# Introduction

*Once there was nothing. Not even time. But it seems that 13.7 billion years ago this nothing became everything when a tiny dot of infinite density spontaneously expanded at a phenomenal rate giving birth to the universe, including time.*

Michio Kaku, Theoretical Physicist (from TV series "Time")

Solving a polynomial equation could be considered as a game of hide-and-seek with a bunch of tiny dots on a painting canvas. We hide the dots behind a polynomial equation, we then seek them using a formula or an algorithm. *Polynomiography* is the algorithmic visualization of the process of searching for the dots, and painting the canvas along the way.

The above is my informal definition into the fields of polynomial root-finding and polynomiography. It stems from the fact that polynomiography has received a wide range of interest because of which I have had the pleasure and honor of speaking before non-technical as well as technical audiences. I have continually strived to find metaphors that would make audiences get a quick feel of what the underlying foundation is, then when appropriate I would get into more technicalities. But through this kind of informal definition I have also tried to suggest to technical audiences well-familiar with some of the underlying mathematical foundation that there are indeed novelties in polynomiography that in particular could change the way they have viewed or utilized polynomials, and for the better.

Often times I have begun my introductory lectures by explaining first what a polynomial is, using the usual definition: "a linear combination of whole powers of a variable." More formally, one defines a polynomial as:

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \qquad (0.1)$$

where $a_n, \ldots, a_0$ are *coefficients*, $n$ is the *degree*, $a_n$ nonzero, and $z$ is a *variable.*

In this book we are interested in the case where the coefficients are *complex numbers* and $z$ a *complex variable*, to be formally defined momentarily. The polynomial is then called a *complex polynomial.* When the coefficients are real numbers and the variable is restricted to the reals, the polynomial may be addressed as a *real polynomial.* In this book we are interested in complex polynomials, whether or not the coefficients are real or complex.

Formally, a polynomial equation is:

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0. \qquad (0.2)$$

A *solution*, or *root*, or *zero* of a polynomial is any specific value of $z$, say $\theta$ that would satisfy the polynomial equation, i.e. when $z$ is assigned the value $\theta$, (0.2) is satisfied.

It would be highly unlikely that in an introductory lecture on polynomials a mathematician, a scientist, or a teacher would give as example a polynomial having a coefficient that is not a real number, even if a complex polynomial is being defined.

Speaking to a non-technical audience it would be a mistake to define a polynomial as in (0.1) and then give as example a polynomial with an imaginary coefficient since in the process this would also force the speaker into defining the square-root of minus one.

Even if one begins by giving an example of a real polynomial, once the corresponding polynomial equation is defined, there is a need to speak of the Fundamental Theorem of Algebra, the complex numbers, and methods to solve the equation. If in between one happen to speak of graphs of real functions, say a quadratic or cubic polynomial and happens to mention Newton's method for finding zeros, undoubtedly one would create a vague and confusing picture of several concepts such as polynomials themselves, polynomial equations, complex numbers, graphs, and Newton's method to say the least.

This route to defining a polynomial equation could sound reasonable if the real roots of real polynomials are being defined. However, from a personal point of view this approach in defining a polynomial equation is not very effective when the main interest lies in polynomiography which deals with the complex plane.

Come to think of which came first - a polynomial or a polynomial equation? Historically, it seems that a polynomial equation came first. Moreover, to introduce complex polynomials it makes more sense to speak first of a polynomial equation than the raw polynomial. The discovery that square-root of two is not a rational number is really a consequence of solving a quadratic equation, as is its approximation. The notion of an abstract function, or a polynomial as a function, is an abstract notion that even college students need much practice to gain a mature feel for. In summary, a polynomial equation and a polynomial do not necessarily provoke very closely related concepts. They could trigger different concepts the realm or domain of which may mean a different thing to different people, even to specialists.

Furthermore, while in general the concept of complex numbers and the corresponding elementary operations are abstract and difficult to comprehend, it is child's play to speak of locations on a map, or points on a canvas. Even with a real polynomial of degree $n$, according to the Fundamental Theorem of Algebra, we are ensured that what gets hidden behind the corresponding polynomial equation is in fact a set of $n$ points, even if some of these points are placed at the same location.

