CHAPTER 1

QUANTUM MECHANICS

Macroscopic quantum phenomena are exceedingly attractive and important in modern physics. A minimum review of quantum mechanics is given from this point of view in this Section. Quantum field theory is constructed on the basis of creation and annihilation operators. An emphasis is placed on the minimum-uncertainty state, that is a coherent state, where the particle number and the conjugate phase are simultaneously measurable most accurately as a quantum system.

1.1 HILBERT SPACES

A state is represented by an element $|\mathfrak{S}\rangle$ in a Hilbert space $\mathbb{H}$. A Hilbert space $\mathbb{H}$ is a vector space characterized by the following properties:

(1) *Principle of superposition*
If $|\mathfrak{S}_1\rangle$ and $|\mathfrak{S}_2\rangle$ are elements of the space $\mathbb{H}$ then so is $\lambda_1|\mathfrak{S}_1\rangle + \lambda_2|\mathfrak{S}_2\rangle$ for arbitrary complex numbers λ_1 and λ_2.

(2) *Inner product*
A complex number $\langle\mathfrak{S}_2|\mathfrak{S}_1\rangle$, called *inner product*, is assigned to any pair of states. It obeys

$$\langle\mathfrak{S}_2|\mathfrak{S}_1\rangle = \langle\mathfrak{S}_1|\mathfrak{S}_2\rangle^*, \tag{1.1.1a}$$

$$\langle\mathfrak{S}_3|\left(\lambda_1|\mathfrak{S}_1\rangle + \lambda_2|\mathfrak{S}_2\rangle\right) = \lambda_1\langle\mathfrak{S}_3|\mathfrak{S}_1\rangle + \lambda_2\langle\mathfrak{S}_3|\mathfrak{S}_2\rangle, \tag{1.1.1b}$$

$$\langle\mathfrak{S}|\mathfrak{S}\rangle \geq 0. \tag{1.1.1c}$$

The inner product $\langle\mathfrak{S}|\mathfrak{S}\rangle$ is called the norm of the state $|\mathfrak{S}\rangle$. It vanishes if and only if $|\mathfrak{S}\rangle = 0$.

When an operator $\mathcal{O}$ is given, a complex number $\langle\mathfrak{S}_2|\mathcal{O}|\mathfrak{S}_1\rangle$ is determined for any pair of states $|\mathfrak{S}_1\rangle$ and $|\mathfrak{S}_2\rangle$ as the inner product of two states $|\mathfrak{S}_2\rangle$ and $\mathcal{O}|\mathfrak{S}_1\rangle$. Conversely, the operator $\mathcal{O}$ is uniquely defined when a complex number $\langle\mathfrak{S}_2|\mathcal{O}|\mathfrak{S}_1\rangle$ is given to any pair of states. Thus, a new operator $\mathcal{O}^\dagger$ is defined from

$\mathcal{O}$ by requiring

$$\langle \mathfrak{S}_2|\mathcal{O}^\dagger|\mathfrak{S}_1\rangle = \langle \mathfrak{S}_1|\mathcal{O}|\mathfrak{S}_2\rangle^* \tag{1.1.2}$$

for any pair of states, where $\langle \mathfrak{S}_1|\mathcal{O}|\mathfrak{S}_2\rangle^*$ is the complex conjugate of $\langle \mathfrak{S}_1|\mathcal{O}|\mathfrak{S}_2\rangle$. The operator $\mathcal{O}^\dagger$ is called the *Hermitian conjugate* of $\mathcal{O}$.

A measurement of the operator $\mathcal{O}$ made on the state $|\mathfrak{S}\rangle$ yields an expectation value $\langle \mathfrak{S}|\mathcal{O}|\mathfrak{S}\rangle$. Since we get a real number from the measurement of a physical quantity, the physical operator should satisfy $\mathcal{O}^\dagger = \mathcal{O}$: such an operator is called a *Hermitian operator*. On the other hand, when an operator $\mathcal{O}$ satisfies $\mathcal{O}^\dagger = \mathcal{O}^{-1}$, where $\mathcal{O}^{-1}$ is the inverse of $\mathcal{O}$, it is called a *unitary operator*.

For any Hermitian operator H, solutions of the eigenvalue problem,

$$H|\mathfrak{S}_i\rangle = \varepsilon_i|\mathfrak{S}_i\rangle, \tag{1.1.3}$$

can be chosen to satisfy the orthonormality condition

$$\langle \mathfrak{S}_j|\mathfrak{S}_k\rangle = \delta_{jk}, \tag{1.1.4}$$

and the completeness condition

$$\sum_j |\mathfrak{S}_j\rangle\langle \mathfrak{S}_j| = 1. \tag{1.1.5}$$

Such a set of states $|\mathfrak{S}_j\rangle$ form a basis of a Hilbert space $\mathbb{H}$. Namely, based on (1.1.4) and (1.1.5), any state is expanded in terms of them as

$$|\mathfrak{S}\rangle = \sum_j |\mathfrak{S}_j\rangle\langle \mathfrak{S}_j|\mathfrak{S}\rangle = \sum_j \lambda_j|\mathfrak{S}_j\rangle, \tag{1.1.6}$$

where $\lambda_j = \langle \mathfrak{S}_j|\mathfrak{S}\rangle$ is the probability amplitude to find the state $|\mathfrak{S}_j\rangle$ in $|\mathfrak{S}\rangle$.

