ldots, c_n$ be the eigenvalues of A. These are the n roots of the characteristic polynomial $p(x)$. Therefore, allowing for the fact that the term of $p(x)$ with highest degree has coefficient $(-1)^n$, one has

$$p(x) = (c_1 - x)(c_2 - x) \cdots (c_n - x).$$

The constant term in this product is evidently just $c_1 c_2 \dots c_n$, while the term in x^{n-1} has coefficient $(-1)^{n-1}(c_1 + \cdots + c_n)$. On the other hand, we previously found these to be $\det(A)$ and $(-1)^{n-1}\mathrm{tr}(A)$ respectively. Thus we arrive at two important relations between the eigenvalues and the entries of A.

Corollary 8.1.3

If A is any complex square matrix, the product of the eigenvalues equals the determinant of A and the sum of the eigenvalues equals the trace of A

Recall from Chapter Six that matrices A and B are said to be similar if there is an invertible matrix S such that $B = SAS^{-1}$. The next result indicates that similar matrices have much in common, and really deserve their name.

Theorem 8.1.4

Similar matrices have the same characteristic polynomial and hence they have the same eigenvalues, trace and determinant.

Proof

The characteristic polynomial of $B = SAS^{-1}$ is

$$\begin{aligned}\det(SAS^{-1} - xI) &= \det(S(A - xI)S^{-1}) \\ &= \det(S)\det(A - xI)\det(S)^{-1} \\ &= \det(A - xI).\end{aligned}$$

Here we have used two fundamental properties of determinants established in Chapter Three, namely 3.3.3 and 3.3.5. The statements about trace and determinant now follow from 8.1.3.

On the other hand, one cannot expect similar matrices to have the same eigenvectors. Indeed the condition for X to be an eigenvector of SAS^{-1} with eigenvalue c is $(SAS^{-1})X = cX$, which is equivalent to $A(S^{-1}X) = c(S^{-1}X)$. Thus X is an eigenvector of SAS^{-1} if and only if $S^{-1}X$ is an eigenvector of A.

Eigenvectors and eigenvalues of linear transformations

Because of the close relationship between square matrices and linear operators on finite-dimensional vector spaces observed in Chapter Six, it is not surprising that one can also define eigenvectors and eigenvalues for a linear operator.

Let $T : V \to V$ be a linear operator on a vector space V over a field of scalars F. An *eigenvector* of T is a non-zero vector $\mathbf{v}$ of V such that $T(\mathbf{v}) = c\mathbf{v}$ for some scalar c in F: here c is the *eigenvalue* of T associated with the eigenvector $\mathbf{v}$.

Suppose now that V is a finite-dimensional vector space over F with dimension n. Choose an ordered basis for V, say $\mathcal{B}$. Then with respect to this ordered basis T is represented by an $n \times n$ matrix over F, say A; this means that

$$[T(\mathbf{v})]_{\mathcal{B}} = A[\mathbf{v}]_{\mathcal{B}}.$$

Here $[\mathbf{u}]_{\mathcal{B}}$ is the coordinate column vector of a vector $\mathbf{u}$ in V with respect to basis $\mathcal{B}$. The condition $T(\mathbf{v}) = c\mathbf{v}$ for $\mathbf{v}$ to be an eigenvector of T with associated eigenvalue c, becomes $A[\mathbf{v}]_{\mathcal{B}} = c[\mathbf{v}]_{\mathcal{B}}$, which is just the condition for $[\mathbf{v}]_{\mathcal{B}}$ to be an eigenvector of the representing matrix A; also the eigenvalues of T and A are the same.

If the ordered basis of V is changed, the effect is to replace A by a similar matrix. Of course any such matrix will have the same eigenvalues as T; thus we have another proof of the fact that similar matrices have the same eigenvalues.

These observations permit us to carry over to linear operators concepts such as characteristic polynomial and trace, which were introduced for matrices.

Example 8.1.4
Consider the linear transformation $T : D_\infty[a,b] \to D_\infty[a,b]$ where $T(f) = f'$, the derivative of the function f. The condition for f to be an eigenvector of T is $f' = cf$ for some constant c. The general solution of this simple differential equation is $f = de^{cx}$ where d is a constant. Thus the eigenvalues of T are all real numbers c, while the eigenvectors are the exponential functions de^{cx} with $d \neq 0$.

Diagonalizable matrices

We wish now to consider the question: when is a square matrix similar to a diagonal matrix? In the first place, why is this an interesting question? The essential reason is that diagonal matrices behave so much more simply than arbitrary matrices. For example, when a diagonal matrix is raised to the nth power, the effect is merely to raise each element on the diagonal to the nth power, whereas there is no simple expression for the nth power of an arbitrary matrix. Suppose that we want to compute A^n where A is similar to a diagonal matrix D, with say $A = SDS^{-1}$. It is easily seen that $A^n = SD^nS^{-1}$. Thus it is possible to calculate A^n quite simply if we have explicit knowledge of S and D. It will emerge in 8.2 and 8.3 that this provides the basis for effective methods of solving systems of linear recurrences and linear differential equations.

Now for the important definition. Let A be a square matrix over a field F. Then A is said to be *diagonalizable* over F if it is similar to a diagonal matrix D over F, that is, there is an invertible matrix S over F such that $A = SDS^{-1}$ or equivalently, $D = S^{-1}AS$. One also says that S *diagonalizes* A. A diagonalizable matrix need not be diagonal: the reader should give an example to demonstrate this. It is an important observation that if A is diagonalizable and its eigenvalues are $c_1, \dots, c_n$, then A must be similar to the diagonal matrix with $c_1, \dots, c_n$ on the principal diagonal. This is because similar matrices have the same eigenvalues and the eigenvalues of a diagonal matrix are just the entries on the principal diagonal – see Example 8.1.2.

What we are aiming for is a criterion which will tell us exactly which matrices are diagonalizable. A key step in the search for this criterion comes next.

Theorem 8.1.5
Let A be an $n \times n$ matrix over a field F and let $c_1, \dots, c_r$ be distinct eigenvalues of A with associated eigenvectors $X_1, \dots, X_r$. Then $\{X_1, \dots, X_r\}$ is a linearly independent subset of F^n.

Proof
Assume the theorem is false; then there is a positive integer i such that $\{X_1, \dots, X_i\}$ is linearly independent, but the addition of the next vector X_{i+1} produces a linearly *dependent* set $\{X_1, \dots, X_{i+1}\}$. So there are scalars $d_1, \dots, d_{i+1}$, not all of them zero, such that

$$d_1X_1 + \cdots + d_{i+1}X_{i+1} = 0.$$

Premultiply both sides of this equation by A and use the equations $AX_j = c_jX_j$ to get

$$c_1d_1X_1 + \cdots + c_{i+1}d_{i+1}X_{i+1} = 0.$$

On subtracting c_{i+1} times the first equation from the second, we arrive at the relation

$$(c_1 - c_{i+1})d_1X_1 + \cdots + (c_i - c_{i+1})d_iX_i = 0.$$

Since $X_1, \ldots, X_i$ are linearly independent, all the coefficients $(c_j - c_{i+1})d_j$ must vanish. But $c_1, \ldots, c_{i+1}$ are all different, so we can conclude that $d_j = 0$ for $j = 1, \ldots, i$; hence $d_{i+1}X_{i+1} = 0$ and so $d_{i+1} = 0$, in contradiction to the original assumption. Therefore the statement of the theorem must be correct.