The following passage from Pan (1997), one of the most authoritative experts in the computational aspects of solving a polynomial equation, eloquently describes the significance of the polynomial root-finding problem:

"*The very ideas of abstract thinking and using mathematical notation are largely due to the study of (0.2). Furthermore, (0.2) has historically motivated the introduction of some fundamental concepts of mathematics (such as rational and complex numbers, algebraic groups, fields, and ideals) and has substantially influenced the earlier development of numerical computing.*"

Since searching for locations on a map from their coordinates is such a familiar task, defining a polynomial equation gives rise to the occasion for defining complex numbers. We may give another informal definition for a polynomial equation:

A polynomial equation is an algebraic encryption of a set of point on a map.

To speak of "hiding points behind a polynomial equation" perhaps also originates or invokes a new perspective at solving a polynomial equation as a task that is not merely that of "seeking the solutions." This together with visualization through polynomiography is the beginning of a new domain of interest and novel applications in the ancient problem of root-finding.

What formalizes the idea of polynomial root-finding as a game of hide-and-seek with dots on a canvas is to convey the fact that a *point* on the *Euclidean plane* is actually a *number*, a *complex number*. This is in the sense that a point on the Euclidean plane with coordinates $(a, b)$, where $a$ stands for the East−West and $b$ for the North−South coordinate, is also a complex number, written as

$$a + ib,$$

where $i$ is the magical number defined as

$$i = \sqrt{-1},$$

i.e. $i * i = -1$. This dual nature is in the sense that a geometric point is actually an algebraic object, a complex number, with respect to which we can perform the four elementary operations on ordinary numbers almost with the same ease. This is a significant discovery behind which lies much brilliance and history due to our forefathers.

Once typical points $(a, b)$ and $(c, d)$ are dressed in their complex number costume, the elementary operations are defined as follows:

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

$$(a + ib) - (c + id) = (a - c) + i(b - d).$$

$$(a + ib) \times (c + id) = (ac - bd) + i(ad + bc).$$

In fact one can easily discover the geometric effect of multiplication of a general number $a + ib$ by the special number $i$ itself, resulting in the clockwise rotation of the point $(a, b)$ by an angle of 90 degrees, arriving at the point $(-b, a)$.

To discover division, it suffices to realize that the reciprocal of a non-zero complex number can easily be derived by conjugation to be:

$$\frac{1}{c + id} = \frac{1}{(c + id)} \frac{(c - id)}{(c - id)} = \left(\frac{c}{c^2 + d^2}\right) - i\left(\frac{d}{c^2 + d^2}\right),$$

another complex number. Then, the definition of division of two complex numbers follows:

$$\frac{a + ib}{c + id} = \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{-ad + bc}{c^2 + d^2}\right).$$

The complex variable $z$ is formally written as

$$z = x + iy$$

with $x$ and $y$ as its *real* and *imaginary* parts, respectively. The *modulus* of $z$, written as

$$r = |z| = \sqrt{x^2 + y^2}$$

is the Euclidean norm of the point $(x, y)$. The *argument* of $z$ is the angle $\theta$ between the point $(x, y)$ as a vector, and the $x$-axis and by convention and for uniqueness of representation is written as an angle satisfying

$$-\pi \leq \theta \leq \pi.$$

The complex variable/number $z$ can then be represented in the *polar* form. Combining the trigonometric polar form with the exponential representation possible through Euler's formula - which embodies De Moivre's formula for powers - one may write a single formula that combines all these for any integer $n$:

$$z^n = (x + iy)^n = (re^{i\theta})^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta).$$

A recommended source for further results on complex operations and geometric interpretations is Wikipedia, the free online encyclopedia. Historical development of complex numbers and the number $i = \sqrt{-1}$ and its interesting history can be traced in such books as, "Imagining Numbers: (particularly square-root of minus fifteen)," Mazur (2003) and "The Story of Square Root of Minus One," Nahin (1998).

The problem of solving a polynomial equation is often truly a task that deals with tiny dots having physical width, as opposed to a task that deals with points which are dimensionless geometric object. This view is justified in the sense that one often needs to approximate the roots of a polynomial equation, rather than computing their exact value which could be an impossible task. Indeed even in the course of looking for the dots we need to approximate the intermediate steps due to round off errors.

