Preface

The genesis of this book dates back to the design of a question for a graduate exam at Rutgers computer science department in the early 1990s. I discovered a method for the approximation of square-root of two, requiring only high school algebra. It coincided with the well-known Newton's method as applied to a corresponding quadratic equation. It appeared interesting and different from the ordinary approach in developing Newton's method. Moreover, it could be easily extended to higher order methods: the third-order method of which for the approximation of square-roots coincided with a method credited to Edmund Halley, the astronomer who has a comet named after him. What I thought would be a matter of a very brief time before I would be sufficiently enlightened about this discovery and then leave it aside has in fact turned out to have taken me beyond my wildest imagination. This book, a by-product of many years of my research, is thus an unexpected fruit of the study of the square-root of two.

I soon came to realize that what stood before me was a mountain. New questions came up and more and more research problems evolved. In short, the polynomial root-finding problem attracted me so much that since then I have never abandoned it. Now I consider it as one of the most fascinating problems of mathematics and science having so much to offer to so many. The mountain that stood before me has no summit, or infinitely many.

Indeed dating back to the ancient civilizations solving algebraic equations has been among the most fascinating and profound intellectual tasks. Even the case of solving quadratic and cubic polynomials - arising naturally in algebraic or geometric settings - it has inspired deep discoveries such as, the irrational numbers, the complex numbers, and even sophisticated algorithms for integer factorization. Perhaps my attraction to the problem during all these years is also due to the very profound nature and gravity

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of the polynomial root-finding problem, but additionally the beauty of its algorithmic visualization by what I have come to call *polynomiography*.

I came to Rutgers as a computer scientists with expertise in mathematical programming and optimization. Not only the study of polynomial root-finding was not among my research areas at the time, in the eyes of many, the problem was and perhaps still is considered to be old-fashioned and done with. Unless one is specialized in specific aspects such as the complexity of root-finding, or in the context of broader fields of study such as algebra, complexity analysis, computer algebra, dynamical systems, numerical analysis, etc., it is not a common practice to study polynomials in their own right. While scientists and mathematicians use polynomials routinely, no one addresses oneself as a "polynomial theorist." From the academic point of view it was thus very risky to dedicate my research and time into the study of polynomial root-finding with so much history behind it.

Not that I would be abandoning doing research on my other areas of interest, but I knew that by doing so I would diminish such opportunities as receiving grant funding which in turn opens up the way for further explorations, graduate student support, support to attend conferences, and receiving more visibility. I also knew that I would subject myself to unpredictable judgements. Despite these risks and drawbacks I continued to work on the problem because it was simply too beautiful to resist. For several years I was merely interested in the theoretical aspects of polynomial root-finding and related problems I encountered or needed to invent along the way. While I continued to produce publications and brought several collaborators into the field, generally speaking for the most part I found doing research in this field somewhat like mountain climbing where one does it for personal satisfaction of different kinds and simply has to accept the risks that come along.

During the first few years I did not even consider computer visualization of the root-finding process. I knew that images coming from iteration functions generally would fall within the category of images known as *fractal*, a term coined by the famed Mandelbrot. Initially, I did not think that images coming from solving polynomial equations would be much different from images already produced by experts and even many more so by amateurs. For instance, a fractal image coming from the approximation of cubic roots of unity based on Newton's method is very familiar and has even been featured on textbook covers.

However, as I became deeply involved in the root-finding problem, dis-

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covering more and more algorithms, it became evident that the computer visualization of the root-finding process through these algorithms would be interesting and worthy of trial. This was partly because of so many choices of algorithms and partly because from the theory I could anticipate the shape of the images coming from some of these algorithms. This was very promising and important since I could sense a reasonable degree of "control" and "design" as opposed to typical fractals.

The initial computer visualization did give rise to striking images and I could foresee that this would just be a beginning of a new set of activities involving visualization. Eventually I felt that this visualization deserved a name of its own. Thus I coined the term *polynomiography* to be the art and science of visualization in the approximation of roots of polynomials (using iteration functions). The word is simply a combination of "polynomial" and the suffix "-graphy." For certain, some would not only find the word hard to pronounce, but perhaps even unnecessary since the term fractal was already so well-known.