The criterion for diagonalizability can now be established.

Theorem 8.1.6

Let A be an $n \times n$ matrix over a field F. Then A is diagonalizable if and only if A has n linearly independent eigenvectors in F^n.

Proof

First of all suppose that A has n linearly independent eigenvectors in F^n, say $X_1, \ldots, X_n$, and that the associated eigenvalues are $c_1, \ldots, c_n$. Define S to be the $n \times n$ matrix whose columns are the eigenvectors; thus

$$S = (X_1 \ldots X_n).$$

The first thing to notice is that S is invertible; for by 8.1.5 its columns are linearly independent. Forming the product of A and S in partitioned form, we find that

$$AS = (AX_1 \ldots AX_n) = (c_1X_1 \ \cdots \ c_nX_n),$$

which equals

$$(X_1 \ \ldots \ X_n) \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & c_n \end{pmatrix} = SD,$$

where D is the diagonal matrix with entries $c_1, \ldots, c_n$. Therefore $S^{-1}AS = D$ and A is diagonalizable.

Conversely, assume that A is diagonalizable and that $S^{-1}AS = D$ is a diagonal matrix with entries $c_1, \dots, c_n$. Then $AS = SD$. This implies that if X_i is the ith column of S, then AX_i equals the ith column of SD, which is c_iX_i. Hence $X_1, \dots, X_n$ are eigenvectors of A associated with eigenvalues $c_1, \dots, c_n$. Since $X_1, \dots, X_n$ are columns of the invertible matrix S, they must be linearly independent. Consequently A has n linearly independent eigenvectors.

Corollary 8.1.7

An $n \times n$ complex matrix which has n distinct eigenvalues is diagonalizable.

This follows at once from 8.1.5 and 8.1.6. On the other hand, it is easy to think of matrices which are not diagonalizable: for example, there is the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Indeed if A were diagonalizable, it would be similar to the identity matrix I_2 since both its eigenvalues equal 1, and $S^{-1}AS = I_2$ for some S; but the last equation implies that $A = SI_2S^{-1} = I_2$, which is not true.

An interesting feature of the proof of 8.1.6 is that it provides us with a method of finding a matrix S which diagonalizes A. One has simply to find a set of linearly independent eigenvectors of A; if there are enough of them, they can be taken to form the columns of the matrix S.

Example 8.1.5

Find a matrix which diagonalizes $A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$.

In Example 8.1.1 we found the eigenvalues of A to be $3+\sqrt{-1}$ and $3-\sqrt{-1}$; hence A is diagonalizable by 8.1.7. We also found eigenvectors for A; these form a matrix

$$S = \begin{pmatrix} (-1+\sqrt{-1})/2 & -(1+\sqrt{-1})/2 \\ 1 & 1 \end{pmatrix}.$$

Then by the preceding theory we may be sure that

$$S^{-1}AS = \begin{pmatrix} 3+\sqrt{-1} & 0 \\ 0 & 3-\sqrt{-1} \end{pmatrix}.$$

Triangularizable matrices

It has been seen that not every complex square matrix is diagonalizable. Compensating for this failure is the fact such a matrix is always similar to an upper triangular matrix; this is a result with many applications.

Let A be a square matrix over a field F. Then A is said to be *triangularizable over* F if there is an invertible matrix S over F such that $S^{-1}AS = T$ is upper triangular. It will also be convenient to say that S *triangularizes* A. Note that the diagonal entries of the triangular matrix T will necessarily be the eigenvalues of A. This is because of Example 8.1.2 and the fact that similar matrices have the same eigenvalues. Thus a necessary condition for A to be triangularizable is that it have n eigenvalues in the field F. When $F = \mathbf{C}$, this condition is always satisfied, and this is the case in which we are interested.

Theorem 8.1.8
Every complex square matrix is triangularizable.

Proof
Let A denote an $n \times n$ complex matrix. We show by induction on n that A is triangularizable. Of course, if $n = 1$, then A is already upper triangular: let $n > 1$. We shall use induction on n and assume that the result is true for square matrices with $n - 1$ rows.

We know that A has at least one eigenvalue c in $\mathbf{C}$, with associated eigenvector X say. Since $X \neq 0$, it is possible to adjoin vectors to X to produce a basis of $\mathbf{C}^n$, say $X = X_1, X_2, \dots, X_n$; here we have used 5.1.4. Next, recall that left multiplication of the vectors of $\mathbf{C}^n$ by A gives rise to linear operator T on $\mathbf{C}^n$. With respect to the basis $\{X_1, \dots, X_n\}$, the linear operator T will be represented by a matrix with the special form

$$B_1 = \begin{pmatrix} c & A_2 \\ 0 & A_1 \end{pmatrix}$$

where A_1 and A_2 are certain complex matrices, A_1 having $n-1$ rows and columns. The reason for the special form is that $T(X_1) = AX_1 = cX_1$ since X_1 is an eigenvalue of A. Notice that the matrices A and B_1 are similar since they represent the same linear operator T; suppose that in fact $B_1 = S_1^{-1}AS_1$ where S_1 is an invertible $n \times n$ matrix.

Now by induction hypothesis there is an invertible matrix S_2 with $n-1$ rows and columns such that $B_2 = S_2^{-1}A_1S_2$ is upper triangular. Write

$$S = S_1 \begin{pmatrix} 1 & 0 \\ 0 & S_2 \end{pmatrix}.$$

This is a product of invertible matrices, so it is invertible. An easy matrix computation shows that $S^{-1}AS$ equals

$$\begin{pmatrix} 1 & 0 \\ 0 & S_2^{-1} \end{pmatrix} (S_1^{-1}AS_1) \begin{pmatrix} 1 & 0 \\ 0 & S_2 \end{pmatrix},$$

which equals

$$\begin{pmatrix} 1 & 0 \\ 0 & S_2^{-1} \end{pmatrix} B_1 \begin{pmatrix} 1 & 0 \\ 0 & S_2 \end{pmatrix}.$$

Replace B_1 by $\begin{pmatrix} c & A_2 \\ 0 & A_1 \end{pmatrix}$ and multiply the matrices together to get

$$S^{-1}AS = \begin{pmatrix} c & A_2S_2 \\ 0 & S_2^{-1}A_1S_2 \end{pmatrix} = \begin{pmatrix} c & A_2S_2 \\ 0 & B_2 \end{pmatrix}.$$

This matrix is clearly upper triangular, so the theorem is proved.

The proof of the theorem provides a method for triangularizing a matrix.