It is a classic result that closed form formulas for solutions of a general polynomial equation is only possible when the degree is less than five. Even for polynomials of degree less than or equal to four, since the closed formulas involve radicals, from the computational point of view even the case of square-root of a number needs to be approximated through iterative methods. Thus, in general we can only approximate the roots and need to be content with any approximation that would fall within disks of prescribed radii, say $\epsilon$, centered at the exact values.

This book is based on solving a polynomial equation using iteration functions, but with major emphasis on the use of a very special and fundamental family of iteration functions called the *Basic Family*, as well as many iteration functions derived from this family or related to the family. The book reveals many mathematical and algorithmic properties of the Basic Family and many significant connections between the family and different mathematical concepts. The goal of the book is not merely to focus on the computation or approximation of the set of roots of a polynomial equation, but much more, including of course polynomiography, a subject turning the task of solving a polynomial equation upside down, potentially bringing wide range of interest into this ancient problem.

Quoting again from Pan (1997), this time with regard to the scope of today's significance in solving a polynomial equation:

"*In fact, as n grows beyond 10 or 20, the present-day practical needs to solve equation (0.2) become more and more sparse, with one major exception: equation (0.2) retains its major role (both as a research problem and a part of practical computational tasks) in the highly important area of computing called computer algebra, which is widely applied to algebraic optimization and algebraic geometry computations.*"

Not only polynomiography would bring novel interest into the root-finding problem, but in the process it would bring interest in the visualization of much higher degree polynomials than degree 10 or 20. Even middle and high school students exposed to polynomiography always seem to want to go to higher and higher degree polynomials. This says that the nature of "application" effectively changes the need for solving polynomial equations.

Before providing a brief description of chapter contents, also offering a guideline to the reader I wish to acknowledge a number of books on polynomials. Essentially, all these books are directed at specialized or advanced audiences, each a very valuable source of information on polynomials, yet the collection gives evidence on how vast and versatile polynomials are. In a sense it would perhaps be very fair to claim that no one can ever master polynomials or polynomial equations.

The present book neither relies on these books, nor is it intended to be complementary to them. It is a book that is hoped to give a completely novel and popular view of polynomials and polynomial equations. No doubt there may lie many imperfections. Yet optimistically it should broaden the scope of polynomial root-finding to a level much beyond its predecessor books.

"Numerical Methods for Roots of Polynomials," McNamee (2007) a recent book with many results on the particular problem of polynomial root-finding from the iterative point of view. The author's monumental work of gathering an online bibliography of publications on root-finding contains over 8000 items (yet non-exhaustive), of which 50 were published in 2005.

"Polynomial and Matrix Computation," Bini and Pan (1994) deals with computations with polynomials and significant underlying algorithms.

"Polynomials," Barbeau (1989) deals with many topics on polynomials, from very elementary to more advanced.

"Complex Polynomials," Sheil-Small (2002), and "Geometry of Polynomials," Marden (1966) both deal with the geometric theory of polynomials and rational functions in the plane, bringing ideas from algebra, topology, and analysis. In particular they consider the location of zeros of polynomials and those of their derivatives.

"Polynomials," Prasolov (2004) deals with classical and modern algebraic point of viewpoint, including Galois theory.

"Polynomials and Polynomial Inequalities," Borwein and Erdélyi (1995) deals with analytic properties of polynomials as well as such topics as geometric properties, orthogonal polynomials, and inequalities.

"Fundamental Problems of Algorithmic Algebra," Yap (1999) though is not just on polynomials deals with topics from the point of view of computer algebra, the study of efficient algorithms for algebraic operations.

"The Fundamental Theorem of Algebra," Fine and Rosenberger (1997) gives several formal proofs of the theorem in the course of which the reader is introduced to complex analysis.

Finally, there are other books that deal with polynomials from the point of view of dynamical systems. We will refer to these later in the book.