1.2 Canonical Formalism

The dynamics of a system is determined when the Lagrangian L is given. It is a function of a collection of "coordinates", $q \equiv \{q_1, q_2, \cdots, q_N\}$, with their "velocities" $\dot{q}$ at time t. The action is defined by

$$S = \int_{t_1}^{t_2} dt\, L[q(t), \dot{q}(t)]. \tag{1.2.1}$$

The principle of the least action implies

$$\delta S = \int_{t_1}^{t_2} dt\ \{L[q(t) + \delta q(t), \dot{q}(t) + \delta\dot{q}(t)] - L[q(t), \dot{q}(t)]\} = 0, \tag{1.2.2}$$

as yields the Euler-Lagrange equation,

$$\frac{\delta S}{\delta q(t)} \equiv \frac{\partial L}{\partial q(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} = 0. \tag{1.2.3}$$

The conjugate "momentum" (canonical momentum) is defined by

$$p_i = \frac{\partial L(q,\dot{q})}{\partial \dot{q}_i}. \tag{1.2.4}$$

The Hamiltonian is then obtained by a Legendre transformation,

$$H(p,q) = \sum_i p_i \dot{q}_i - L\left[q, \dot{q}(p,q)\right], \tag{1.2.5}$$

which is a function of the coordinate and the momentum.[1] The physical meaning of the Hamiltonian is the energy of the system. It is a Hermitian operator.

The system is quantized when the canonical commutation relation is postulated between the coordinate and the canonical momentum,

$$[q_i, p_j] = i\hbar\delta_{ij}. \tag{1.2.7}$$

The Hamiltonian is promoted to an operator acting on states $|\mathfrak{S}\rangle$. The dynamics of a quantum system is determined when the Hamiltonian and the canonical commutation relation are given. This procedure is customarily called the *first quantization*.

The Schrödinger equation is

$$i\hbar \frac{d}{dt}|\mathfrak{S}(t)\rangle = H|\mathfrak{S}(t)\rangle. \tag{1.2.8}$$

The time-evolution operator $U(t)$ is defined by

$$|\mathfrak{S}(t)\rangle = U(t)|\mathfrak{S}(0)\rangle, \tag{1.2.9}$$

and obeys the differential equation

$$i\hbar \frac{d}{dt}U(t) = HU(t) \tag{1.2.10}$$

together with the boundary condition $U(0) = 1$. It is a unitary operator. When the Hamiltonian is independent of time, it is formally solved as

$$U(t) = e^{-iHt/\hbar}. \tag{1.2.11}$$

The state evolves according to the formula (1.2.9), while the operator remains unchanged. This is the *Schrödinger picture*.

Although the Schrödinger picture is familiar in quantum mechanics, the *Heisenberg picture* is more convenient in quantum field theory. These two pictures are

[1] To see this, we take the external derivative. Since the Lagrangian is a function of q and $\dot{q}$, we have $dL = (\partial_q L)dq + (\partial_{\dot{q}} L)d\dot{q}$. Now,

$$dH = d(p\dot{q}) - dL = \dot{q}dp + pd\dot{q} - (\partial_q L)dq - (\partial_{\dot{q}} L)d\dot{q} = \dot{q}dp - \dot{p}dq, \tag{1.2.6}$$

where use was made of the Euler-Lagrange equation (1.2.3) and the definition of the momentum (1.2.4). It implies that H is a function of q and p with the canonical equations of motion, $\dot{q} = \partial_p H$ and $\dot{p} = -\partial_q H$.

related by way of the time-evolution operator $U(t)$. The operator $A_{\rm H}(t)$ and the state $|\mathfrak{S}_{\rm H}\rangle$ in the Heisenberg picture are defined by

$$A_{\rm H}(t) \equiv U^\dagger(t) A U(t), \qquad |\mathfrak{S}_{\rm H}\rangle \equiv U^\dagger(t)|\mathfrak{S}(t)\rangle = |\mathfrak{S}(0)\rangle, \tag{1.2.12}$$

from the operator A and the state $|\mathfrak{S}(t)\rangle$ in the Schrödinger picture. The Heisenberg state $|\mathfrak{S}_{\rm H}\rangle$ is independent of time,

$$\frac{d}{dt}|\mathfrak{S}_{\rm H}\rangle = 0, \tag{1.2.13}$$

and the Heisenberg operator $A_{\rm H}(t)$ obeys the differential equation

$$i\hbar\frac{d}{dt}A_{\rm H}(t) = U^\dagger(t)[A, H(\mathcal{O})]U(t) = [A_{\rm H}(t), H(\mathcal{O}_{\rm H}(t))]. \tag{1.2.14}$$

Here, $\mathcal{O}$ stands for a generic operator (such as a momentum) involved in the Hamiltonian. The Hamiltonian $H(\mathcal{O}_{\rm H}(t))$ is given simply by replacing $\mathcal{O}$ with the Heisenberg operator $\mathcal{O}_{\rm H}(t)$ in $H(\mathcal{O})$, since

$$\begin{aligned} U^\dagger H(\mathcal{O})U &= U^\dagger(b_0 + b_1\mathcal{O} + b_2\mathcal{O}^2 + \cdots)U \\ &= b_0 + b_1\mathcal{O}_{\rm H} + b_2\mathcal{O}_{\rm H}^2 + \cdots = H(\mathcal{O}_{\rm H}), \end{aligned} \tag{1.2.15}$$

where $H(\mathcal{O})$ is expanded in polynomials of $\mathcal{O}$ with coefficients b_i. Equation (1.2.14) is called the Heisenberg equation of motion. In the Heisenberg picture the state remains unchanged while the operator evolves as time passes.

In the Schrödinger picture, the observed value of the operator A in the state $|\mathfrak{S}(t)\rangle$ is

$$\langle A\rangle = \langle\mathfrak{S}(t)|A|\mathfrak{S}(t)\rangle, \tag{1.2.16}$$

which reads

$$\langle A\rangle = \langle\mathfrak{S}_{\rm H}|U^\dagger U A_{\rm H}(t) U^\dagger U|\mathfrak{S}_{\rm H}\rangle = \langle\mathfrak{S}_{\rm H}|A_{\rm H}(t)|\mathfrak{S}_{\rm H}\rangle \tag{1.2.17}$$

in the Heisenberg picture. Namely, the observed value $\langle A\rangle$ of the operator A is identical both in the Schrödinger and Heisenberg pictures, as should be the case. We omit the index H of the Heisenberg operator $A_{\rm H}$ in what follows.