Example 8.1.6

Triangularize the matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

The characteristic polynomial of A is $x^2 - 4x + 4$, so both eigenvalues equal 2. Solving $(A - 2I_2)X = 0$, we find that all the eigenvectors of A are scalar multiples of $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence A is not diagonalizable by 8.1.6.

Let T be the linear operator on $\mathbf{C}^2$ arising from left multiplication by A. Adjoin a vector to X_2 to X_1 to get a basis $\mathcal{B}_2 = \{X_1,\ X_2\}$ of $\mathbf{C}^2$, say $X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Denote by $\mathcal{B}_1$ the standard basis of $\mathbf{C}^2$. Then the change of basis $\mathcal{B}_1 \rightarrow \mathcal{B}_2$ is described by the matrix $S_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Therefore by 6.2.6 the matrix A which represents T with respect to the basis $\mathcal{B}_2$ is

$$S_1AS_1^{-1} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Hence $S = S_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ triangularizes A.

Exercises 8.1

1. Find all the eigenvectors and eigenvalues of the following matrices:

$$\begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}; \quad \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}; \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 4 \end{pmatrix}.$$

2. Prove that $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$ and $\operatorname{tr}(cA) = c \operatorname{tr}(A)$ where A and B are $n \times n$ matrices and c is a scalar.

3. If A and B are $n \times n$ matrices, show that AB and BA have the same eigenvalues. [Hint: let c be an eigenvalue of AB and prove that it is an eigenvalue of BA].

4. Suppose that A is a square matrix with real entries and real eigenvalues. Prove that every eigenvalue of A has an associated *real* eigenvector.

5. If A is a real matrix with distinct eigenvalues, then A is diagonalizable over **R**: true or false?

6. Let $p(x)$ be the polynomial

$$(-1)^n(x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0).$$

Show that $p(x)$ is the characteristic polynomial of the following matrix (which is called the *companion matrix* of $p(x)$):

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}.$$

7. Find matrices which diagonalize the following:

(a) $\begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$.

8. For which values of a and b is the matrix $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ diagonalizable over $\mathbf{C}$?

9. Prove that a complex 2×2 matrix is *not* diagonalizable if and only if it is similar to a matrix of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ where $b \neq 0$.

10. Let A be a diagonalizable matrix and assume that S is a matrix which diagonalizes A. Prove that a matrix T diagonalizes A if and only if it is of the form $T = CS$ where C is a matrix such that $AC = CA$.

11. If A is an invertible matrix with eigenvalues $c_1, \ldots, c_n$, show that the eigenvalues of A^{-1} are $c_1^{-1}, \ldots, c_n^{-1}$.

12. Let $T : V \to V$ be a linear operator on a complex n-dimensional vector space V. Prove that there is a basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ of V such that $T(\mathbf{v}_i)$ is a linear combination of $\mathbf{v}_i, \ldots, \mathbf{v}_n$ for $i = 1, \ldots, n$.

13. Let $T : P_n(\mathbf{R}) \to P_n(\mathbf{R})$ be the linear operator corresponding to differentiation. Show that all the eigenvalues of T are zero. What are the eigenvectors?

14. Let $c_1, \ldots, c_n$ be the eigenvalues of a complex matrix A. Prove that the eigenvalues of A^m are $c_1^m, \ldots, c_n^m$ where m is any positive integer. [Hint: A is triangularizable].

15. Prove that a square matrix and its transpose have the same eigenvalues.

8.2 Applications to Systems of Linear Recurrences

A *recurrence relation* is an equation involving a function y of a non-negative integral variable n, the value of y at n being written y_n. The equation relates the values of the function at certain consecutive integers, typically $y_{n+1}, y_n, \ldots, y_{n-r}$. In addition there may be some initial conditions to be satisfied, which specify certain values of y_i. If the equation is linear in y, the recurrence relation is said to be *linear*. The problem is to solve the recurrence, that is, to find the most general function which satisfies the equation and the initial conditions. Linear recurrence relations, and more generally systems of linear recurrence relations, occur in many real-life problems. We shall see that the theory of eigenvalues provides an effective means for solving such problems.

To understand how systems of linear relations can arise we consider a predator-prey problem.

Example 8.2.1

In a population of rabbits and weasels it is observed that each year the number of rabbits is equal to four times the number of rabbits less twice the number of weasels in the previous year. The number of weasels in any year equals the sum of the numbers of rabbits and weasels in the previous year. If the initial numbers of rabbits and weasels were 100 and 10 respectively, find the numbers of each species after n years.

Let r_n and w_n denote the respective numbers of rabbits and weasels after n years. The information given in the statement of the problem translates into the equations

$$\begin{cases} r_{n+1} = 4r_n - 2w_n \\ w_{n+1} = r_n + w_n \end{cases}$$

together with the initial conditions $r_0 = 100$, $w_0 = 10$. Thus we have to solve a system of two linear recurrence relations for r_n and w_n, subject to two initial conditions.

At first sight it may not seem clear how eigenvalues enter into this problem. However, let us put the system of linear recurrences in matrix form by writing

$$X_n = \begin{pmatrix} r_n \\ w_n \end{pmatrix} \text{ and } A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}.$$

Then the two recurrences are equivalent to the single matrix equation

$$X_{n+1} = AX_n,$$

while the initial conditions assert that

$$X_0 = \begin{pmatrix} 100 \\ 10 \end{pmatrix}.$$

These equations enable us to calculate successive vectors X_n; thus $X_1 = AX_0$, $X_2 = A^2X_0$, and in general

$$X_n = A^nX_0.$$

In principle this equation provides the solution of our problem. However the equation is difficult to use since it involves calculating powers of A; these soon become very complicated and there is no obvious formula for A^n.

The key observation is that powers of a *diagonal* matrix are easy to compute; one simply forms the appropriate power of each diagonal element. Fortunately the matrix A is diagonalizable since it has distinct eigenvalues 2 and 3. Corresponding eigenvectors are found to be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; therefore the matrix $S = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ diagonalizes A, and

$$D = S^{-1}AS = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

It is now easy to find X_n; for $A^n = (SDS^{-1})^n = SD^nS^{-1}$. Therefore

$$\begin{aligned} X_n = A^n X_0 &= SD^nS^{-1}X_0 \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 100 \\ 10 \end{pmatrix}, \end{aligned}$$

which leads to

$$X_n = \begin{pmatrix} 180 \cdot 3^n - 80 \cdot 2^n \\ 90 \cdot 3^n - 80 \cdot 2^n \end{pmatrix}.$$

The solution to the problem can now be read off:

$$r_n = 180 \cdot 3^n - 80 \cdot 2^n \text{ and } w_n = 90 \cdot 3^n - 80 \cdot 2^n.$$

Let us consider for a moment the implications of these equations. Notice that r_n and w_n both increase without limit as $n \to \infty$ since 3^n is the dominant term; however

$$\lim_{n\to\infty} \left(\frac{r_n}{w_n}\right) = 2.$$

The conclusion is that, while both populations explode, in the long run there will be twice as many rabbits as weasels.