The present book offers modern and novel perspectives into the theory and practice of the historical subject of polynomial root-finding, rejuvenating the field via polynomiography, a creative and novel computer visualization that renders spectacular images of a polynomial equation. Polynomiography will not only pave the way for new applications of polynomials in science and mathematics, but also in art and education. The book presents a thorough development of the Basic Family, arguably the most fundamental family of iteration functions for root-finding, deriving many surprising and novel theoretical results and practical applications such as: algorithms for approximation of roots of polynomials and analytic functions, polynomiography, bounds on zeros of polynomials, formulas for the approximation of $\pi$, characterizations of solutions of homogeneous recurrence relation, their polyhedral representation, visualizations associated with a homogeneous linear recurrence relation, connections to Voronoi regions, continued fractions, Fibonacci and Lucas numbers and their generalizations, even novel views into classical theorems such as the Gauss-Lucas theorem and the maximum modulus principle. These discoveries and a set of beautiful images provide new visions and appreciations to polynomials and recurrence relations. The book also describes some polynomiography-related experiences with educators, students, and artists, including middle and high school teachers and students, and a summary of their feedback with respect to the utility of polynomiography.

This book is for all mathematicians, scientists, advanced undergraduates and graduate students, any one who has come to deal with polynomials in formal settings. However, chapters or parts of this book are also for anyone with an appreciation for the connections between a fantastically creative art form and medium and its ancient mathematical foundations.

Many chapters can be read independently of each other, and relevant formulas for the Basic Family are repeated in different chapters. Some chapters include a list of research problems.

Chapter 1, derives algorithms for approximation of square roots and offers their visualization in the complex plane.

Chapter 2, makes use of the Fundamental Theorem of Algebra to derive a special case of Taylor's Theorem that is the genesis of iteration functions for polynomial root-finding algorithms and the Basic Family.

Chapter 3, offers an introduction to the Basic Family of iteration functions and to polynomiography.

Chapter 4, derives several equivalent formulations of the Basic Family.

Chapter 5, deals in much detail with the iterations of the Basic Family from the point of view of dynamical systems, developing the essential parts of the theory of iterations of rational functions, but keeping in view the Basic Family itself. In particular, this chapter should be a useful chapter for anyone interested in the essentials of the dynamics of iterations of rational functions.

Chapter 6, analyzes the properties of the fixed-points of the Basic Family.

Chapter 7, gives an algebraic derivation of the Basic Family and proves many characterizations and optimality results.

Chapter 8, defines and analyzes the Truncated Basic Family and a special case, called Halley Family.

Chapter 9, develops characterizations of solutions of homogeneous linear recurrence relations via the Basic Family and gives polyhedral representations.

Chapter 10, derives a significant determinantal generalization of Taylor's Theorem and Newton's method and its applications in approximation theory.

Chapter 11, develops a multipoint version of the Basic Family and analyzes their order of convergence.

Chapter 12, presents a computational comparison of some of the multipoint Basic Family members.

Chapter 13, develops a general determinantal lower bound and describes its specific applications in root-finding.

Chapter 14, develops new formulas for approximation of pi based on root-finding algorithms, specifically the use of the Basic Family.

Chapter 15, makes use of the Basic Family to derive a unique family of bounds on the roots of polynomials and analytic functions.

Chapter 16, defines a single geometric optimization related to polynomials and derives as its algebraic offsprings the Gauss-Lucas Theorem, the Maximum Modulus Principle, and novelties such as problems in computational geometry, and also introduces the Gauss-Lucas iteration function.

Chapter 17, describes polynomiography as algorithms for visualization of polynomial equations.

Chapter 18, offers techniques for the visualization of homogeneous linear recurrence relations via polynomiography, also develops a whole new family of iteration functions from the Basic Family, the "Induced Basic Family."

Chapter 19, offers applications of polynomiography in art, education, science and mathematics.

Chapter 20, revisits the approximation of square-roots via the Basic Family and develops connections to continued fractions and factorization of integers.

Chapter 21, offers further applications and extensions of the Basic Family and polynomiography.

Whenever possible the book complements the concepts within each chapter via polynomiography images.

