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MORPHISMS OF PSEUDOGROUPS AND FOLIATED MAPS

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1 Introduction

In [2], we have introduced the concept of *morphism* of pseudogroups generalizing the *étalé morphisms* of Haefliger [11]. With our definition, any continuous map between foliated spaces mapping leaves to leaves (a *foliated map*) induces a morphism between the corresponding holonomy pseudogroups. This concept can be interpreted as a *morphism* of S-*atlases*, defined by Van Est [17], and is more general than a homomorphism of étalé groupoids: only transverse foliated maps induce homomorphisms of holonomy groupoids. The main result of [2] states that any morphism between complete pseudogroups of local isometries is *complete*, has a *closure* and its maps are C^∞ along the orbit closures. Here, *completeness* and *closure* are obvious versions for morphisms of concepts introduced by Haefliger for pseudogroups. The proof of this theorem only involves basic techniques, but it is rather complicated. It is applied to approximate foliated maps by smooth ones in the case of *complete Riemannian foliations*, yielding the *foliated homotopy* invariance of their *spectral sequence*.

The goal of this paper is to clarify that theorem by giving a simplified proof for the case of dense orbits, recalling the main ideas without many details. Applications, examples and related open problems are also given.

2 Morphisms of pseudogroups

A *pseudogroup* $\mathcal{H}$ of local transformations of a space T (or *acting* on T) is a collection of homeomorphisms between open subsets of T which contains the identity map id_T, and is closed under the operations of composition (wherever defined), inversion, restriction to open sets and combination [11, Section 1.4]. The pseudogroup $\mathcal{H}$ is *generated* by a subset $S \subset \mathcal{H}$ if any map in $\mathcal{H}$ can be obtained from S by using these operations. The *restriction* of $\mathcal{H}$ to a subspace $T_0 \subset T$ is the pseudogroup $\mathcal{H}|_{T_0}$ of local transformations of T_0 that can be locally extended to maps in $\mathcal{H}$. If T is a C^∞ manifold and the maps in $\mathcal{H}$ are C^∞, then $\mathcal{H}$ is called C^∞. The basic dynamical concepts can be generalized to pseudogroups: *orbits*, *saturations*, *invariant* or *saturated* sets, etc. The quotient space of $\mathcal{H}$-orbits is denoted by $\mathcal{H}\backslash T$. For any open $U \subset T$, let $\mathcal{H}_U = \{h \in \mathcal{H} \mid \mathrm{dom} h = U\}$.

Let $\mathcal{H}$ and $\mathcal{H}'$ be pseudogroups acting on spaces T and T'. According to [11], an *étalé morphism* $\Phi : \mathcal{H} \to \mathcal{H}'$ is a maximal collection of homeomorphisms of open subsets of T to open subsets of T' satisfying the following properties.

(i) If $\phi \in \Phi$, $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$, then $h' \circ \phi \circ h \in \Phi$.

(ii) The domains of elements of Φ cover T.

(iii) If $\phi, \psi \in \Phi$, then $\psi \circ \phi^{-1} \in \mathcal{H}'$.

If $\Phi^{-1} = \{\phi^{-1} \mid \phi \in \Phi\}$ is an étalé morphism too, then Φ is called an *equivalence* and $\mathcal{H}$ is said to be *equivalent* to $\mathcal{H}'$. In this case, $\mathcal{H}'' = \mathcal{H} \cup \mathcal{H}' \cup \Phi \cup \Phi^{-1}$ is a pseudogroup acting on the topological sum $T'' = T \sqcup T'$ whose orbits cut T and T', and so that $\mathcal{H}''|_T = \mathcal{H}$ and $\mathcal{H}''|_{T'} = \mathcal{H}'$. Reciprocally, if a pseudogroup $\mathcal{H}''$ acting on T'' satisfies these conditions, then $\{\phi \in \mathcal{H}'' \mid \mathrm{dom}\phi \subset T,\ \mathrm{im}\phi \subset T'\}$ is an equivalence $\mathcal{H} \to \mathcal{H}'$.

We want to generalize étalé morphisms by involving arbitrary local maps, and thus the above condition (iii) must be modified accordingly.

Definition 2.1 A *morphism* $\Phi : \mathcal{H} \to \mathcal{H}'$ is a maximal collection of continuous maps of open subsets of T to T' satisfying the following properties

(i) If $\phi \in \Phi$, $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$, then $h' \circ \phi \circ h \in \Phi$.

(ii) The domains of elements of Φ cover T.

(iii) If $\phi, \psi \in \Phi$ and $x \in \mathrm{dom}\phi \cap \mathrm{dom}\psi$, then there is some $h' \in \mathcal{H}'$ with $\phi(x) \in \mathrm{dom}h'$ and so that $h' \circ \phi = \psi$ on some neighborhood of x.

A morphism $\mathcal{H} \to \mathcal{H}'$ induces a continuous map $\mathcal{H}\backslash T \to \mathcal{H}'\backslash T'$. When $\mathcal{H}$ and $\mathcal{H}'$ are C^∞, a morphism $\mathcal{H} \to \mathcal{H}'$ is said to be C^∞ if it consists of C^∞ maps.

Let Φ_0 be a family of continuous maps of open subsets of T to T' satisfying the following properties:

(ii') The $\mathcal{H}$-saturations of the domains of elements of Φ_0 cover T.

(iii') There is a set S of generators of $\mathcal{H}$, such that, if $\phi, \psi \in \Phi_0$, $h \in S$ and $x \in \mathrm{dom}\phi \cap \mathrm{dom}(\psi \circ h)$, then there is some $h' \in \mathcal{H}'$ with $\phi(x) \in \mathrm{dom}h'$ and so that $h' \circ \phi = \psi \circ h$ on some neighborhood of x.

Then there is a unique morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ containing Φ_0, which is said to be *generated* by Φ_0. Observe that morphisms consisting of local homeomorphisms are precisely those generated by étalé morphisms.