Having seen that eigenvalues provide a satisfactory solution to the rabbit-weasel problem, we proceed to consider systems of linear recurrences in general.

Systems of first order linear recurrence relations

A *system of first order (homogeneous) linear recurrence relations* in functions $y_n^{(1)}, \dots, y_n^{(m)}$ of an integral variable n is a set of equations of the form

$$\begin{cases} y_{n+1}^{(1)} &= a_{11}y_n^{(1)} & + & \cdots & + & a_{1m}y_n^{(m)} \\ y_{n+1}^{(2)} &= a_{21}y_n^{(1)} & + & \cdots & + & a_{2m}y_n^{(m)} \\ \cdot & \cdot & & \cdots & & \cdot \\ y_{n+1}^{(m)} &= a_{m1}y_n^{(1)} & + & \cdots & + & a_{mm}y_n^{(m)} \end{cases}$$

We shall only consider the case where the coefficients a_{ij} are constants. One objective might be to find all the functions $y_n^{(1)}, \dots, y_n^{(m)}$ which satisfy the equations of the system, i.e., the general solution. Alternatively, one might want to find a solution which satisfies certain given conditions,

$$y_0^{(1)} = b_1,\ y_0^{(2)} = b_2,\ \dots,\ y_0^{(m)} = b_m$$

where $b_1, \dots, b_m$ are constants. Clearly the rabbit and weasel problem is of this type.

The method adopted in Example 8.2.1 can be applied with advantage to the general case. First convert the given system of recurrences to matrix form byintroducing the matrix $A = [a_{ij}]_{m,m}$, the *coefficient matrix*, and defining

$$Y_n = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \\ \vdots \\ y_n^{(m)} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then the system of recurrences becomes simply

$$Y_{n+1} = AY_n,$$

with the initial condition $Y_0 = B$. The general solution of this is

$$Y_n = A^n B_0.$$

Now assume that A is diagonalizable: suppose that in fact $D = S^{-1}AS$ is diagonal with diagonal entries $d_1, \dots, d_m$. Then $A = SDS^{-1}$ and $A^n = SD^nS^{-1}$, so that

$$Y_n = SD^nS^{-1}B.$$

Here of course D^n is the diagonal matrix with entries

${d_1}^n, {d_2}^n \dots, {d_m}^n$. Since we know how to find S and D, all we need do is compute the product Y_n, and read off its entries to obtain the functions ${y_n}^{(1)}, \dots, {y_n}^{(m)}$.

At this point the reader may ask: what if A is not diagonalizable? A complete discussion of this case would take us too far afield. However one possible approach is to exploit the fact that the coefficient matrix A is certainly triangularizable by 8.1.8. Thus we can find S such that $S^{-1}AS = T$ is upper triangular. Now write $U_n = S^{-1}Y_n$, so that $Y_n = SU_n$. Then the recurrence $Y_{n+1} = AY_n$ becomes $SU_{n+1} = ASU_n$, or $U_{n+1} = (S^{-1}AS)U_n = TU_n$. In principle this "triangular" system of recurrence relations can be solved by a process of back substitution: first solve the last recurrence for $u_n^{(m)}$, then substitute for $u_n^{(m)}$ in the second last recurrence and solve for $u_n^{(m-1)}$, and so on. What makes the procedure effective is the fact that powers of a triangular matrix are easier to compute than those of an arbitrary matrix.

Example 8.2.2

Consider the system of linear recurrences

$$\begin{cases} y_{n+1} &= y_n + z_n \\ z_{n+1} &= -y_n + 3z_n \end{cases}$$

The coefficient matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ is not diagonalizable, but it was triangularized in Example 8.1.6; there it was found that

$$T = S^{-1}AS = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ where } S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Put $U_n = S^{-1}Y_n$; here the entries of U_n and Y_n are written u_n, v_n and y_n, z_n respectively. The recurrence relation $Y_{n+1} = AY_n$ becomes $U_{n+1} = TU_n$. This system of linear recurrences is in triangular form:

$$\begin{cases} u_{n+1} &= 2u_n \quad + v_n \\ v_{n+1} &= \qquad\quad 2v_n \end{cases}$$

The second recurrence has the obvious solution $v_n = d_2 2^n$ with d_2 constant. Substitute for v_n in the first equation to get $u_{n+1} = 2u_n + d_2 2^n$. This recurrence can be solved in a simple-minded fashion by calculating successively $u_1, u_2, \ldots$ and looking for the pattern. It turns out that $u_n = d_1 2^n + d_2 n\, 2^{n-1}$ where d_1 is another constant. Finally, y_n and z_n can be found from the equation $Y_n = SU_n$; the general solution is therefore

$$\begin{cases} y_n = d_1 2^n + d_2 n 2^{n-1} \\ z_n = d_1 2^n + d_2 (n+2) 2^{n-1} \end{cases}$$

Higher order recurrence relations

A system of recurrence relations for $y_n^{(1)}, \ldots, y_n^{(m)}$ which expresses each $y_{n+1}^{(i)}$ in terms of the $y_j^{(k)}$ for $j = n-r+1, \ldots, n$, is said to be of *order* r. When $r \geq 2$, such a system can be converted into a first order system by introducing more unknowns. The method works well even for a single recurrence relation, as the next example shows.

Example 8.2.3 (*The Fibonacci sequence*)
The sequence of integers 0, 1, 1, 2, 3, 5,. .. is generated by adding pairs of consecutive terms to get the next term. Thus, if the terms are written $y_0, y_1, y_2, \ldots$, then y_n satisfies

$$y_{n+1} = y_n + y_{n-1}, \quad n \geq 1,$$

which is a second order recurrence relation.

To convert this into a first order system we introduce the new function $z_n = y_{n-1}$, $(n \geq 1)$. This results in an equivalent system of first order recurrences

$$\begin{cases} y_{n+1} = y_n + z_n \\ z_{n+1} = y_n \end{cases}$$

with initial conditions $y_0 = 0$ and $z_0 = 1$. The coefficient matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, so it is diagonalizable. Diagonalizing A as in Example 8.1.5, we find that

$$D = S^{-1}AS = \begin{pmatrix} (1+\sqrt{5})/2 & 0 \\ 0 & (1-\sqrt{5})/2 \end{pmatrix}$$

where

$$S = \begin{pmatrix} (1+\sqrt{5})/2 & (1-\sqrt{5})/2 \\ 1 & 1 \end{pmatrix}.$$

Then $Y_n = A^n Y_0 = (SDS^{-1})^n Y_0 = SD^n S^{-1} Y_0$. This yields the rather unexpected formula

$$y_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}$$

for the $(n+1)$th Fibonacci number.

Markov processes

In order to motivate the concept of a Markov process, we consider a problem about population movement.

Example 8.2.4

Each year 10% of the population of California leave the state for some other part of the United States, while 20% of the U.S. population outside California enter the state. Assuming a constant total population of the country, what will the ultimate population distribution be?