1.3 Creation and Annihilation Operators

We introduce the creation and annihilation operators of quanta, that is, particles or excitation modes. Quanta obey either the Bose statistics or the Fermi statistics in the 3-dimensional space.[2] *Bose* quanta are characterized by the property that one quantum state can contain an arbitrary number of identical quanta. *Fermi* quanta obey the Pauli exclusion principle stating that no two quanta of the same type can

[2] A generalized statistics is possible in the 2-dimensional space, as will be reviewed in Chapter 8.

occupy a single quantum state. They are essential ingredients of quantum field theory.

Bose Quanta

Bose quanta are described by a pair of the operator a and its Hermitian conjugate $a^\dagger$ satisfying the commutation relation

$$[a, a^\dagger] = 1. \tag{1.3.1}$$

We construct a Hilbert space $\mathbb{H}$ realizing this commutation relation. We define the number operator N by

$$N = a^\dagger a, \tag{1.3.2}$$

and the vacuum state $|0\rangle$ by

$$a|0\rangle = 0. \tag{1.3.3}$$

It contains no quanta, $N|0\rangle = 0$, and is normalized as $\langle 0|0\rangle = 1$.

It follows from (1.3.1) and (1.3.2) that

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \qquad a|n\rangle = \sqrt{n}|n-1\rangle. \tag{1.3.4}$$

Thus, $a^\dagger$ and a are the creation and annihilation operators. One Bose quanta is created by operating $a^\dagger$ onto the vacuum state $|0\rangle$, $a^\dagger|0\rangle = |1\rangle$. In general, the normalized state containing n Bose quanta is

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \tag{1.3.5}$$

where

$$N|n\rangle = n|n\rangle, \qquad \langle n|m\rangle = \delta_{nm}. \tag{1.3.6}$$

All the states, $\{|n\rangle; n = 0, 1, 2, 3, \cdots\}$, constitute a Hilbert space $\mathbb{H}$ together with the orthonormality condition (1.3.6) and the completeness condition

$$\sum_n |n\rangle\langle n| = 1. \tag{1.3.7}$$

It is called the occupation-number space or the *Fock space*. The vacuum $|0\rangle$ is called the *Fock vacuum*.

If one quantum carries the energy ε, the Hamiltonian is postulated as

$$H = \varepsilon N = \varepsilon a^\dagger a, \tag{1.3.8}$$

so that the energy of the state $|n\rangle$ is $n\varepsilon$,

$$H|n\rangle = n\varepsilon|n\rangle. \tag{1.3.9}$$

The Heisenberg equation reads

$$i\hbar \frac{\partial}{\partial t} a = [a, H] = \varepsilon a. \tag{1.3.10}$$

Its solution is given by

$$a(t) = e^{-i\omega t} a(0), \tag{1.3.11}$$

where

$$\varepsilon = \hbar\omega. \tag{1.3.12}$$

It describes a noninteracting system of one kind of Bose quanta.

When the system contains several kinds of quanta, the annihilation and creation operators are introduced with respect to each of them,

$$[a_j, a_k^\dagger] = \delta_{jk}, \qquad [a_j, a_k] = 0. \tag{1.3.13}$$

They satisfies

$$N_j a_k^\dagger = a_k^\dagger (N_j + \delta_{jk}), \qquad N_j a_k = a_k (N_j - \delta_{jk}), \tag{1.3.14}$$

with the number operator of the jth quantum being

$$N_j = a_j^\dagger a_j. \tag{1.3.15}$$

The Fock vacuum $|0\rangle$ satisfies $a_j|0\rangle = 0$ with $\langle 0|0\rangle = 1$. The state $|n_1, n_2, \cdots\rangle$ containing these quanta is given by a product of the states (1.3.5),

$$|n_1, n_2, \cdots\rangle = \prod_j \frac{1}{\sqrt{n_j!}} (a_j^\dagger)^{n_j} |0\rangle. \tag{1.3.16}$$

When the energy of the jth quantum is ε_j, the Hamiltonian is given by

$$H = \sum_j \varepsilon_j a_j^\dagger a_j. \tag{1.3.17}$$

It satisfies

$$H|n_1, n_2, \cdots\rangle = E|n_1, n_2, \cdots\rangle, \tag{1.3.18}$$

with

$$E = \sum_j \varepsilon_j n_j. \tag{1.3.19}$$

This is the energy of the state $|n_1, n_2, \cdots\rangle$.

For later convenience we define *normal ordering* of operators. Normal ordering, denoted by a double-dot symbol, dictates that all creation operators are to be placed to the left of all annihilation operators. An example reads

$$:a_1 a_2 a_3^\dagger a_1^\dagger: = :a_1^\dagger a_1 a_2 a_3^\dagger: = a_1^\dagger a_3^\dagger a_1 a_2. \tag{1.3.20}$$

Note that the conjugate operators a_1 and $a_1^\dagger$ commute freely within the normal ordering symbol.

Fermi Quanta

Fermi quanta are described by a set of operators c and its Hermitian conjugate $c^\dagger$ satisfying the anticommutation relations

$$\{c, c^\dagger\} = 1, \qquad \{c, c\} = \{c^\dagger, c^\dagger\} = 0. \tag{1.3.21}$$

Here, $\{A, B\} \equiv AB + BA$. We define the number operator by

$$N = c^\dagger c, \tag{1.3.22}$$

and the Fock vacuum $|0\rangle$ by

$$c|0\rangle = 0. \tag{1.3.23}$$

Because

$$N^2 = c^\dagger c c^\dagger c = c^\dagger (1 - c^\dagger c) c = c^\dagger c = N, \tag{1.3.24}$$

the eigenvalues of N are 0 and 1. Namely, there are only two states $|0\rangle$ and $|1\rangle = c^\dagger|0\rangle$ in the Hilbert space $\mathbb{H}$. No two quanta can occupy the same state. Indeed, $c^\dagger|1\rangle = c^\dagger c^\dagger|0\rangle = 0$. It is referred to as the Pauli exclusion principle.