The *composition* of two consecutive morphisms is the morphism generated by the composites of the corresponding maps (wherever defined). With this operation, these morphisms form a category PsGr, whose isomorphisms are the morphisms generated by equivalences. Notice that id_T generates the *identity morphism* $\mathrm{id}_\mathcal{H}$ at $\mathcal{H}$ in PsGr, and $\mathcal{H} \subset \mathrm{id}_\mathcal{H}$. The *restriction* of a morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ to a subspace $T_0 \subset T$ is the morphism $\Phi|_{T_0} : \mathcal{H}|_{T_0} \to \mathcal{H}'$ consisting of all maps of open subsets of T_0 to T' that can be locally extended to maps in Φ. The inclusion map $T_0 \hookrightarrow T$ generates a morphism $\mathcal{H}|_{T_0} \to \mathcal{H}$, whose composition with any morphism Φ is $\Phi|_{T_0}$.

The *product* of two pseudogroups, $\mathcal{H}_1$ and $\mathcal{H}_2$ acting on spaces T_1 and T_2, respectively, is the pseudogroup $\mathcal{H}_1 \times \mathcal{H}_2$ acting on $T_1 \times T_2$ generated by the product of maps in $\mathcal{H}_1$ and $\mathcal{H}_2$. The *product* of two morphisms $\Phi_i : \mathcal{H}_i \to \mathcal{H}'_i$, $i = 1, 2$, is the morphism $\Phi_1 \times \Phi_2 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}'_1 \times \mathcal{H}'_2$ generated by the products $\phi_1 \times \phi_2$ with $\phi_1 \in \Phi_1$ and $\phi_2 \in \Phi_2$. The *pair* of two morphisms $\Phi_i : \mathcal{H} \to \mathcal{H}'_i$, $i = 1, 2$, is the morphism $(\Phi_1, \Phi_2) : \mathcal{H} \to \mathcal{H}'_1 \times \mathcal{H}'_2$ generated by the pairs (ϕ_1, ϕ_2), where $\phi_1 \in \Phi_1$ and $\phi_2 \in \Phi_2$ have the same domain.

Let Top be the category of continuous maps between topological spaces. There is a canonical injective covariant functor Top $\to$ PsGr which assigns the pseudogroup generated by id_T to each space T, and assigns the morphism generated by f to each continuous map f. We will consider Top as a subcategory of PsGr in this way. Many topological concepts can be generalized to pseudogroups by using morphisms and orbits instead of continuous maps and points: *path connectedness*, *homotopies*, *homotopy equivalences*, *singular (co)homology*, *homotopy groups* (see [10, 17]), etc.

3 Holonomy pseudogroups of foliated spaces

A *foliated structure* $\mathcal{F}$ *of dimension* $n \in \mathbb{N}$ on a space X can be described by a *defining cocycle* [12], which is a collection $\{U_i, p_i\}$, where $\{U_i\}$ is an open cover of X and each p_i is a topological submersion of U_i onto some space T_i whose fibers are connected open subsets of $\mathbb{R}^n$, such that the following *compatibility condition* is satisfied: for every $x \in U_i \cap U_j$, there is an open neighborhood $U^x_{i,j}$ of x in $U_i \cap U_j$ and a homeomorphism $h^x_{i,j} : p_i(U^x_{i,j}) \to p_j(U^x_{i,j})$ so that $p_j = h^x_{i,j} \circ p_i$ on $U^x_{i,j}$. Another defining cocycle $\{U'_a, p'_a\}$ determines the same foliated structure when $\{U_i, p_i\} \cup \{U'_a, p'_a\}$ is a defining cocycle. The space X endowed with $\mathcal{F}$ is called a *foliated space*. The usual terminology of foliations can be generalized to foliated spaces: *foliated chart*, *foliated atlas*, *plaques*, *leaf topology*, *leaves*, *local transversals*, *simple open sets*, etc. The quotient space of leaves is denoted by $X/\mathcal{F}$. Notice that $\mathcal{F}$ can be identified with its *canonical* defining cocycle consisting of all simple open subsets of X and the canonical projections onto the corresponding quotient spaces of plaques. Foliated spaces with *boundary* or *corners* can be defined similarly. Indeed, we will only need the connectedness and local path connectedness of the fibers of the submersions p_i. Many interesting examples of foliated spaces are given in [5].

For a defining cocycle $\{U_i, p_i\}$ of $\mathcal{F}$, the homeomorphisms $h^x_{i,j}$, given by its compatibility condition, generate a pseudogroup $\mathcal{H}$ acting on $T = \bigsqcup_i T_i$, and $\{p_i\}$ generates a morphism $\mathcal{P} : X \to \mathcal{H}$. Let $\{U'_a, p'_a\}$ be another defining cocycle of $\mathcal{F}$ with $p'_a : U'_a \to T'_a$, which induces a pseudogroup $\mathcal{H}'$ acting on $T' = \bigsqcup_a T'_a$ and a morphism $\mathcal{P}' : X \to \mathcal{H}'$. Then $\{U_i, p_i\} \cup \{U'_a, p'_a\}$ induces a pseudogroup $\mathcal{H}''$ acting on $T'' = T \sqcup T'$, defining a canonical isomorphism $\Phi : \mathcal{H} \to \mathcal{H}'$ so that $\Phi \circ \mathcal{P} = \mathcal{P}'$. The "transverse dynamics" or "transverse structure" of $\mathcal{F}$ is described by the equivalence class of $\mathcal{H}$, which is called *holonomy pseudogroup*. It has a canonical representative induced by the canonical defining cocycle, which is denoted by $\mathrm{Hol}(\mathcal{F})$ and will be called *holonomy pseudogroup* too.

Let $\mathcal{F}$ and $\mathcal{G}$ be foliated structures on spaces X and Y. Their *product* is the foliated structure $\mathcal{F} \times \mathcal{G}$ on $X \times Y$ whose leaves are the products of leaves of $\mathcal{F}$ and $\mathcal{G}$. Observe that $\mathrm{Hol}(\mathcal{F} \times \mathcal{G})$ is equivalent to $\mathrm{Hol}(\mathcal{F}) \times \mathrm{Hol}(\mathcal{G})$.