Let y_n and z_n be the numbers of people inside and outside California after n years; then the information given translates into the system of linear recurrences

$$\begin{cases} y_{n+1} = .9y_n + .2z_n \\ z_{n+1} = .1y_n + .8z_n \end{cases}$$

Writing

$$X_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix} \text{ and } A = \begin{pmatrix} .9 & .2 \\ .1 & .8 \end{pmatrix},$$

we have $X_{n+1} = AX_n$. The matrix A has eigenvalues 1 and .7, so we could proceed to solve for y_n and z_n in the usual way. However this is unnecessary in the present example since it is only the ultimate behavior of y_n and z_n that is of interest.

Assuming that the limits exist, we see that the real object of interest is the vector

$$X_\infty = \lim_{n\to\infty} X_n = \begin{pmatrix} \lim_{n\to\infty} y_n \\ \lim_{n\to\infty} z_n \end{pmatrix}.$$

Taking the limit as $n \to \infty$ of both sides of the equation $X_{n+1} = AX_n$, we obtain $X = AX$; hence X is an eigenvector of A associated with the eigenvalue 1. An eigenvector is quickly found to be $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Thus X_∞ must be a scalar multiple of this vector. Now the sum of the entries of X_∞ equals the total U.S. population, p say, and it follows that

$$X_\infty = \frac{p}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

So the (alarming) conclusion is that ultimately two thirds of the U.S. population will be in California and one third elsewhere. This can be confirmed by explicitly calculating y_n and z_n and taking the limit as $n \to \infty$.

The preceding problem is an example of what is known as a Markov process. For an understanding of this concept some knowledge of elementary probability is necessary. A *Markov process* is a system which has a finite set of states $S_1, \dots, S_n$. At any instant the system is in a definite state and over a fixed period of time it changes to another state. The probability that the system changes from state S_j to state S_i over one time period is assumed to be a constant p_{ij}. The matrix

$$P = [p_{ij}]_{n,n}$$

is called the *transition matrix* of the system. In Example 8.2.4 there are two states: a person is either in or not in California. The transition matrix is the matrix A.

Clearly all the entries of P lie in the interval [0, 1]; more importantly P has the property that *the sum of the entries in any column equals* 1. Indeed $\sum_{i=1}^{n} p_{ij} = 1$ since it is certain that the system will change from state S_j to *some* state S_i. This property guarantees that 1 *is an eigenvalue of* P; indeed $\det(P - I) = 0$ because the sum of the entries in any column of the matrix $P - I$ is equal to zero, so its determinant is zero.

Suppose that we are interested in the behavior of the system over two time periods. For this we need to know the probability of going from state S_j to state S_i over two periods. Now the probability of the system going from S_j to S_i via S_k is $p_{ik}p_{kj}$, so the probability of going from state S_j to S_i over two periods is

$$\sum_{k=1}^{n} p_{ik}p_{kj}.$$

But this is immediately recognizable as the (i, j) entry of P^2; therefore the transition matrix for the system over two time periods is P^2. More generally the transition matrix for the system over k time periods is seen to be P^k by similar considerations.

The interesting problem for a Markov process is to determine the ultimate behavior of the system over a long period of time, that is to say, $\lim_{k\to\infty}(P^k)$. For the (i,j) entry of this matrix is the probability that the system will go from state S_i to state S_j in the long run.

The first question to be addressed is whether this limit always exists. In general the answer is negative, as a very simple example shows: if $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then P^k equals either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, according to whether k is even or odd; so the limit does not exist in this case. Nevertheless it turns out that under some mild assumptions about the matrix the limit does exist. Let us call a transition matrix P *regular* if some positive power of P has all its entries positive. For example, the matrix $\begin{pmatrix} 0 & .5 \\ 1 & .5 \end{pmatrix}$ is regular; indeed all powers after the first have positive entries. But, as we have seen, the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not regular. A Markov system is said to be *regular* if its transition matrix is regular.

The fundamental theorem about Markov processes can now be stated. A proof may be found in [15], for example.

Theorem 8.2.1
Let P be the transition matrix of a regular Markov system. Then $\lim_{k\to\infty}(P^k)$ exists and has the form $(X \ X \ \dots \ X)$ where X is the unique eigenvector of P associated with the eigenvalue 1 which has entry sum equal to 1.

Our second example of a Markov process is the library book problem from Chapter One (see Exercise 1.2.12).

Example 8.2.5
A certain library owns 10,000 books. Each month 20% of the books in the library are lent out and 80% of the books lent out

are returned, while 10% remain lent out and 10% are reported lost. Finally, 25% of books listed as lost the previous month are found and returned to the library. How many books will be in the library, lent out, and lost in the long run?

Here there are three states that a book may be in: $S_1 =$ in the library: $S_2 =$ lent out: $S_3 =$ lost. The transition matrix for this Markov process is

$$P = \begin{pmatrix} .8 & .8 & .25 \\ .2 & .1 & 0 \\ 0 & .1 & .75 \end{pmatrix}.$$

Clearly P^2 has positive entries, so P is regular. Of course P has the eigenvalue 1; the corresponding eigenvector with entry sum equal to 1 is found to be

$$X = \frac{1}{59}\begin{pmatrix} 45 \\ 10 \\ 4 \end{pmatrix}.$$

So the probabilities that a book is in states S_1, S_2, S_3 after a long period of time are 45/59, 10/59, 4/59 respectively. Therefore the expected numbers of books in the library, lent out, and lost, *in the long run*, are obtained by multiplying these probabilities by the total number of books, 10,000. These numbers are therefore 7627, 1695, 678 respectively.

Exercises 8.2

1. Solve the following systems of linear recurrences with the specified initial conditions:

(a) $\begin{cases} y_{n+1} & = & & -\ 12x_n \\ x_{n+1} & = y_n & + & 7z_n \end{cases}$ where $y_0 = 0,\ z_0 = 1;$

(b) $\begin{cases} y_{n+1} & = 2y_n & + & 10z_n \\ z_{n+1} & = 2y_n & + & 3z_n \end{cases}$ where $y_0 = 0,\ z_0 = 1.$

2. In a certain nature reserve there are two competing animal species A and B. It is observed that the number of species A equals three times the number of A last year less twice the number of species B last year. Also the number of species B is twice the number of B last year less the number of species A last year. Write down a system of linear recurrence relations for a_n and b_n, the numbers of each species after n years, and solve the system. What are the long term prospects for each species?

3. A pair of newborn rabbits begins to breed at age one month, and each successive month produces one pair of offspring (one of each sex). Initially there were two pairs of rabbits. If r_n is the total number of pairs of rabbits at the beginning of the nth month, show that r_n satisfies $r_{n+1} = r_n + r_{n-1}$ and $r_1 = 2 = r_2$. Solve this second order recurrence relation for r_n.

4. A tower n feet high is to be built from red, white and blue blocks. Each red block is 1 foot high, while the white and blue blocks are 2 feet high. If u_n denotes the number of different designs for the tower, show that the recurrence relation $u_{n+1} = u_n + 2u_{n-1}$ must hold. By solving this recurrence, find a formula for u_n.