When there are several kinds of Fermi quanta we introduce the creation operator $c_j^\dagger$ and the annihilation operator c_j to each of them,

$$\{c_j, c_k^\dagger\} = \delta_{jk}, \qquad \{c_j, c_k\} = 0. \tag{1.3.25}$$

Note that $(c_j c_k)^\dagger = c_k^\dagger c_j^\dagger$. Many-quantum states are given by

$$|n_1, n_2, \cdots\rangle = (c_1^\dagger)^{n_1} (c_2^\dagger)^{n_2} \cdots |0\rangle, \tag{1.3.26}$$

where $n_j = 0$ or $n_j = 1$.

Normal ordering of fermion operators is defined in the same way as boson operators. However, to be consistent with the anticommutation relations, it is understood that any two operators anticommute freely within the normal ordering symbol. An example reads

$$:c_1 c_2 c_3^\dagger c_1^\dagger: = \; - :c_1^\dagger c_1 c_2 c_3^\dagger: = -c_1^\dagger c_3^\dagger c_1 c_2. \tag{1.3.27}$$

Here, we first moved $c_1^\dagger$. The minus sign arises since it was anticommuted with three fermion operators including c_1. We then moved $c_3^\dagger$ by making it anticommute with two fermion operators.

1.4 UNCERTAINTY PRINCIPLE

We define the Hermitian operators p and q by

$$a = \frac{1}{\sqrt{2M\hbar\omega}}(M\omega q + ip), \qquad a^\dagger = \frac{1}{\sqrt{2M\hbar\omega}}(M\omega q - ip). \tag{1.4.1}$$

The commutation relation (1.3.1) is equivalent to

$$[q, p] = i\hbar. \tag{1.4.2}$$

Comparing this with (1.2.7) we identify q and p as the coordinate and the momentum. The Hamiltonian (1.3.8) is rewritten as

$$H = \frac{1}{2M}p^2 + \frac{M\omega^2}{2}q^2, \tag{1.4.3}$$

with $\varepsilon = \hbar\omega$. This is the Hamiltonian of the harmonic oscillator with frequency ω.

In classical theory a simultaneous measurement is possible of coordinate q and momentum p with arbitrary accuracy. This is impossible in quantum theory. The commutation relation (1.4.2) leads to the Heisenberg uncertainty principle, which is expressed as

$$\Delta q \Delta p \geq \frac{\hbar}{2}. \tag{1.4.4}$$

Its precise meaning is

$$\langle (\Delta q)^2 \rangle \langle (\Delta p)^2 \rangle \geq \frac{\hbar^2}{4}, \tag{1.4.5}$$

where $\langle \cdots \rangle$ stands for the expectation value with respect to a generic state, with

$$\Delta q = q - \langle q \rangle, \qquad \Delta p = p - \langle p \rangle. \tag{1.4.6}$$

Here, $\langle q \rangle$ and $\langle p \rangle$ are the averages, while $\sqrt{\langle (\Delta q)^2 \rangle}$ and $\sqrt{\langle (\Delta p)^2 \rangle}$ are the variances. When the coordinate is fixed, $\Delta q = 0$, the momentum is completely ambiguous, $\Delta p = \infty$, from (1.4.4). The momentum p is not a good quantum number in an eigenstate of the coordinate q.

The uncertainly relation (1.4.4) makes a simultaneous measurement of the coordinate and the momentum impossible. Nevertheless, there exist quantum systems where it is practically possible. They are *coherent states* and *squeezed coherent states*. These states are very important in the study of quantum Hall (QH) effects. QH states are coherent states with the spin coherence, when the Zeeman gap is not too large. Bilayer QH states are squeezed coherent states with the interlayer coherence, when the tunneling gap is not too large.

1.5 Coherent States and von Neumann Lattice

There exists a class of states where the minimum uncertainty holds,

$$\langle(\Delta q)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{4}. \tag{1.5.1}$$

They are the ones closest to the classical states.

We introduce the *coherent state*. By definition it is an eigenstate of the bosonic annihilation operator. We consider an eigenstate $|v\rangle$ of the annihilation operator a,

$$a|v\rangle = v|v\rangle, \qquad \langle v|a^\dagger = \langle v|v^* \tag{1.5.2}$$

The eigenvalue v is a complex number. An intriguing feature is that a has a classical counterpart $v = \langle v|a|v\rangle$ on the coherent state $|v\rangle$ though it is not a Hermitian operator.[3]

We then introduce the operator η by

$$a = v + \eta, \tag{1.5.3}$$

obeying $[\eta, \eta^\dagger] = 1$. The coherent state $|v\rangle$ is the Fock vacuum of the new operator η,

$$\eta|v\rangle = 0. \tag{1.5.4}$$

We evaluate the variances of the operators q and p defined by (1.4.1). It is easy to see

$$\Delta q = \sqrt{\frac{\hbar}{2\omega}}(\eta^\dagger + \eta), \qquad \Delta p = i\sqrt{\frac{\hbar\omega}{2}}(\eta^\dagger - \eta), \tag{1.5.5}$$

and

$$\begin{aligned} \langle(\Delta q)^2\rangle &= \frac{\hbar}{2\omega}\langle(\eta^\dagger + \eta)^2\rangle = \frac{\hbar}{2\omega}, \\ \langle(\Delta p)^2\rangle &= -\frac{\hbar\omega}{2}\langle(\eta^\dagger - \eta)^2\rangle = \frac{\hbar\omega}{2}, \end{aligned} \tag{1.5.6}$$

where $\langle\cdots\rangle$ stands for the expectation value with respect to the state $|v\rangle$. Hence, the minimum uncertainty (1.5.1) holds on the coherent state.