However, not only a polynomiography image, called *polynomiograph* is not necessarily a fractal image, even when it is a *fractal polynomiograph* it is very distinct from a typical fractal image and has a precise foundation. Moreover, fractal polynomiographs result in new class of visualizations and thereby dramatically enhance the horizon of fractals. In contrast to polynomiography, the word fractal is so broad and could be interpreted so vaguely that anything from an image of a Julia set coming from the iterations of a quadratic or cubic, to an image from Newton's method, to a complete binary or ternary tree, to a decimal expansion of a rational number, to an ordinary tree, to a mountain, and even to a galaxy, it could all be considered to be fractal. Repetition does not necessarily lead to fractal.

In contrast even in the domain of rational iteration functions on the complex plane, the word polynomiograph is very distinct from a typical fractal image due to such iterations. Indeed even if in such a fractal image we sufficiently zoom into a Fatou component of the underlying rational iteration, there is nothing fractal about the image. In summary, the term polynomiography is quite a logical term and a deserving one, immediately putting a face behind the image, namely a polynomial equation. Moreover, after a few iterations in pronouncing the term, it turns into a memorable and a meaningful one.

After many years of doing polynomial root-finding and polynomiography and numerous experiences that include national and international presentation, I am now convinced that the field of polynomiography will

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change the view of polynomials and dramatically extend their usage. I feel that polynomiography has the potential to turn into a creative activity that would popularize math among the youth, bring innovations to art and design, bridge art and math, inspire mathematicians and educators, but also engage the general public.

While polynomials remain to be fundamental entities in science and math and in education, never before in their history has there been a systematic development of algorithms to reveal the magnificent visual beauty behind solving polynomial equations. More so to liberate polynomials and to widen their scope of utility to a scale never before imagined. Polynomiography is the algorithmic visualization of polynomial equations. While polynomiography uses sophisticated mathematical algorithms on a computer to create a polynomiograph, with proper software development it turns the polynomial root-finding problem upside down and into a medium of expression, art, design, science, math, education, innovation, discovery, creativity and more.

I have delivered many lectures on polynomiography or the theory that has inspired it, at many levels, at many locations, and to many different audiences. These include, from presentation at theoretical conferences on mathematics or computer science, such as numerical analysis, number theory, computational geometry, computer graphics, art-math conferences, to presentations with audiences that consisted of middle and high school students, university students of science and math or art and graphics, mathematicians and computer scientists, K-12 teachers, general artists, art curators, and even the general public, and in several different countries.

I have had solo or group exhibitions at museums and art galleries. My images have appeared on various covers or inside of magazines and books. Also, articles about polynomiography have appeared in some national and international media. I have even developed a few courses on polynomiography, and with the help of some collaborators have conducted separate teacher and student workshops. During these activities a demo polynomiography software has been tested by the respective middle or high school students, college students, and K-12 teachers. It has never failed to arouse the curiosity and interest of the participants, followed by interesting and deep questions and more importantly much interest in wanting to learn more about polynomiography and its foundation. Because of polynomiography, I have witnessed more interest in polynomials from middle school students than college student who have only been exposed to polynomials abstractly.

Ironically, despite the very multidisciplinary and innovative nature of

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polynomiography and numerous claimed potentials (that continue to become more and more evident), or the tremendously rich and profound underlying theoretical foundation, at times I have been perceived as an artist when expecting to be considered a computer scientist or a mathematician and educator, while at other times viewed as a scientist or a mathematician when expecting to be considered an artist or an educator. Claiming to be all is unimaginable to some, and a contradiction in terms. Though I had never considered doing art seriously before polynomiography - nor did I try to label myself as an artist after - I am pleased to say that through polynomiography I have come to learn a few things about art and have received the recognition and appreciation of artists of different kinds - from traditional artists, to digital artists, to the general public. Thanks to polynomiography I have also dared to consider myself an artist in addition to being a computer scientist and a mathematician, and given the opportunity I would fill up a large gallery with my personal artwork. Furthermore, I feel that the invention of polynomiography technology for which I hold a U.S. patent will also create many new artists.

As an “algorithmic artist” I consider myself a “polynomiographer,” not a “fractal artist.” Personally, I enjoy creating a beautiful polynomiograph as much as proving a beautiful theorem, realizing well that beauty is in the eyes of beholder. Indeed I can no longer distinguish between the two kinds of creative activities. Mathematics and art are closely related and computer science as algorithms and technology allows mathematics to unveil its visual beauty. Polynomiography is a set of unique algorithms that unveils the beauty of polynomial equations. I foresee that polynomiography would become well-utilized someday, especially as a powerful medium in education and in art.