4 Holonomy morphisms of foliated maps

Let X and Y be foliated spaces with foliated structures $\mathcal{F}$ and $\mathcal{G}$. A *foliated map* $f : \mathcal{F} \to \mathcal{G}$ is a map $f : X \to Y$ which maps leaves of $\mathcal{F}$ to leaves of $\mathcal{G}$; it induces a map $X/\mathcal{F} \to Y/\mathcal{G}$. The set of continuous foliated maps $\mathcal{F} \to \mathcal{G}$ is denoted by $C(\mathcal{F}, \mathcal{G})$; the notation $C^\infty(\mathcal{F}, \mathcal{G})$ is used for the set of C^∞

foliated maps when $\mathcal{F}$ and $\mathcal{G}$ are C^∞ foliations. Continuous foliated maps, with the operation of composition, form a category denoted by Fol. The concept of foliated map can be similarly defined for singular foliations.

Let $\{U_i, p_i\}$ and $\{V_a, p'_a\}$ be defining cocycles of $\mathcal{F}$ and $\mathcal{G}$ with $p_i : U_i \to T_i$ and $p'_a : V_a \to T'_a$, which induce pseudogroups $\mathcal{H}$ and $\mathcal{H}'$, and morphisms $\mathcal{P} : X \to \mathcal{H}$ and $\mathcal{P}' : Y \to \mathcal{H}'$. Given any $f \in C(\mathcal{F}, \mathcal{G})$, we can choose $\{U_i, p_i\}$ and $\{V_a, p'_a\}$ such that f maps each fiber of p_i to a fiber of p'_{a_i} for some mapping $i \mapsto a_i$. So there are continuous maps $\phi_i : T_i \to Q_{a_i}$ satisfying $\phi_i \circ p_i = p'_{a_i} \circ f|_{U_i}$. Then $\{\phi_i\}$ generates a morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ such that $\mathcal{P}' \circ f = \Phi \circ \mathcal{P}$. By composing Φ with canonical isomorphisms, we get a morphism $\mathrm{Hol}(f) : \mathrm{Hol}(\mathcal{F}) \to \mathrm{Hol}(\mathcal{G})$ called the *holonomy morphism* of f; we may also say that Φ *represents* $\mathrm{Hol}(f)$. This defines a covariant functor $\mathrm{Hol} : \mathrm{Fol} \to \mathrm{PsGr}$ called the *holonomy functor*.

Any topological space X can be considered as a foliated space X_{pt} whose leaves are its points, and any map between topological spaces is a foliated map in this sense. This defines an injective functor Top $\to$ Fol whose composition with the holonomy functor is the canonical injective functor Top $\to$ PsGr.

On the other hand, any connected manifold M can be considered as a foliated space with one leaf, also denoted by M, and any map between connected manifolds is a foliated map in this sense.

A product $f_1 \times f_2$ of foliated maps $f_i : \mathcal{F}_i \to \mathcal{G}_i$, $i = 1, 2$, is a foliated map $\mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{G}_1 \times \mathcal{G}_2$. When f_1 and f_2 are continuous, $\mathrm{Hol}(f_1) \times \mathrm{Hol}(f_2)$ represents $\mathrm{Hol}(f_1 \times f_2)$. A pair (f_1, f_2) of foliated maps $f_i : \mathcal{F} \to \mathcal{G}_i$, $i = 1, 2$, is a foliated map $\mathcal{F} \to \mathcal{G}_1 \times \mathcal{G}_2$. When f_1 and f_2 are continuous, $(\mathrm{Hol}(f_1), \mathrm{Hol}(f_2))$ represents $\mathrm{Hol}(f_1, f_2)$.

5 Leafwise homotopies and foliated homotopies

Let X and Y be foliated spaces with foliated structures $\mathcal{F}$ and $\mathcal{G}$. A *leafwise homotopy* between foliated maps $f, g : \mathcal{F} \to \mathcal{G}$ is a foliated map $\mathcal{F} \times I \to \mathcal{G}$ ($I = [0, 1]$) which is a homotopy between f and g. In this case, f and g are said to be *leafwisely homotopic*, and we get $\mathrm{Hol}(f) = \mathrm{Hol}(g)$. From this, we can define *leafwise homotopy equivalences* in a standard way. If f is a leafwise homotopy equivalence, then $\mathrm{Hol}(f)$ is an isomorphism.

A *foliated homotopy* between f and g is a homotopy $H : X \times I \to Y$ between f and g which is a foliated map $\mathcal{F} \times I_{\mathrm{pt}} \to \mathcal{G}$. In this case, f and g are said to be *foliatedly homotopic*. Any leafwise homotopy is a foliated homotopy. We can also define *foliated homotopy equivalences* in the standard way. Since $\mathrm{Hol}(\mathcal{F} \times I_{\mathrm{pt}})$ is equivalent to $\mathrm{Hol}(\mathcal{F}) \times I$, if

$H : \mathcal{F} \times I_{\mathrm{pt}} \to \mathcal{G}$ is a foliated homotopy between f and g, then $\mathrm{Hol}(H)$ defines a homotopy between $\mathrm{Hol}(f)$ and $\mathrm{Hol}(g)$. Therefore $\mathrm{Hol}(f)$ is a homotopy equivalence if f is a foliated homotopy equivalence.

6 Complete pseudogroups and complete morphisms

For any map $h : T \to T'$ between topological spaces, let $\gamma(h, x)$ denote its germ at any $x \in T$. If $\mathcal{H}$ is a pseudogroup acting on a space T, let $\gamma(\mathcal{H})$ denote the topological groupoid of all germs of the maps in $\mathcal{H}$ with the operation induced by composition and the étalé topology.