To construct explicitly the coherent state, we make use of the displacement operator,[4]

$$D(v) \equiv e^{va^\dagger - v^* a} = e^{-|v|^2/2}e^{va^\dagger}e^{-v^* a}. \tag{1.5.7}$$

Since it has the property $D^\dagger(v) = D(-v) = D^{-1}(v)$, it is a unitary operator,

$$D(v)D^\dagger(v) = D^\dagger(v)D(v) = 1. \tag{1.5.8}$$

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[3] In physical applications we use the quantum field $\phi(t, \boldsymbol{x})$ for the operator a. It is important that the classical field $\phi^{\rm cl}(t, \boldsymbol{x})$ is defined on the coherent state by $\phi^{\rm cl}(t, \boldsymbol{x}) = \langle v|\phi(t, \boldsymbol{x})|v\rangle$. See Chapter 4 and Chapter 7 for details.

[4] We use the relation $e^{A+B} = e^A e^B e^{-[A,B]/2}$ with $A = va^\dagger$ and $B = v^* a$ to derive the last term.

We take the derivative of $D^\dagger(v)aD(v)$ with respect to v,

$$\frac{\partial}{\partial v}\left[D^\dagger(v)\hat{a}D(v)\right] = D^\dagger(v)\left[\hat{a},\hat{a}^\dagger\right]D(v) = D^\dagger(v)D(v) = 1,$$

noticing that v and v^* are independent variables. We then integrate it over v from $v = 0$ to $v = 1$,

$$D^\dagger(v)\hat{a}D(v) = \hat{a} + v.$$

Similarly we can derive

$$D^\dagger(v)a^\dagger D(v) = a^\dagger + v^*. \tag{1.5.9}$$

Thus it shifts the annihilation operator a by v, and the creation operator $a^\dagger$ by v^*. We now take the state

$$|v\rangle \equiv D(v)|0\rangle = e^{-|v|^2/2}e^{va^\dagger}|0\rangle, \tag{1.5.10}$$

and find

$$a|v\rangle = D(v)D^\dagger(v)aD(v)|0\rangle = D(v)\,(a+v)\,|0\rangle = v|v\rangle. \tag{1.5.11}$$

Hence the state $|v\rangle \equiv D(v)|0\rangle$ is the coherent state by the definition (1.5.2).

The coherent state (1.5.10) is expanded in terms of the eigenstates $|n\rangle$ of the number operator N,

$$|v\rangle = e^{-|v|^2/2}\sum_{n=0}^{\infty}\frac{1}{n!}(va^\dagger)^n|0\rangle = e^{-|v|^2/2}\sum_{n=0}^{\infty}\frac{v^n}{\sqrt{n!}}|n\rangle. \tag{1.5.12}$$

Using the orthonormality condition $\langle n|m\rangle = \delta_{nm}$, we find

$$\langle n|v\rangle = e^{-|v|^2/2}\frac{v^n}{\sqrt{n!}}, \tag{1.5.13}$$

or

$$P_n \equiv |\langle n|v\rangle|^2 = \frac{\bar{n}^n}{n!}e^{-\bar{n}^2}, \tag{1.5.14}$$

where $\bar{n} \equiv \langle v|N|v\rangle = v^2$ is the average number of bosons in $|v\rangle$. Namely, the coherent state is such a state where the probability of finding n bosons obeys the Poisson distribution.

Since a is not a Hermitian operator, there is no reason that its eigenstates span an orthonormal complete set. Indeed, the scalar product of two coherent states $|u\rangle$ and $|v\rangle$ are

$$\langle u|v\rangle = \exp\left(-\frac{1}{2}|u|^2 - \frac{1}{2}|v|^2 + u^*v\right). \tag{1.5.15}$$

They are not orthogonal one to another. On the other hand, the completeness condition holds,

$$\frac{1}{\pi}\int d^2v\ |v\rangle\langle v| = 1, \tag{1.5.16}$$

because, setting $v = re^{i\theta}$, we find

$$\frac{1}{\pi}\int d^2v\,\langle n|v\rangle\langle v|m\rangle = \frac{1}{\sqrt{n!}\sqrt{m!}}\int_0^\infty dr^2\, r^{n+m}e^{-r^2}\int_0^{2\pi}\frac{d\theta}{2\pi}\,e^{i(m-n)\theta} = \delta_{nm} \quad (1.5.17)$$

for any two states $|m\rangle$ and $|n\rangle$, where we have used (1.5.13) and $\int_0^\infty s^n e^{-s}ds = n!$.

It is notable that all coherent states $|v\rangle$ participate to form a complete set in (1.5.16). However, the number of coherent states $|v\rangle$ is "much larger" than that of the occupation-number states $|n\rangle$ because the coherent state $|v\rangle$ is defined for an arbitrary complex number v. Thus, the set of all coherent states must be "overcomplete". It is expected that a countable number of coherent states exist to form a complete set, as was first addressed by von Neumann.[369]

Let us construct a subset by choosing only those states $|v_{nm}\rangle$ with discrete values of v_{mn} of the parameter v in (1.5.2) such that

$$v_{mn} = \sqrt{\pi}\ell(m + in), \quad (1.5.18)$$

where m and n are integers, and ℓ is a real number. They form a lattice with the lattice spacing $\sqrt{\pi}\ell$ in the v-space. From (1.4.1) and (1.5.2) we find

$$\langle q\rangle = \sqrt{\frac{2\pi\hbar}{M\omega}}\ell m, \qquad \langle p\rangle = \sqrt{2\pi M\hbar\omega}\ell n. \quad (1.5.19)$$