**Acknowledgements.** I would like to thank many people who - in one form or another - have been helpful in the course of completion of this book over several years, or with respect to polynomiography activities addressed in this book, or with respect to the motivation of these activities, or for participation in these activities, or simply for their encouragements. Also, those who have motivated, inspired, collaborated, or have helped me in different ways. All have made me even more determined to advance the topic of polynomial root-finding and polynomiography, hoping to extend the popularity of the latter to the point of bringing it to K-16 education and beyond, perhaps even to the general public. Specifically, I mention the following individuals.

I would like to thank Michio Kaku for permission to quote him.

I would like to thank Curt McMullen for several electronic communi-

cations on the subject of dynamical systems and for clarifications of some technical issues, also Tan Lei and Mitsuhiro Shishikura.

I would like to thank Doron Zeilberger for his encouragements, inspirations, and enthusiasm over several years, and for his appreciation of both the technical contents and images in this book.

I would like to thank Yi Jin who as my Ph.D. student wrote a very nice thesis on the combinatorial aspects of polynomial root-finding. This book could only afford to highlight some the results which certainly deserve further attention.

I would like to thank Iraj Kalantari for having been a collaborator in several different ways and for being a significant catalyst and advisor in making possible some very fruitful activities, such as running teacher workshops in New Jersey and in Illinois, and more activities to come.

I would like to thank Fedor Andreev, an ongoing collaborator on some polynomiography-related activities that includes his expert programming of a version of polynomiography software.

I would like to thank Ali Maher for his support that made it possible to organize a very successful teacher workshop at his center CAIT at Rutgers.

I would like to thank several people who in the past few years have made it possible to give special presentations on polynomiography outside of Rutgers. They include Vasek Chvátal, Claude Bruter, Alfred Vendl, Rudolph Taschner, Steven Wolfram, and Iraj Kalantari for presentations at Condordia University (Montreal), Henrie Poincaré Institute (Paris), Angewandte Kunts (Vienna), MuseumsQuartier (Vienna), NKS conference (Boston), and University of Western Illinois (Macomb), respectively.

I also like to thank Reza Emamy-K for several invitations to University of Puerto Rico over the years and for his interest to bring polynomiography to Puerto Rico.

I would like to thank all those at Rutgers who have been supportive of my polynomiography activities over the years. In particular, I thank Kathleen Hull, Victoria Ukachukwu and Sarolta Takács for their encouragements and arrangements in the teaching of polynomiography courses such as First-Year seminars (now Byrne Seminars) and honors courses. Also, Kathleen Hull, Sara Harrington, and Joseph Consoli who made possible a very nice polynomiography exhibition at the Art Library of Rutgers, a photograph of which is featured in the book.

I also thank several K-12 mathematics educators for their invitation to make presentations at their middle and high schools, or their enthusiasm that has brought me closer to K-12 educators and a participant at K-12

teacher conferences in New Jersey, Illinois, Pennsylvania and beyond. They include Carol Ann Altamura (Randolph High School), Jan Gebert (Warren Hills Regional High School), and Mike Stern (Montgomery High School) in New Jersey, Candy Rosene (Western Illinois University) and Beth Shyrock (Macomb High School) in Illinois. It is a pleasure to thank some of their students as well for their enthusiasm which has given me more incentive and determination to proceed with polynomiography in the field of education. In particular, Elizabeth Sergison in New Jersey and Alexandria Munger in Illinois.

I would like to thank Lillian Schwartz with whom I have had many fruitful electronic communications regarding art, digital art, and polynomiography as art.

I would like to thank Hossein Talakoub for his efforts in arranging the production of a beautiful hand-woven carpet based on a polynomiography design featured in this book.

I would like to thank Jihui Zhao for doing a nice job in preparing most of the non-polynomiography figures of this book and for his expert technical assistance with Latex and Beamer.

I would like to thank Ms. Chionh, the Editor of World Scientific for countless communications and for her assistance in bringing this book to publication, and with the level of perfection I had hoped for.

I also thank Aileen Goh of World Scientific who used my desirable foreground and background polynomiography images to come up with the nice cover design.

Finally, I would like to thank my siblings, Iraj, Laleh and Khosrow who have given me much encouragements and moral support in pursuing my goals and obsessions with polynomiography. Even my niece and nephews who ended up encouraging me in their own ways. These include Sara, Leo, and Arshaun who assured me that even a four year old could learn to say "polynomiography."