5. Solve the system of recurrence relations $y_{n+1} = 3y_n - 2z_n$, $z_{n+1} = 2y_n - z_n$, with the initial conditions $y_0 = 1$, $z_0 = 0$.

6. Solve the second order system $y_{n+1} = y_{n-1}$, $z_{n+1} = y_n + 4z_n$, with the initial conditions $y_0 = 0$, $y_1 = 1 = z_1$.

7. In a certain city 90% of employed persons retain their jobs at the end of each year, while 60% of the unemployed find a job during the year. Assuming that the total employable population remains constant, find the unemployment rate in the long run.

8. A certain species of bird nests in three locations A, B and C. It is observed that each year half of the birds at A and half of the birds at B move their nests to C, while the others stay in the same nesting place. The birds nesting at C are evenly split between A and B. Find the ultimate distribution of birds among the three nesting sites, assuming that the total bird population remains constant.

9. There are three political parties in a certain city, conservatives, liberals and socialists. The probabilities that someone who voted conservative last time will vote liberal or socialist at the next election are .3 and .2 respectively. The probabilities of a liberal voting conservative or socialist are .2 and .1. Finally, the probabilities of a socialist voting conservative or liberal are .1 and .2. What percentages of the electorate will vote for the three parties in the long run, assuming that everyone votes and the number of voters remains constant?

8.3 Applications to Systems of Linear Differential Equations

In this section we show how the theory of eigenvalues developed in 8.1 can be applied to solve systems of linear differential equations. Since there is a close analogy between linear recurrence relations and linear differential equations, the reader will soon notice a similarity between the methods used here and in 8.2.

For simplicity we consider initially a system of *first order linear* (*homogeneous*) *differential equations* for functions $y_1, \dots, y_n$ of x. This has the general form

$$\begin{cases} y_1' &= a_{11}y_1 & + & \cdots & + & a_{1n}y_n \\ y_2' &= a_{21}y_1 & + & \cdots & + & a_{2n}y_n \\ \cdot & \quad\cdot & & \cdots & & \cdot \\ y_n' &= a_{n1}y_1 & + & \cdots & + & a_{nn}y_n \end{cases}$$

Here the a_{ij} are assumed to be constants. The object is to find the most general functions $y_1, \ldots, y_n$, differentiable in some interval $[a\ , b]$, which satisfy the equations of the system. Alternatively one may wish to find functions which satisfy in addition a set of initial conditions of the form

$$y_1(x_0) = b_1,\ y_2(x_0) = b_2,\ \ldots,\ y_n(x_n) = b_n.$$

Here the b_i are certain constants and x_0 is in the interval $[a, b]$.

Let $A = [a_{ij}]$, the *coefficient matrix* of the system and write

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Then we define the *derivative* of Y to be

$$Y' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix}.$$

With this notation the given system of differential equations can be written in matrix form

$$Y' = AY.$$

By a *solution* of this equation we shall mean any column vector Y of n functions in $D[a, b]$ which satisfies the equation. The set of all solutions is a subspace of the vector space of all n-column vectors of differentiable functions; this is called the *solution space*. It can be shown that *the dimension of the solution space equals* n, so that there are n linearly independent solutions, and every solution is a linear combination of them.

If a set of n initial conditions is given, there is in fact a unique solution of the system satisfying these conditions. For an account of the theory of systems of differential equations the reader may consult a book on differential equations such as [15] or [16]. Here we are concerned with methods of finding solutions, not with questions of existence and uniqueness of solutions.

Suppose that the coefficient matrix A is diagonalizable, so there is an invertible matrix S such that $D = S^{-1}AS$ is diagonal, with diagonal entries $d_1, \ldots, d_n$ say. Here of course the d_i are the eigenvalues of A. Define

$$U = S^{-1}Y.$$

Then $Y = SU$ and $Y' = SU'$ since S has constant entries. Substituting for Y and Y' in the equation $Y' = AY$, we obtain $SU' = ASU$, or

$$U' = (S^{-1}AS)U = DU.$$

This is a system of linear differential equations for $u_1, \ldots, u_n$, the entries of U. It has the very simple form

$$\begin{cases} u_1' & = d_1u_1 \\ u_2' & = d_2u_2 \\ \cdot & \cdot \\ u_n' & = d_nu_n \end{cases}$$

The equation $u_i' = d_iu_i$ is easy to solve since its differential form is

$$d(\ln\ u_i) = d_i.$$

Thus its general solution is $u_i = c_ie^{d_ix}$ where c_i is a constant. The general solution of the system of linear differential equations for $u_1, \ldots, u_n$ is therefore

$$u_1 = c_1e^{d_1x},\ \ldots\ , u_n = c_ne^{d_nx}.$$

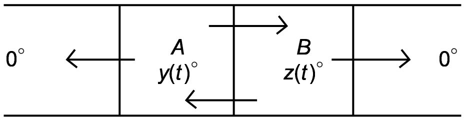
To find the original functions y_i, simply use the equation $Y = SU$ to get

$$y_i = \sum_{j=1}^{n} s_{ij}u_j = \sum_{j=1}^{n} s_{ij}c_j e^{d_j x}.$$

Since we know how to find S, this procedure provides an effective method of solving systems of first order linear differential equations in the case where the coefficient matrix is diagonalizable.

Example 8.3.1

Consider a long tube divided into four regions along which heat can flow. The regions on the extreme left and right are kept at $0°C$, while the walls of the tube are insulated. It is assumed that the temperature is uniform within each region. Let $y(t)$ and $z(t)$ be the temperatures of the regions A and B at time t. It is known that the rate at which each region cools equals the sum of the temperature differences with the surrounding media. Find a system of linear differential equations for $y(t)$ and $z(t)$ and solve it.



According to the law of cooling

$$\begin{cases} y' &= (z - y) + (0 - y) \\ z' &= (y - z) + (0 - z) \end{cases}$$

Thus we are faced with the linear system of differential equations

$$\begin{cases} y' = -2y + z \\ z' = y - 2z \end{cases}$$

Here

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y \\ z \end{pmatrix}.$$

Now the matrix A is diagonalizable; indeed

$$D = S^{-1}AS = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \text{ where } S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Setting $U = S^{-1}Y$, we obtain from $Y' = AY$ the equation $U' = DU$. This yields two very simple differential equations

$$\begin{cases} u_1' = -u_1 \\ u_2' = -3u_2 \end{cases}$$

where u_1 and u_2 are the entries of U. Hence $u_1 = ce^{-t}$ and $u_2 = de^{-3t}$, with arbitrary constants c and d. Finally

$$Y = SU = \begin{pmatrix} ce^{-t} + de^{-3t} \\ ce^{-t} - de^{-3t} \end{pmatrix}.$$

The general solution of the original system of differential equations is therefore

$$\begin{cases} y = ce^{-t} + de^{-3t} \\ z = ce^{-t} - de^{-3t} \end{cases}$$

Thus the temperatures of both regions A and B tend to zero as $t \to \infty$.

In the next example complex eigenvalues arise, which causes a change in the procedure.