The unit cell has the area $2\pi\hbar\ell^2$ in the (q,p) phase space. It has been proved[389,62] that the condition for the states $|v_{mn}\rangle$ to form a complete set is $\ell \le 1$. Furthermore, a minimum complete set of coherent states is obtained by setting $\ell = 1$, though it is still overcomplete by having just one linear dependent state, $\sum_{m,n}|v_{mn}\rangle = 0$. However, the removal of this dependent state turns out to change the classical character and spoil the coherence.[74]

We have taken a square lattice as a simplest example. We may consider a wide class of lattices spanned by two complex numbers ℓ_q and ℓ_p,

$$v_{nm} = \sqrt{\pi}\,(\ell_q m + \ell_p n)\,. \quad (1.5.20)$$

The area spanned by a unit cell is

$$\left|\vec{\ell}_q \times \vec{\ell}_p\right| = \left|\ell_q^x\ell_p^y - \ell_q^y\ell_p^x\right| = \left|\mathrm{Im}[\ell_q^*\ell_p]\right|, \quad (1.5.21)$$

where we have set $\ell_j = \ell_j^x + i\ell_j^y$ and $\vec{\ell}_j = (\ell_j^x, \ell_j^y)$ for $j = q$ and p. We require the unit cell to have the area $2\pi\hbar$, as corresponds to the uncertainty relation $[q,p] = i\hbar$. Such a lattice is called a von Neumann lattice.[74] The states $|v_{nm}\rangle$ on a von Neumann lattice form a minimum complete set of coherent states. See also Section 10.5.

1.6 Squeezed Coherent State

We construct a general class of coherent states by redistributing quantum fluctuations between the conjugate variables q and p [Fig.1.1]. For this purpose we make a *Bogoliubov transformation*, which is a linear transformation from a set of the annihilation and creation operators into another set, $(\eta, \eta^\dagger) \to (\zeta, \zeta^\dagger)$,

$$\zeta = \eta \cosh\tau + \eta^\dagger e^{i\beta} \sinh\tau,$$
$$\zeta^\dagger = \eta^\dagger \cosh\tau + \eta e^{-i\beta} \sinh\tau, \tag{1.6.1}$$

where β and τ are real constants. It is a canonical transformation keeping the commutation relation $[\zeta, \zeta^\dagger] = 1$. It is essentially a "rotation" with an angle τ. The inverse transformation is a rotation with the angle $-\tau$,

$$\eta = \zeta \cosh\tau - \zeta^\dagger e^{i\beta} \sinh\tau,$$
$$\eta^\dagger = \zeta^\dagger \cosh\tau - \zeta e^{-i\beta} \sinh\tau. \tag{1.6.2}$$

The Bogoliubov transformation (1.6.1) is expressed as (see Appendix G),

$$\zeta = e^{-iG}\eta e^{iG}, \qquad \zeta^\dagger = e^{-iG}\eta^\dagger e^{iG} \tag{1.6.3}$$

with the generator

$$G = \frac{i}{2}\tau(e^{-i\beta}\eta\eta - e^{i\beta}\eta^\dagger\eta^\dagger). \tag{1.6.4}$$

We consider the state defined by

$$|v\rangle\!\rangle = e^{-iG}|v\rangle, \tag{1.6.5}$$

and the operator defined by

$$b = e^{-iG}ae^{iG} = v + \zeta. \tag{1.6.6}$$

It is easy to see that

$$b|v\rangle\!\rangle = e^{-iG}ae^{iG}e^{-iG}|v\rangle = v|v\rangle\!\rangle, \tag{1.6.7}$$

and

$$\zeta|v\rangle\!\rangle = 0. \tag{1.6.8}$$

The state $|v\rangle\!\rangle$ is a coherent state of the b quantum and the Fock vacuum of the ζ quantum. It will turn out to be a squeezed state.

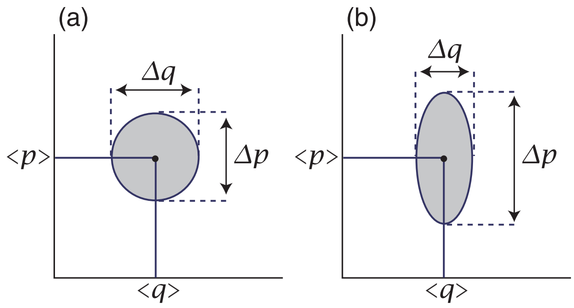

Fig. 1.1 (a) Due to the commutation relation $[q,p]=i\hbar$, we can determine position q and momentum p only with certain uncertainties Δp and Δq. The minimum uncertainty holds in the coherent state, where $(\Delta p)(\Delta q)=\hbar/2$. We have illustrated an uncertainty domain with $\Delta p\simeq\Delta q$. (b) In the squeezed state, an uncertainty domain may be squeezed without spoiling the minimum uncertainty, $(\Delta p)(\Delta q)=\hbar/2$. The position can be determined more accurately than the momentum in this instance.

We evaluate the variance (1.5.5) of the operators q and p. Since

$$\Delta q=\sqrt{\frac{\hbar}{2\omega}}(\eta^\dagger+\eta)=\sqrt{\frac{\hbar}{2\omega}}\left[(\cosh\tau-e^{i\beta}\sinh\tau)\zeta^\dagger+(\cosh\tau-e^{-i\beta}\sinh\tau)\zeta\right],$$
$$\Delta p=i\sqrt{\frac{\hbar\omega}{2}}(\eta^\dagger-\eta)=i\sqrt{\frac{\hbar\omega}{2}}\left[(\cosh\tau+e^{i\beta}\sinh\tau)\zeta^\dagger-(\cosh\tau+e^{-i\beta}\sinh\tau)\zeta\right],\tag{1.6.9}$$

we find

$$\langle\!\langle(\Delta q)^2\rangle\!\rangle=\frac{\hbar}{2\omega}\langle\!\langle(\eta^\dagger+\eta)^2\rangle\!\rangle=\frac{\hbar}{2\omega}(\cosh\tau-e^{i\beta}\sinh\tau)(\cosh\tau-e^{-i\beta}\sinh\tau),$$
$$\langle\!\langle(\Delta p)^2\rangle\!\rangle=-\frac{\hbar\omega}{2}\langle\!\langle(\eta^\dagger-\eta)^2\rangle\!\rangle=\frac{\hbar\omega}{2}(\cosh\tau+e^{i\beta}\sinh\tau)(\cosh\tau+e^{-i\beta}\sinh\tau),\tag{1.6.10}$$

and

$$\langle\!\langle(\Delta q)^2\rangle\!\rangle\langle\!\langle(\Delta p)^2\rangle\!\rangle=\frac{\hbar^2}{4}(\cosh^2\tau-e^{2i\beta}\sinh^2\tau)(\cosh^2\tau-e^{-2i\beta}\sinh^2\tau).\tag{1.6.11}$$