Recall from [11] that a pseudogroup $\mathcal{H}$ acting on a space T is said to be *complete* if, for all $x, y \in T$, there are open neighborhoods U and V of x and y such that, for any $h \in \mathcal{H}$ and any $z \in U \cap \mathrm{dom} h$ with $h(z) \in V$, there is some $\tilde{h} \in \mathcal{H}$ so that $U \subset \mathrm{dom}\tilde{h}$ and $\gamma(\tilde{h}, z) = \gamma(h, z)$; in this case, (U, V) is called a *completeness pair*.

Definition 6.1 Let $\mathcal{H}$ and $\mathcal{H}'$ be pseudogroups acting on topological spaces T and T'. A morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ is said to be *complete* when, given any $\phi, \psi \in \Phi$, any $x \in \mathrm{dom}\phi$ and any $y \in \mathrm{dom}\psi$, there are open subsets $U, V \subset T$ with $x \in U \subset \mathrm{dom}\phi$, $y \in V \subset \mathrm{dom}\psi$, and such that, for all $h \in \mathcal{H}$ and every $z \in U \cap \mathrm{dom} h$ with $h(z) \in V$, there is some $\tilde{h} \in \mathcal{H}$ and some $h' \in \mathcal{H}'$ so that $U \subset \mathrm{dom}\tilde{h}$, $\tilde{h}(U) \subset \mathrm{dom}\psi$, $\gamma(\tilde{h}, z) = \gamma(h, z)$, and $h' \circ \phi = \psi \circ \tilde{h}$ on U.

Observe that a pseudogroup $\mathcal{H}$ is complete if and only if the identity morphism $\mathrm{id}_{\mathcal{H}}$ is complete (see Section 2).

7 Pseudogroups of local isometries

Let $\mathcal{H}$ be a pseudogroup of local isometries of an n-dimensional Riemannian manifold T. Then $\gamma(\mathcal{H})$ is a Hausdorff manifold of dimension n, the topological groupoid $J^1(T)$ of 1-jets of local diffeomorphisms of T is a manifold of dimension $n^2 + 2n$, and the 1-jet homomorphism $j^1 : \gamma(\mathcal{H}) \to J^1(T)$ is continuous and injective.

Theorem 7.1 (Haefliger [11, Proposition 3.1]) *With the above notation, suppose $\mathcal{H}$ is complete. Then there is a unique pseudogroup $\overline{\mathcal{H}}$ of local isometries of T such that $j^1(\gamma(\overline{\mathcal{H}})) = \overline{j^1(\gamma(\mathcal{H}))}$ in $J^1(T)$. Moreover $\overline{\mathcal{H}}$ is complete, its orbits are the closures of the $\mathcal{H}$-orbits, and $\overline{\mathcal{H}}\backslash T$ is Hausdorff.*

In Theorem 7.1, $\overline{\mathcal{H}}$ is called the *closure* of $\mathcal{H}$, and $\mathcal{H}$ is said to be *closed* if $\mathcal{H} = \overline{\mathcal{H}}$. In this case, the maps in $\mathcal{H}$ that are, roughly speaking, close enough to identity maps generate a closed complete pseudogroup $\mathcal{H}_0$ whose orbits are the connected components of the orbits of $\mathcal{H}$. The following is a generalization of the theorem of Myers-Steenrod.

Theorem 7.2 (Salem [15]) *With the above notation, if $\mathcal{H}$ is closed, then $\mathcal{H}_0$ is defined by an effective isometric local action of some local Lie group G.*

Suppose that $\mathcal{H}$ is not closed, and consider an effective isometric local action of a local Lie group G on T defining $\overline{\mathcal{H}}_0$. Then the elements of G defining maps in $\mathcal{H}$ form a dense local subgroup $\Lambda \subset G$.

8 Riemannian foliations

A C^∞ foliation $\mathcal{F}$ on a manifold M is called *Riemannian* when its holonomy pseudogroup consists of local isometries for some Riemannian metric [14]. In this case, there is a defining cocycle of $\mathcal{F}$ consisting of Riemannian submersions for some Riemannian metric on M, which is called a *bundle-like metric*. A characteristic property of bundle-like metrics is that any geodesic is orthogonal to the leaves at every point if it is so at one point. This condition can be considered for singular foliations too, obtaining the definition of *singular Riemannian foliations* [14].

A Riemannian foliation $\mathcal{F}$ on a manifold M is called *transversely complete* when there is a bundle-like metric so that the geodesics orthogonal to the leaves are complete. In this case, $\mathrm{Hol}(\mathcal{F})$ is complete [14].

By Theorem 7.2, the connected components of the orbits closures of a complete pseudogroup of local isometries are the leaves of a singular Riemannian foliation [11]. Thus the leaf closures of any transversely complete Riemannian foliation $\mathcal{F}$ are the leaves of a singular Riemannian foliation $\overline{\mathcal{F}}$.

9 Morphisms of complete pseudogroups of local isometries

Theorem 9.1 *The following properties hold for any morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ between complete pseudogroups of local isometries*

(*i*) *Φ is complete.*

(*ii*) *Φ generates a morphism $\overline{\Phi} : \overline{\mathcal{H}} \to \overline{\mathcal{H}'}$.*

(*iii*) *The maps in Φ are C^∞ along the leaves of the singular foliation defined by the orbit closures, with continuous leafwise derivatives of arbitrary order.*

In Theorem 9.1, the morphism $\overline{\Phi}$ is called the *closure* of Φ. The continuity of leafwise derivatives, used in (iii), makes sense even for singular foliations!

Properties (i) and (ii) of Theorem 9.1 can be reduced to the following result.