Since I make strong claims on the potentials of polynomiography, it is perhaps fair to give the reader a sample of the evaluations and reviews I have received - positive and negative - in order for the reader to judge for himself/herself as to which point of view they would tend to agree with, but also exposing the reader to the vast possibilities in polynomial root-finding and polynomiography.

The reviews to be considered here are limited to the cases of submission of few grant proposals to funding agencies, or submission of articles and proposals to publishers.

On an interdisciplinary proposal to a funding agency regarding polynomiography applications in art, science, math, and education - when the subject was in its earlier stages - a reviewer wrote:

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"*Root-finding is indeed a hard field to make a splash in. Since the Sumerians, ancient Greeks, Isaac Newton, through Hermann Weyl and Steve Smale, the best minds have given it a shot. I have taught the material in the classroom and have avoided the glitz associated with a lot of fractal geometry for fear of giving students less than what they need. Well, I am now turning my head! This proposal represents a serious contribution to global root-finding algorithms, computer generated art, and bringing it all to the masses.*"

Ironically, a second anonymous and independent reviewer on the same proposal - though seemingly impressed with the mathematical developments - merely summarized the worth of the proposal as

"*My impression is, 'who cares!'* "

On the journal submission of a theoretical article on the subject of iteration functions for root-finding while a reviewer considered the work among other things "*significant and scholarly*," another reviewer wrote, "*there is little that would be of value to practising numerical analysts who are already equipped with all the iteration functions they need.*"

A polynomiography proposal focused on applications in K-12 math education with several enthusiastic participating consultants and collaborators from diverse locations in the country, including mathematicians with expertise in K-12 education, and with reasonable initial experimental evidence - to the extent possible without funding - despite the fact that its goals were considered to address a significant problem of national interest was judged to have insufficient experimental evidence!

An anonymous referee of a proposal for a popular or non-technical book on polynomiography suggested that such book could be billed as an elementary introduction to the Riemann Hypothesis. He justified this by saying that though the Riemann Hypothesis (the most famous unsolved problem of mathematics with a one-millon dollar prize for a solution) is not about polynomials, like polynomiography it is about location of complex roots of a function. The reviewer however went on to claim that the availability of computer and specialized software has greatly reduced the level of interest in how one goes about finding roots of polynomials, or solutions of equations in general.

These reviews are revealed here because I would like to raise several important questions regarding the study of polynomial root-finding and polynomiography. Questions that deserve deeper scrutiny than a response based on impulse.

To whom do polynomials belong? Is polynomial root-finding a task

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exclusive to numerical analysts? Is Newton's method sufficient for root-finding? If the study of iteration functions is pointless why do experts of dynamical systems ("dynamicists") study iterations of rational functions?

A numerical analyst, a mathematician, or a scientist familiar with Newton's method and its quadratic order of convergence may consider the study of higher order iteration functions for root-finding pointless because he/she may argue as follows: if we consider every other iterate in Newton's method we get a sequence which if convergent, it will result in a fourth order method. And if we consider every third iterate, the sequence if convergent, it will result in an eight order method, and so on. Indeed I have heard this argument several times as a quick way to dismiss the study of iteration functions for root-finding. But this view completely misses the point of iteration functions and at best only makes sense when in a small neighborhood of a polynomial root. Moreover, such view should also find pointless the work or Fatou, Julia, and even Halley - a contemporary of Newton - whose iteration function was inspired by Newton's, yet in turn it inspired the celebrated Taylor's Theorem. In summary, such view should also find pointless the entire theory of iteration of rational functions, and polynomiography. Thus it is a superficial view and should be discarded.

Can one ever claim that polynomials are so well understood that need no further research? Can any educator claim that no further worthy curricula can be developed based on polynomials and their applications? What if students acquire a liking of polynomiography and working with polynomials to the extent that they would end up raising deep mathematical questions? What if they would get exposed to deep mathematical concepts through exciting and fun visualizations? How much evidence is needed to convince a reviewer or an educator to approve of the use of a novel medium in the classroom?

Clearly, the utility of polynomials is not restricted to scientists and mathematicians or teachers of math and science. Nor are specialists dealing with polynomials the only ones who can truly appreciate or judge the utility of these fundamental objects of mathematics. Indeed polynomial root-finding through polynomiography has a chance of becoming a subject of interest to many, from K-12 education, to higher education, and even the general public.