Here, $\langle\!\langle\cdots\rangle\!\rangle$ stands for the expectation value with respect to the state $|v\rangle\!\rangle$. The uncertainties Δq and Δp in the conjugate variables q and p are controlled by angles τ and β. The minimum uncertainty holds only when $\beta=0,\pm\pi$.

We need to use the Bogoliubov transformation with $\beta=0$ in later chapters. In this case, (1.6.9) yields

$$\zeta = \sqrt{\frac{\omega}{2\hbar}} e^{\tau} \Delta q + i \sqrt{\frac{1}{2\hbar\omega}} e^{-\tau} \Delta p. \tag{1.6.12}$$

Compare this with (1.5.5), or

$$\eta = \sqrt{\frac{\omega}{2\hbar}} \Delta q + i \sqrt{\frac{1}{2\hbar\omega}} \Delta p. \tag{1.6.13}$$

The Bogoliubov transformation from the field η to the field ζ is to insert weights $e^{\pm\tau}$ to the canonical conjugate variables Δq and Δp. The uncertainties are

$$\langle\!\langle (\Delta q)^2 \rangle\!\rangle = \frac{\hbar}{2\omega} e^{-2\tau}, \qquad \langle\!\langle (\Delta p)^2 \rangle\!\rangle = \frac{\hbar\omega}{2} e^{2\tau}, \tag{1.6.14}$$

and

$$\langle\!\langle (\Delta q)^2 \rangle\!\rangle \langle\!\langle (\Delta p)^2 \rangle\!\rangle = \frac{\hbar^2}{4}. \tag{1.6.15}$$

The state $|v\rangle\!\rangle$ is called a *squeezed state*,[35,34] since one of the uncertainties is squeezed. By definition (1.5.2), it is not a coherent state of the a quantum. Nevertheless, it has a classical counterpart,

$$\langle\!\langle v|a|v \rangle\!\rangle = v, \tag{1.6.16}$$

which is the key property of the coherent state. Hence, it is also called a *squeezed coherent state*.

1.7 Particle Number and Phase

The phase operator Θ is defined by[117]

$$a \equiv e^{i\Theta} \sqrt{N}, \qquad a^\dagger \equiv \sqrt{N} e^{-i\Theta}, \tag{1.7.1}$$

where $N = a^\dagger a$ is the number operator. From the commutation relation between a and $a^\dagger$, we find that

$$[e^{i\Theta}, N] = e^{i\Theta}, \tag{1.7.2}$$

from which we obtain[5]

$$[N, \Theta] = i. \tag{1.7.3}$$

The Heisenberg uncertainty principle follows from (1.7.3),

$$\Delta n \Delta \theta \geq \frac{1}{2}, \tag{1.7.4}$$

[5] Equation (1.7.3) is the first order term in (1.7.2) when the power expansion, $e^{i\Theta} = 1 + i\Theta + \cdots$, is made. Conversely, (1.7.2) is derived from (1.7.3) by setting $A = i\Theta$ and $B = N$ in (B.1) in Appendix A.

where $\Delta n = \sqrt{\langle(\Delta N)^2\rangle}$ and $\Delta\theta = \sqrt{\langle(\Delta\Theta)^2\rangle}$. They act on the eigenstates of the number operator as

$$N|n\rangle = n|n\rangle, \qquad e^{i\Theta}|n\rangle = |n-1\rangle, \tag{1.7.5}$$

and hence

$$\langle n|e^{i\Theta}|n\rangle = 0. \tag{1.7.6}$$

It tells physically that no measurement of the phase Θ is possible on the eigenstate $|n\rangle$ of the number operator N: The phase oscillates so rapidly that its exponentiation $e^{i\Theta}$ vanishes.

The eigenstate $|\theta\rangle$ of the phase operator Θ may be formally constructed,

$$|\theta\rangle = \sum_n e^{i\theta n}|n\rangle, \tag{1.7.7}$$

as a superposition of the eigenstates $|n\rangle$ of the number operator. To see this, we realize the commutation relation (1.7.3) by

$$\Theta = -i\frac{\partial}{\partial n} \tag{1.7.8}$$

in the representation where N is diagonalized. It acts on $|\theta\rangle$ as

$$\Theta|\theta\rangle = -i\sum_n \frac{\partial}{\partial n}e^{i\theta n}|n\rangle = \theta|\theta\rangle. \tag{1.7.9}$$

It is normalized as

$$\langle\theta'|\theta\rangle = \sum_{n,m} e^{i\theta n - i\theta' m}\langle m|n\rangle = \sum_n e^{in(\theta-\theta')} = 2\pi\delta(\theta'-\theta). \tag{1.7.10}$$

The inverse relation is

$$|n\rangle = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{-i\theta n}|\theta\rangle. \tag{1.7.11}$$

The state $|\theta\rangle$ has a periodic property, $|\theta+2\pi\rangle = |\theta\rangle$, as is clear in the definition (1.7.7).