Proposition 9.2 *For all $\phi \in \Phi$ and all $x \in \mathrm{dom}\phi$, there is an open neighborhood U of x in $\mathrm{dom}\phi$ satisfying the following properties*

(*i*) *There is a (compact-open) neighborhood $\mathcal{O}$ of id_U in $\overline{\mathcal{H}}_U$ such that, for all $h \in \mathcal{O}$, we have $h(U) \subset \mathrm{dom}\phi$, and there is some $h' \in \overline{\mathcal{H}'}$ with $\phi(U) \subset \mathrm{dom}h'$ and $h' \circ \phi = \phi \circ h$ on U.*

(*ii*) *For all $h'_1, h'_2 \in \overline{\mathcal{H}'}$ with $\phi(U) \subset \mathrm{dom}h'_1 \cap \mathrm{dom}h'_2$, if $h'_1 \circ \phi = h'_2 \circ \phi$ on some neighborhood of x, then $h'_1 \circ \phi = h'_2 \circ \phi$ on U.*

Let us show that Proposition 9.2 implies (i) and (ii) of Theorem 9.1. To prove that Φ generates $\overline{\Phi}$, it is enough to prove the condition (iii') of generation. Take $\phi, \psi \in \Phi$, $h \in \overline{\mathcal{H}}$ and $x \in \mathrm{dom}\phi \cap \mathrm{dom}(\psi \circ h)$. Consider the open neighborhood U of x given by Proposition 9.2. We can also assume that (U, U) is a completeness pair for $\overline{\mathcal{H}}$ by taking U small enough. Then there is a sequence of maps $h_n \in \mathcal{H}$, defined on some open neighborhood of x in U, so that $h_n \to h$ with respect to the compact-open topology. On the one hand, the sequence $g_n = h_n^{-1} \circ h \in \overline{\mathcal{H}}$ is defined in some fixed neighborhood of x, where it converges to the identity. Since (U, U) is a completeness pair, there is a sequence $\tilde{g}_n \in \overline{\mathcal{H}}_U$ so that $\gamma(\tilde{g}_n, x) = \underline{\gamma(g_n, x)}$, and thus $\tilde{g}_n \to \mathrm{id}_U$. By Proposition 9.2-(i), there is some $\tilde{g}'_n \in \overline{\mathcal{H}'}$ for n large enough such that $\phi(U) \subset \mathrm{dom}h'_n$ and $\tilde{g}'_n \circ \phi = \phi \circ \tilde{g}_n$ on U. On the other hand, since Φ is a morphism, there is a sequence $h'_n \in \mathcal{H}'$ with $\phi(x) \in \mathrm{dom}h'_n$ and $\psi \circ h_n = h'_n \circ \phi$ around x. Then (iii') is satisfied with $h' = h'_n \circ \tilde{g}'_n \in \overline{\mathcal{H}'}$ for n large enough.

Completeness is proved first for $\overline{\Phi}$. Take $\phi, \psi \in \overline{\Phi}$, $x \in \mathrm{dom}\phi$ and $y \in \mathrm{dom}\psi$. Let P and Q be open connected neighborhoods of x and y with compact closures, and so that $\overline{P} \subset P_1$ and $\overline{Q} \subset Q_1$ for some completeness pair (P_1, Q_1) of $\overline{\mathcal{H}}$ consisting of connected relatively compact sets. By Proposition 9.2-(i), every $h \in \overline{\mathcal{H}}_{P_1}$ with $h(\overline{P}) \cap \overline{Q} \neq \emptyset$ has a neighborhood $\mathcal{G}_h$ in $\overline{\mathcal{H}}_{P_1}$ so that any $g \in \mathcal{G}_h$ satisfies the completeness condition with some open neighborhoods U_h and V_h of x and y. Since P_1 is connected with compact closure, the mapping $\underline{(h, z)} \mapsto j^1(\gamma(h, z))$ defines a homeomorphism between $\mathcal{F} = \{(h, z) \in \overline{\mathcal{H}}_{P_1} \times \overline{P} \mid h(z) \in \overline{Q}\}$ and the compact space of elements of $j^1(\gamma(\overline{\mathcal{H}}))$ with range in $\overline{P}$ and target in $\overline{Q}$. The sets $\mathcal{F}_h = \mathcal{F} \cap (\mathcal{G}_h \times \overline{P})$ with $h \in \overline{\mathcal{H}}_{P_1}$ form an open covering of $\mathcal{F}$, and thus there is a finite subcovering $\mathcal{F}_{h_1}, \dots, \mathcal{F}_{h_n}$. Then the condition of completeness holds with $U = U_{h_1} \cap \dots \cap U_{h_n}$ and any open neighborhood V of y with $\overline{V} \subset Q$.

Now, let us prove that Φ is complete. Take $\phi, \psi \in \Phi$, $x \in \mathrm{dom}\phi$ and $y \in \mathrm{dom}\psi$. With these data, the completeness condition of $\overline{\Phi}$ is satisfied for some open neighborhoods U and V of x and y. We can assume that Proposition 9.2 holds with this U, and that $\phi(U)$ and $\psi(V)$ are contained in the sets of some completeness pair of $\overline{\mathcal{H}'}$. Take any $h \in \mathcal{H}$ with $x \in \mathrm{dom}h$ and $h(x) \in V$. We can suppose that $\mathrm{dom}h = U$. Then there is some $h'_0 \in \overline{\mathcal{H}'}$ so that $\phi(U) \subset \mathrm{dom}h'_0$ and $\psi \circ h = h'_0 \circ \phi$ on U. On the other hand, since Φ is a morphism, there is some $h' \in \mathcal{H}'$ such that $\phi(x) \subset \mathrm{dom}h'$ and $h' \circ \phi = \psi \circ h$ around x. So $h' \circ \phi = h'_0 \circ \phi$ around x. We can assume that $\phi(U) \subset \mathrm{dom}h'$, obtaining $h' \circ \phi = h'_0 \circ \phi$ on U by Proposition 9.2-(ii), and thus $\psi \circ h = h' \circ \phi$ on U.

10 Ideas of the proof of Proposition 9.2

In both T and T', the distance function will be denoted by d, $B(y,R)$ will denote the ball of radious R centered at a point y, and $\exp_y$ the exponential map at y.