Polynomial root-finding is not solely about computing or approximating the roots. It is a process and understanding it. Polynomiography enhances this process and reverses the root-finding problem. One can learn to play with the roots: place them as one pleases and then find the roots

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of the corresponding polynomial using polynomiography techniques. The Fundamental Theorem of Algebra, one of the most celebrated theorems of mathematics - through polynomiography - suddenly becomes visible, obvious, and as believable as the force of gravity. It becomes discoverable even by children and consequently popular and appreciable. At the same time while the foundation of polynomiography relies on the Fundamental Theorem of Algebra - through polynomiography - it becomes evident that despite many existential proofs, algorithmic attempts in proving the theorem poses a whole new set of mysteries and interesting problems.

In fact suppose that one takes just a few points on the Euclidean plane, say three or more, then forms a polynomial having those as its roots, then selects an arbitrary point. Now consider the question: Does Newton's method gravitate the point toward any of the roots? Physicists may get closer and closer to the understanding of the true nature of gravity, but neither theoretical physicists, nor mathematicians would be able to make a definitive decision on this simple-looking question. It is undecidable!

It is thus fair to say that despite the fact that some "experts" of various fields may have found or may find the root-finding problem old-fashioned, or too specialized, etc., it is in fact an inexhaustible problem and through polynomiography it could become interesting to many different groups and audiences of diverse backgrounds. Polynomials are foundational to math and education. Indeed whenever students are introduced to notions such as functions, derivatives, integrals, solution of equations, graphs and much more, a polynomial is the first example to be considered. Polynomiography opens the way for bringing new views and applications into these topics and even more importantly providing a platform to offer new teaching curricula.

It would of course be tremendously rewarding if polynomiography would ever serve as a medium that would bring any attention or introduction to the Riemann Hypothesis. But it would be at least equally rewarding if polynomiography would help bring recognition and popularity to polynomials themselves. Perhaps like Fermat's Last Theorem, the Riemann Hypothesis too will someday be concurred. But it is plausible that long after a solution to the Riemann Hypothesis is found, polynomiography would still remain to be useful and interesting to many, not just to specialists.

During the years since introducing polynomiography, I have received many interesting questions in one form or another. Here is a sample of such questions from K-12 students. A 7-th grader wrote,

"*I like polynomiography, what are polynomials?* "

An 8-th grader and her classmates wrote,

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"*We love your work, are the polynomials you use complex?*"

She and her teacher eventually invited me to their school in New Jersey for a presentation to many students. At a summer math camp for 11−13 year old girls in Illinois who were given the chance to play with a simple demo polynomiography software a student wrote,

"*I love the polynomiography software! I'm into art so to see that math is related to art was really cool. Thank you for that opportunity.*"

She went on to request the camp teachers to allow older girls to attend the following year. The reason: she "*loved it so much*" that she wanted to attend again, but she would be 14 the following year. This type of enthusiasm about working with polynomials is quite novel.

In addition to students, educators from middle schools, high schools, and colleges have expressed excitement and interest in polynomiography. Detailed feedback from both students and teachers will be given in the book. These and many other examples and experiences signify an unprecedented level of interest in polynomiography. And the evidence is mounting. Polynomiography is here to stay and there is really no limit in the extent of its applications.

This book is not intended for the high school student or even a typical undergraduate student - such books are subject of future projects. However, there are chapters or topics in this book that could appeal to an advanced undergraduate student, a high school mathematics teacher, or even an art teacher, or an artist. The general reader of this book could include graduate students of mathematics or science, mathematicians and scientists, science or math teachers. Particular courses where some material from this book can definitely be used include algebra, calculus, numerical analysis, complex analysis, dynamical systems, number theory, computer graphics, and specialized courses dealing with patterns, symmetry, algorithmic art, or honors courses as in interdisciplinary courses. But some parts of this book could perhaps even inspire a high school student to learn about such notions as square-root of minus one, complex numbers, polynomial equations, root-finding algorithms, functions, geometry, Voronoi regions, Fibonacci numbers, homogeneous recurrences, iterations, Newton's method, convergence, limit, the fundamental theorem of algebra, the Gauss-Lucas theorem, the maximum modulus principle, fractals, Julia sets, algorithms and art.

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