A careful treatment is needed for the phase operator. It follows from (1.7.1) that $e^{i\Theta} = aN^{-1/2}$ and $e^{-i\Theta} = N^{-1/2}a^\dagger$. On one hand, we have

$$e^{i\Theta}e^{-i\Theta}|n\rangle = aN^{-1}a^\dagger|n\rangle = \sqrt{n+1}aN^{-1}|n+1\rangle = |n\rangle \tag{1.7.12}$$

for any state $|n\rangle$. On the other hand, we have

$$e^{-i\Theta}e^{i\Theta}|n\rangle = N^{-1/2}a^\dagger aN^{-1/2}|n\rangle = \frac{1}{\sqrt{n}}N^{-1/2}a^\dagger a|n\rangle = |n\rangle \tag{1.7.13}$$

for any state $|n\rangle \neq |0\rangle$, but

$$e^{-i\Theta}e^{i\Theta}|0\rangle = 0 \tag{1.7.14}$$

by (1.7.5). Hence, we conclude

$$e^{i\Theta}e^{-i\Theta} = 1, \qquad e^{-i\Theta}e^{i\Theta} = 1 - |0\rangle\langle 0|. \tag{1.7.15}$$

Strictly speaking, the phase Θ is not a Hermitian operator, though it is practically so in a macroscopic system. See literature[466,238,387,80,179] for more details on this point.

1.8 MACROSCOPIC COHERENCE

We evaluate the number and phase operators on the coherent state. See Section 4.5 with respect to the squeezed state. The coherent state is characterized by

$$\langle v|a|v\rangle = v, \qquad \langle v|N|v\rangle = |v|^2 \equiv n. \tag{1.8.1}$$

Thus, the particle number is measurable. However, it is not an eigenstate of the number operator,

$$N|v\rangle = \left(n + v\eta^\dagger\right)|v\rangle. \tag{1.8.2}$$

It is an eigenstate of the annihilation operator,

$$a|v\rangle = e^{i\theta}\sqrt{n}|v\rangle, \tag{1.8.3}$$

where we have set $v = e^{i\theta}\sqrt{n}$ with a certain phase θ.

The standard deviation Δn is

$$(\Delta n)^2 \equiv \langle(\Delta N)^2\rangle = \langle v|(a^\dagger a - n)^2|v\rangle = n, \tag{1.8.4}$$

as is easily verified by using $a = v + \eta$. Because of this large uncertainty in a macroscopic system ($n \to \infty$), the conjugate phase is measurable simultaneously,

$$\langle v|e^{i\Theta}|v\rangle \simeq e^{i\theta}. \tag{1.8.5}$$

We can derive this as follows. It follows from (1.7.1) that

$$e^{i\Theta} = aN^{-1/2} = \frac{v}{\sqrt{n}}\left(1 + \frac{\eta}{v}\right)\left(1 - \frac{\eta^\dagger}{v^*}\right)^{-1/2} = e^{i\theta}\left(1 + e^{-i\theta}\frac{\eta}{\sqrt{n}}\right)\left(1 - e^{i\theta}\frac{\eta^\dagger}{\sqrt{n}}\right)^{-1/2}. \tag{1.8.6}$$

We evaluate it by making a power expansion. Since (1.5.4) implies $\langle v|\eta|v\rangle = \langle v|(\eta^\dagger)^k|v\rangle = 0$ for an arbitrary integer k, we obtain

$$\langle v|e^{i\Theta}|v\rangle = e^{i\theta}\langle v|\left(1 + \frac{\eta\eta^\dagger}{2n}\right)|v\rangle = e^{i\theta}\left(1 + \frac{1}{2n}\right) \simeq e^{i\theta} \tag{1.8.7}$$

in a macroscopic system ($n \to \infty$).

We make a heuristic argument to show that the minimum uncertainty (1.7.4) holds on a macroscopic coherent state. In (1.5.3) we regard η as a small deviation of a from the average value, $v = \langle a \rangle \equiv \langle v|a|v\rangle$. Using the parametrization (1.7.1), we find

$$a = v + \eta = e^{i(\langle\Theta\rangle+\Delta\Theta)}\sqrt{\langle N\rangle + \Delta N} = \left(1 + i\Delta\Theta + \frac{\Delta N}{2\langle N\rangle}\right) v + \cdots, \tag{1.8.8}$$

with

$$v = e^{i\theta}\sqrt{n}, \qquad \eta \simeq \left(i\Delta\Theta + \frac{\Delta N}{2n}\right) v. \tag{1.8.9}$$

We solve this equation for ΔN and $\Delta\Theta$,

$$\Delta N \simeq v\eta^\dagger + v^*\eta, \qquad \Delta\Theta \simeq \frac{i}{2n}\left(v\eta^\dagger - v^*\eta\right), \tag{1.8.10}$$

and obtain

$$(\Delta n)^2 \equiv \langle(\Delta N)^2\rangle \simeq n, \tag{1.8.11a}$$

$$(\Delta\theta)^2 \equiv \langle(\Delta\Theta)^2\rangle \simeq \frac{1}{4n} \tag{1.8.11b}$$

by the use of (1.5.3). Actually, (1.8.11a) is a rigorous formula given by (1.8.4). The minimum uncertainty follows,

$$\Delta n\Delta\theta \simeq \frac{1}{2}. \tag{1.8.12}$$

It is concluded that

$$\frac{\Delta n}{n} \simeq \frac{1}{\sqrt{n}} \to 0, \qquad \Delta\theta \simeq \frac{1}{\sqrt{n}} \to 0 \tag{1.8.13}$$

as $n \to \infty$. Both the particle number and the phase are measurable very accurately on macroscopic coherent states. Indeed, if $n \simeq 10^{22}$, the uncertainties are only $\Delta n/n \simeq 10^{-11}$ and $\Delta\theta \simeq 10^{-11}$, which are negligible experimentally. Both the particle number and the phase are practically good quantum numbers for macroscopic coherent states.