For any $\phi \in \Phi$, there are relatively compact connected open subsets $U_1 \subset U_0 \subset T$ and $U'_1 \subset U'_0 \subset T'$, and numbers $R, R' > 0$ such that

(A) $\overline{U_0} \subset \mathrm{dom}\phi$ and $\phi(\overline{U_0}) \subset U'_0$;

(B) (U_0, U_0) and (U'_0, U'_0) are completeness pairs of $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}'}$;

(C) $\phi(U_1) \subset U'_1$;

(D) $\mathrm{diameter}(\overline{U_1}) < R$ and $d(\overline{U_1}, T \setminus U_0) > R$;

(E) $d(U'_1, T' \setminus U'_0) > R'$;

(F) $\phi(\overline{B(y,R)}) \subset B(\phi(y), R')$ for all $y \in \overline{U_1}$;

(G) $\exp_{y'}$ is a well defined diffeomorphism on the ball of radius R' around zero in the tangent space at every $y' \in \overline{U'_1}$, whose inverse is denoted by $\log_{y'}$;

(H) $U_1 \cap \overline{\mathcal{H}}(y)$ is connected for all $y \in U_1$;

(I) $U_1 \cap h_1(U_1) \cap h_1(h_2(U_1)) \cap \overline{\mathcal{H}}(y) \neq \emptyset$ for all $y \in U_1$ and all $h_1, h_2 \in \overline{\mathcal{H}}_{U_0}$ close enough to id_{U_0}.

According to (G), for $0 < r \leq R$ and $y \in \overline{U_1}$, let $E(y,r)$ be the linear span of $\log_{y'}(\phi(B(y,R)))$ in the tangent space of T' at $y' = \phi(y) \in \overline{U'_1}$. Set also $E(y) = \bigcap_{0<r\leq R} E(y,r)$ and $E(X) = \bigcup_{y \in X} E(y)$ for any $X \subset \overline{U_1}$. Notice that $E(y) = E(y,r)$ for some r.

Lemma 10.1 *For $h'_1, h'_2 \in \overline{\mathcal{H}'}_{U'_0}$, we have $h'_1 \circ \phi = h'_2 \circ \phi$ on $B(y,r)$ if and only if $h'_{1*} = h'_{2*}$ on $E(y,r)$.*

Lemma 10.2 *For $h \in \overline{\mathcal{H}}_{U_0}$ and $h' \in \overline{\mathcal{H}'}_{U'_0}$, if $\phi \circ h = h' \circ \phi$ around y, then $h'_* : E(y) \to E(h(y))$ is an isomorphism.*

Lemmas 10.1 and 10.2 are easy consequences of the fact that a local isometry locally corresponds to its differential map via the exponential map. The following kind of semi-continuity of $E(y)$ is more difficult to prove.

Lemma 10.3 *If V is an open subset of U_1 and $z \in \overline{V \cap \mathcal{H}(y)}$, then $E(z) \subset \bigcap_W \overline{E(V \cap W \cap \mathcal{H}(y))}$, where W runs in the open neighborhoods of z.*

Corollary 10.4 *Let $h \in \mathcal{H}_{U_0}$ and $h' \in \mathcal{H}'_{U'_0}$. If $h(\overline{U_1}) \subset U_0$ and $\phi \circ h = h' \circ \phi$ on some neighborhood of y, then $\phi \circ h = h' \circ \phi$ on some neighborhood of $U_1 \cap \overline{\mathcal{H}}(y)$.*

Proof. Let A be the set of points $z \in U_1$ such that $\phi \circ h = h' \circ \phi$ around z. Clearly, this set contains y and is open. Then, by (H), it is enough to prove that $A \cap \overline{\mathcal{H}}(y)$ is closed in $U_1 \cap \overline{\mathcal{H}}(y)$.

Take any $z \in U_1 \cap \overline{\mathcal{H}}(y) \cap \overline{A \cap \overline{\mathcal{H}}(y)} = U_1 \cap \overline{A \cap \mathcal{H}(y)}$. Since Φ is a morphism and by (B), there is some $h'' \in \mathcal{H}'_{U'_0}$ such that $\phi \circ h = h'' \circ \phi$ on some neighborhood P of z. For $V = P \cap A$, we get $z \in \overline{V \cap \mathcal{H}(y)}$ and $h' \circ \phi = \phi \circ h = h'' \circ \phi$ on V. Hence $h'_* = h''_*$ on $E(V)$ by Lemma 10.1, yielding $h'_* = h''_*$ on $E(z)$ by Lemma 10.3. So $h' \circ \phi = h'' \circ \phi$ around z by Lemma 10.1, and thus $z \in A$. □

Corollary 10.5 *If $h(\overline{U_1}) \subset U_0$ for some $h \in \overline{\mathcal{H}}_{U_0}$, then there is some $h' \in \overline{\mathcal{H}'}_{U'_0}$ such that $\phi \circ h = h' \circ \phi$ on some neighborhood of $U_1 \cap \overline{\mathcal{H}}(y)$.*

Proof. By (B), there is a sequence $h_n \in \mathcal{H}_{U_0}$ converging to h in $\overline{\mathcal{H}}_{U_0}$. We can assume that $h_n(\overline{U_1}) \subset U_0$ for all n. Since Φ is a morphism and by (B), there is another sequence $h'_n \in \mathcal{H}'_{U'_0}$ so that $\phi \circ h_n = h'_n \circ \phi$ around y. Therefore $\phi \circ h_n = h'_n \circ \phi$ on $U_1 \cap \overline{\mathcal{H}(y)}$ by Corollary 10.4. Because $j^1(\gamma(h'_n, \phi(y)))$ approaches in $j^1(\gamma(\overline{\mathcal{H}'}))$ the compact subset of elements with source $\phi(y)$ and target $\phi \circ h(y)$, we can assume by (B) that $j^1(\gamma(h'_n, \phi(y)))$ is convergent to some element of the form $j^1(\gamma(h', \phi(y)))$ with $h' \in \overline{\mathcal{H}'}_{U'_0}$. Hence $h'_n \to h'$ in $\overline{\mathcal{H}'}_{U'_0}$ because this is a space of local isometries and U'_0 is connected, obtaining $\phi \circ h = h' \circ \phi$ on $U_1 \cap \overline{\mathcal{H}(y)}$. □

Lemma 10.1 yields Proposition 9.2-(ii). Proposition 9.2-(i) follows from Corollary 10.5 when the $\mathcal{H}$-orbits are dense; the general case is much more involved.