Example 8.3.2
Solve the linear system of differential equations

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = y_1 + y_2 \end{cases}$$

The coefficient matrix here is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

which has complex eigenvalues $1 + i$ and $1 - i$; we are using the familiar notation $i = \sqrt{-1}$ here. The corresponding eigenvectors are

$$\begin{pmatrix} i \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -i \\ 1 \end{pmatrix},$$

respectively. Let S be the 2×2 matrix which has these vectors as its columns; then $S^{-1}AS = D$, the diagonal matrix with diagonal entries $1 + i$ and $1 - i$. If we write $U = S^{-1}Y$, the system of equations becomes $U' = DU$, that is,

$$\begin{cases} u_1' = (1 + i)u_1 \\ u_2' = (1 - i)u_2 \end{cases}$$

where u_1 and u_2 are the entries of U.

The first equation has the solution $u_1 = e^{(1+i)x}$, while the second has the obvious solution $u_2 = 0$. Using these values for u_1 and u_2, we obtain a complex solution of the system of differential equations

$$Y = SU = \begin{pmatrix} ie^{(1+i)x} \\ e^{(1+i)x} \end{pmatrix}.$$

Of course we are looking for real solutions, but these are in fact at hand. For the real and imaginary parts of Y will also be solutions of the system $Y' = AY$. Thus we obtain two real solutions from the single complex solution Y, by taking the real and imaginary parts of Y; these are respectively

$$Y_1 = \begin{pmatrix} -e^x \sin\ x \\ e^x \cos\ x \end{pmatrix} \text{ and } Y_2 = \begin{pmatrix} e^x \cos\ x \\ e^x \sin\ x \end{pmatrix}.$$

Now Y_1 and Y_2 are easily seen to be linearly independent solutions; therefore the general solution of the system is obtained by taking an arbitrary linear combination of these:

$$Y = c_1Y_1 + c_2Y_2 = e^x \begin{pmatrix} -c_1 \sin\ x + c_2 \cos\ x \\ c_1 \cos\ x + c_2 \sin\ x \end{pmatrix}$$

where c_1 and c_2 are arbitrary real constants. Hence

$$\begin{cases} y_1 = e^x(-c_1 \sin\ x + c_2 \cos\ x) \\ y_2 = e^x(c_1 \cos\ x + c_2 \sin\ x) \end{cases}$$

Of course the success of the method employed in the last two examples depended entirely upon the fact that A is diagonalizable. However, should this not be the case, one can still treat the system of differential equations by triangularizing the coefficient matrix and solving the resulting triangular system using back substitution, rather as was done for systems of linear recurrences in 8.2.

Example 8.3.3

Solve the linear system of differential equations

$$\begin{cases} y_1' &= y_1 &+ y_2 \\ y_2' &= -y_1 &+ 3y_2 \end{cases}$$

In this case the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

is not diagonalizable, but it can be triangularized. In fact it was shown in Example 8.1.6 that

$$T = S^{-1}AS = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

where $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Put $U = S^{-1}Y$ and write u_1, u_2 for the entries of U. Then $Y = SU$ and $Y' = SU'$. The equation $Y' = AY$ now becomes $U' = TU$. This yields the triangular system

$$\begin{cases} u_1' = 2u_1 + u_2 \\ u_2' = 2u_2 \end{cases}$$

Solving the second equation, we find that $u_2 = c_2e^{2x}$ with c_2 an arbitrary constant. Now substitute for u_2 in the first equation to get

$$u_1' - 2u_1 = c_2e^{2x}.$$

This is a first order linear equation which can be solved by a standard method: multiply both sides of the equation by the "integrating factor"

$$e^{\int -2dx} = e^{-2x}.$$

The equation then becomes $(u_1e^{-2x})' = c_2$, whence $u_1e^{-2x} = c_2x + c_1$, with c_1 another arbitrary constant. Thus $u_1 = c_2xe^{2x} + c_1e^{2x}$. To find the original functions y_1 and y_2, we form the product

$$Y = SU = e^{2x}\begin{pmatrix} c_1 + c_2x \\ c_1 + c_2(x+1) \end{pmatrix}.$$

Thus the general solution of the system is

$$\begin{cases} y_1 = (c_1 + c_2x)e^{2x}, \\ y_2 = (c_1 + c_2(x+1))e^{2x} \end{cases}$$

Finally, suppose that initial conditions $y_1(0) = 1$ and $y_2(0) = 0$ are given. We can find the correct values of c_1 and c_2 by substituting $t = 0$ in the expressions for y_1 and y_2, to get $c_1 = 1$ and $c_2 = -1$. The required solution is $y_1 = (1 - x)e^{2x}$ and $y_2 = -x\ e^{2x}$.

The next application is one of a military nature.

Example 8.3.4

Two armored divisions A and B engage in combat. At time t their respective numbers of tanks are $a(t)$ and $b(t)$. The rate at which tanks in a division are destroyed is proportional to the number of intact enemy tanks at that instant. Initially A and B have a_0 and b_0 tanks where $a_0 > b_0$. Predict the outcome of the battle.

According to the information given, the functions a and b satisfy the linear system

$$\begin{cases} a' &= -kb \\ b' &= -ka \end{cases}$$

where k is some positive constant. Here the coefficient matrix is

$$A = \begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix}.$$

The characteristic equation is $x^2 - k^2 = 0$, so the eigenvalues are k and $-k$ and A is diagonalizable. It turns out that

$$S^{-1}AS = D = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix},$$

where $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. If we set $Y = \begin{pmatrix} a \\ b \end{pmatrix}$, the system of differential equations becomes $Y' = AY$. On writing $U = S^{-1}Y$, we get $U' = DU$. This is the system

$$\begin{cases} u' &= ku \\ v' &= -kv \end{cases}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$. Hence $u = ce^{kx}$ and $v = de^{-kx}$, with c and d arbitrary constants. The general solution is $Y = SU$, which yields

$$\begin{cases} a = \quad ce^{kt} + de^{-kt} \\ b = -ce^{kt} + de^{-kt} \end{cases}$$

Now the initial conditions are $a(0) = a_0$ and $b(0) = b_0$, so

$$\begin{cases} \quad c + d = a_0 \\ -c + d = b_0 \end{cases}$$

Solving we obtain $c = (a_0 - b_0)/2,\ d = (a_0 + b_0)/2$. Therefore the numbers of tanks surviving at time t in Divisions A and B are respectively

$$\begin{cases} a \quad = \quad \left(\dfrac{a_0 - b_0}{2}\right) e^{kt} + \left(\dfrac{a_0 + b_0}{2}\right) e^{-kt} \\ b \quad = -\left(\dfrac{a_0 - b_0}{2}\right) e^{kt} + \left(\dfrac{a_0 + b_0}{2}\right) e^{-kt} \end{cases}$$

It is more convenient to write $a(t)$ and $b(t)$ in terms of the hyperbolic functions $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. Then the solution becomes

$$\begin{cases} a = a_0\cosh(kt) - b_0\sinh(kt) \\ b = b_0\cosh(kt) - a_0\sinh(kt) \end{cases}$$

Now Division B will have lost all its tanks when $b = 0$, i.e., after time

$$t = \frac{1}{k}\tanh^{-1}(\frac{b_0}{a_0}).$$

Observe also that

$$a^2 - b^2 = a_0^2 - b_0^2$$

because of the identity $\cosh^2(kt) - \sinh^2(kt) = 1$. Therefore at the time when Division B has lost all of its tanks, Division A still has a tanks where $a^2 - 0 = a_0^2 - b_0^2$. Hence the number of tanks that Division A has left at the end of the battle is

$$\sqrt{a_0^2 - b_0^2}.$$

Not surprisingly, since it had more tanks to start with, Division A wins the battle.