11 Smoothness along the orbit closures

To prove Theorem 9.1-(iii), let us continue with the notation and arguments of the above section. Let $X_y = U_1 \cap \overline{\mathcal{H}}(y)$ and $X'_y = \phi(X_y)$.

Corollary 11.1 *In Corollary 10.5, if h is close enough to id_{U_0}, we can choose h' as close as desired to $\mathrm{id}_{U'_0}$.*

Proof. Consider a sequence $h_n \in \overline{\mathcal{H}}_{U_0}$ such that $h_n \to \mathrm{id}_{U_0}$. We can assume $h_n\left(\overline{U_1}\right) \subset U_0$ for all n. By Corollary 10.5, there is a sequence $h'_n \in \overline{\mathcal{H}'}_{U'_0}$ such that $\phi \circ h_n = h'_n \circ \phi$ on X_y. We have $h'_n \circ \phi(y) = \phi \circ h_n(y) \to y'$, and thus $\sigma_n = j^1(\gamma(h'_n, y'))$ approaches the compact set $j^1(\gamma(\overline{\mathcal{H}'}))^{y'}_{y'}$. Hence some subsequence σ_{n_m} is convergent to some $\tau \in j^1(\gamma(\overline{\mathcal{H}'}))^{y'}_{y'}$. There is some $f' \in \overline{\mathcal{H}'}_{U'_0}$ such that $\tau = j^1(\gamma(f', y'))$. Furthermore $h'_{n_m} \to f'$ on U'_0, and thus $f' \circ \phi = \phi$ on X_y, yielding that f' is the identity on X'_y. Hence $h'_{n_m} \circ f'^{-1}$ is defined on some neighbourhood of X'_y for all m, and we have $j^1(\gamma(h'_{n_m} f'^{-1}, y')) = \sigma_{n_m}\tau^{-1} \longrightarrow 1_{y'}$. There is some $h''_m \in \overline{\mathcal{H}'}_{U'_0}$ such that $\sigma_n\tau^{-1} = j^1(\gamma(h''_m, y'))$, and we get $h''_m \to \mathrm{id}_{U'_0}$. Then $h''_m = h'_{n_m} \circ f'^{-1}$ on some neighbourhood of X'_y because this space is connected by (H). Therefore $h''_m \circ \phi = h'_{n_m} \circ f'^{-1} \circ \phi = h'_{n_m} \circ \phi = \phi \circ h_{n_m}$ on X_y because f' equals the identity on X'_y. □

Corollary 11.2 *If $h \in \overline{\mathcal{H}}_{U_0}$ and $h' \in \overline{\mathcal{H}'}_{U'_0}$ are close enough to the corresponding identity maps, and $\phi \circ h = h' \circ \phi$ around X_y, then $\phi \circ h^{-1} = h'^{-1} \circ \phi$ around X_y.*

Proof. Suppose that $h(\overline{U_1}) \subset U_0$, $\overline{U_1} \subset h(U_0)$, $h(X_y) \cap X_y \neq \emptyset$ and $\overline{U'_1} \subset h'(U_0)$. These conditions satisfied if h and h' are in neighbourhoods of id_{U_0} and $\mathrm{id}_{U'_0}$ by (I). We get $U_1 \subset \mathrm{dom} h^{-1}$, $U'_1 \subset \mathrm{dom} h'^{-1}$, and $\phi \circ h^{-1} = h'^{-1} \circ \phi$ around $h(X_y) \cap X_y$. Take $f \in \overline{\mathcal{H}}_{U_0}$ and $f' \in \overline{\mathcal{H}'}_{U'_0}$ equal to h^{-1} and h'^{-1} on U_1 and U'_1, respectively. Thus $\phi \circ f = f' \circ \phi$ around $h(X_y) \cap X_y$. Moreover $h(X_y) \cap X_y \neq \emptyset$ and $f(\overline{U_1}) = h^{-1}(\overline{U_1}) \subset U_0$. So $\phi \circ h^{-1} = \phi \circ f = f' \circ \phi = h'^{-1} \circ \phi$ around X_y by Corollary 10.4. □

Corollary 11.3 *If $h_1, h_2 \in \overline{\mathcal{H}}_{U_0}$ and $h'_1, h'_2 \in \overline{\mathcal{H}'}_{U'_0}$ are close enough to the corresponding identity maps, and $\phi \circ h_i = h'_i \circ \phi$ around X_y for $i = 1, 2$, then $\phi \circ h_1 \circ h_2 = h'_1 \circ h'_2 \circ \phi$ around X_y.*

Proof. Assume $Y_y = X_y \cap h_1(X_y) \cap h_1 \circ h_2(X_y) \neq \emptyset$, $h_1(\overline{U_1}) \subset U_0$, $h_2 \circ h_1(\overline{U_1}) \subset U_0$ and $h'_i(\overline{U'_1}) \subset U'_0$ for $i = 1, 2$. These conditions hold for h'_i and h'_i in some neighborhoods of the corresponding identity maps by (I). We get $U_1 \subset \mathrm{dom}(h_1 \circ h_2)$, $U'_1 \subset \mathrm{dom}(h'_1 \circ h'_2)$ and $\phi \circ h_1 \circ h_2 = h'_1 \circ h'_2 \circ \phi$ around Y_y. Take some $f \in \overline{\mathcal{H}}_{U_0}$ which equals $h_1 \circ h_2$ on U_1, and some $f' \in \overline{\mathcal{H}'}_{U'_0}$ which equals $h'_1 \circ h'_2$ on U'_1. Thus $\phi \circ f = f' \circ \phi$ around $h_2^{-1} \circ h_1^{-1}(Y_y)$. Moreover $f(\overline{U_1}) = h_1 \circ h_2(\overline{U_1}) \subset U_0$. Therefore $\phi \circ h_1 \circ h_2 = \phi \circ f = f' \circ \phi = h'_1 \circ h'_2 \circ \phi$ around X_y by Corollary 10.4. □