However, there is a way in which Division B could conceivably win. Suppose that

$$\frac{1}{\sqrt{2}} a_0 \; < \; b_0 \; < a_0.$$

Suppose further that Division A consists of two columns with equal numbers of tanks, and that Division B manages to attack one column of Division A before the other column can come to its aid. Since $b_0 > \frac{1}{2} a_0$, Division B defeats the first column of Division A, and it still has $\sqrt{b_0^2 - \frac{1}{4} a_0^2}$ tanks left. Then Division B attacks the second column and wins with

$$\sqrt{b_0^2 - \frac{1}{4} a_0^2 - \frac{1}{4} a_0^2} = \sqrt{b_0^2 - \frac{1}{2} a_0^2}$$

tanks left.

Thus Division B wins the battle despite having fewer tanks than Division A: but it must have more than $a_0/\sqrt{2}$ or 71% of the strength of the larger division for the plan to work. This explains the frequent success of the “divide and conquer strategy”.

Higher order equations

Systems of linear differential equations of order 2 or more can be converted to first order systems by introducing additional functions. Once again the procedure is similar to that adopted for systems of linear recurrences.

Example 8.3.5
Solve the second order system

$$\begin{cases} y_1'' &= -2y_2 &+ \ y_1' &+ 2y_2' \\ y_2'' &= \ \ 2y_1 &+ 2y_1' &- \ y_2' \end{cases}$$

The system may be converted to a first order system by introducing two new functions

$$y_3 = y_1' \quad \text{and} \quad y_4 = y_2'.$$

Thus $y_1'' = y_3'$ and $y_2'' = y_4'$. The given system is therefore equivalent to the first order system

$$\begin{cases} y_1' = y_3 \\ y_2' = y_4 \\ y_3' = -2y_2 + y_3 + 2y_4 \\ y_4' = 2y_1 + 2y_3 - y_4 \end{cases}$$

The coefficient matrix here is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 2 \\ 2 & 0 & 2 & -1 \end{pmatrix}.$$

Its eigenvalues turn out to be $1, -1, 2, -2$, with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \\ 2 \end{pmatrix}.$$

Therefore, if S denotes the matrix with these vectors as its columns, we have $S^{-1}AS = D$, the diagonal matrix with diagonal entries $1, -1, 2, -2$. Now write $U = S^{-1}Y$. Then the equation $Y' = AY$ becomes $U' = (S^{-1}AS)U = DU$, which is equivalent to

$$u_1' = u_1, \quad u_2' = -u_2, \quad u_3' = 2u_3, \quad u_4' = -2u_4.$$

Solving these simple equations, we obtain

$$u_1 = c_1e^x, \quad u_2 = c_2e^{-x}, \quad u_3 = c_3e^{2x}, \quad u_4 = c_4e^{-2x}.$$

The functions y_1 and y_2 may now be read off from the equation $Y = SU$ to give the general solution

$$\begin{cases} y_1 = c_1e^x + 2c_2e^{-x} + c_3e^{2x} + c_4e^{-2x} \\ y_2 = 2c_1e^x - c_2e^{-x} + c_3e^{2x} - c_4e^{-2x} \end{cases}$$

Exercises 8.3

1. Find the general solutions of the following systems of linear differential equations:

$$\text{(a)} \begin{cases} y_1' = -y_1 + y_2 \\ y_2 = 2y_1 - 3y_2 \end{cases} \qquad \text{(b)} \begin{cases} y_1' = 3y_1 - 2y_2 \\ y_2' = -2y_1 + 3y_2 \end{cases}$$

$$\text{(c)} \begin{cases} y_1' = y_1 + y_2 + y_3 \\ y_2' = y_2 \\ y_3' = y_2 + y_3 \end{cases}$$

2. Find the general solution (in real terms) of the system of differential equations

$$\begin{cases} y_1' = y_1 + y_2 \\ y_2' = -2y_1 + 3y_2 \end{cases}$$

Then find a solution satisfying the initial conditions $y_1(0) = 1$, $y_2(0) = 2$.

3. By triangularizing the coefficient matrix solve the system of differential equations

$$\begin{cases} y_1' = 5y_1 + 3y_2 \\ y_2' = -3y_1 - y_2 \end{cases}$$

Then find a solution satisfying the initial conditions $y_1(0) = 0$, $y_2(0) = 2$.

4. Solve the second order linear system

$$\begin{cases} y_1'' = 2y_1 + y_2 + y_1' + y_2' \\ y_2'' = -5y_1 + 2y_2 + 5y_1' - y_2' \end{cases}$$

5. Given a system of n (homogeneous) linear differential equations of order k, how would you convert this to a system of first order equations? How many equations will there be in the first order system?

6. Describe a general method for solving a system of second order linear differential equations of the form $Y'' = AY$, where A is diagonalizable.

7. Solve the systems of differential equations

$$\text{(a)} \begin{cases} y_1'' = y_1 - y_2 \\ y_2'' = 3y_1 + 5y_2 \end{cases} \quad \text{(b)} \begin{cases} y_1'' = -4y_2 \\ y_2'' = y_1 + 5y_2 \end{cases}$$

[Note that the general solution of the differential equation $u'' = a^2u$ is $u = c_1\cosh(ax) + c_2\sinh(ax)$].

8. (*The double pendulum*) A string of length $2l$ is hung from a rigid support. Two weights each of mass m are attached to the midpoint and lower end of the string, which is then allowed to execute small vibrations subject to gravity only. Let y_1 and y_2 denote the horizontal displacements of the two weights from the equilibrium position at time t.

(a) (optional) By using Newton's Second Law of Motion, show that y_1 and y_2 satisfy the differential equations

$y_1'' = a^2(-3y_1 + y_2)$, $y_2'' = a^2(y_1 - y_2)$ where $a = \sqrt{g/l}$ and g is the acceleration due to gravity.

(b) Solve the linear system in (a) for y_1 and y_2. [Note: the general solution of the differential equation $y'' + a^2y = 0$ is $y = c_1 \cos\ ax + c_2 \sin\ ax$].

9. In Example 8.3.4 assume that Division A consists of m equal columns. Suppose that Division B is able to attack each column of A in turn. Show that Division B will win the battle provided that $b_0 > \frac{a_0}{\sqrt{m}}$.