Corollary 11.4 *There are neighborhoods of id_{U_0} in $\overline{\mathcal{H}}_{U_0}$ and of $\mathrm{id}_{U'_0}$ in $\overline{\mathcal{H}'}_{U'_0}$ such that, for maps h and h' in such neighborhoods, if h' is close enough to $\mathrm{id}_{U'_0}$ and $\phi \circ h = h' \circ \phi$ around X_y, then there is some $f \in \overline{\mathcal{H}}_{U_0}$, as close as desired to id_{U_0}, such that $\phi \circ f = h' \circ \phi$ around X_y.*

Proof. Take sequences $h_n \in \overline{\mathcal{H}}_{U_0}$ and $h'_n \in \overline{\mathcal{H}'}_{U'_0}$ such that $\phi \circ h_n = h'_n \circ \phi$ on X_y and $h'_n \to \mathrm{id}_{U'_0}$. Suppose that these sequences are taken in some shrinkings of neighborhoods satisfying the conditions of Corollary 11.2. The sequence $j^1(\gamma(h_n, y))$ approaches the compact set $j^1(\gamma(\overline{\mathcal{H}}))_y^{\overline{U_0}}$, and thus some subsequence h_{n_m} is convergent to some h in $\overline{\mathcal{H}}_{U_0}$. We get $\phi \circ h = \phi$ around X_y, yielding $U_1 \subset \mathrm{dom} h^{-1}$ and $\phi = \phi \circ h^{-1}$ around X_y by Corollary 11.2. Then $h^{-1} \circ h_{n_m}$ is defined on U_1 for m large enough, and thus there is a unique $f_m \in \overline{\mathcal{H}}_{U_0}$ which equals $h^{-1} \circ h_{n_m}$ on U_1. Moreover $f_m \to \mathrm{id}_{U_0}$ and $\phi \circ f_m = \phi \circ h^{-1} \circ h_{n_m} = \phi \circ h_{n_m} = h'_{n_m} \circ \phi$ around X_y for m large enough. □

According to Theorem 7.2, let G and G' be local Lie groups with local actions on T and T' generating the pseudogroups $\overline{\mathcal{H}_0}$ and $\overline{\mathcal{H}'_0}$. Since U_0 and U'_0 are relatively compact, we can suppose that G and G' are small enough around its identity elements e and e' so that the local action of all of their elements is defined on the whole U_0 and U'_0. For $g \in G$ and $g' \in G'$, we shall also denote by g and g' the corresponding maps in $\overline{\mathcal{H}}_{U_0}$ and $\overline{\mathcal{H}'}_{U'_0}$ defined by the local action. Again, we can assume G and G' are so small that the corresponding maps in $\overline{\mathcal{H}}_{U_0}$ and $\overline{\mathcal{H}'}_{U'_0}$ belong to the neighbourhoods of the identity maps given by Corollaries 11.2, 11.3 and 11.4.

Let G_y be the set of elements $g \in G$ such that there is some $g' \in G'$ so that $g' \circ \phi = \phi \circ g$ around X_y, and let G'_y be the set of elements $g' \in G'$ satisfying $g' \circ \phi = \phi \circ g$ around X_y for some $g \in G_y$.

Lemma 11.5 *G_y is an open local subgroup of G.*

Proof. This follows easily from Corollaries 11.1, 11.2 and 11.3. □

Lemma 11.6 *G'_y is a local Lie subgroup of G'.*

Proof. From Corollaries 11.2, 11.3 and 11.4, it follows easily that G'_y is a locally compact local subgroup of G', and thus a local Lie subgroup of G' by [4, page 227, Théorème 2]. □

Lemma 11.7 *X'_y is an open subset of the G'_y-orbit of y', and thus a C^∞ submanifold of $\overline{\mathcal{H}'}y'$.*

Proof. Since G_y is an open neighbourhood of e in G, we get that $\overline{\mathcal{H}_0 y}$ is the orbit of the local action of G_y on T that contains y. Take any $z \in X_y$. Because X_y is a connected open subset of $\overline{\mathcal{H}_0 y}$, there are $g_1, \dots, g_k \in G_y$ such that $z = g_1 \dots g_k y$ and $g_i g_{i+1} \dots g_k y \in X_y$ for $i \in \{1, \dots, k\}$. Then there are $g'_1, \dots, g'_k \in G'_y$ such that $g'_i \circ \phi = \phi \circ g_i$ around X_y. Hence $\phi(z) = g'_1 \dots g'_k y'$ is in the G'_y-orbit of y'.

On the other hand, for all $g' \in G'_y$, there is some $g \in G_y$ such that $g' \circ \phi = \phi \circ g$ on X_y; in particular, $g'y' = \phi(gy)$. Furthermore, by Corollary 11.4, the above g can be chosen to approach e in G_y as g' approaches e' in G'_y. So $gy \in X_y$ if g' is close enough e' in G'_y, yielding $g'y' = \phi(gy) \in X'_y$. Therefore X'_y contains a nontrivial open subset of the G'_y-orbit of y'. It follows that X'_y is open in that orbit as desired. □

We can assume that G' is an open local subgroup of a local Lie group $\widetilde{G}'$ such that the product $g'h'$ is defined in $\widetilde{G}'$ for all $g', h' \in G'$, and the local action of G' on T' can be extended to a local action of $\widetilde{G}'$ on T'. Two elements of G'_y are called equivalent if, as elements of $\overline{\mathcal{H}'_{U'_0}}$, they have the same restriction to X'_y. The corresponding quotient space will be denoted by G''_y.

Lemma 11.8 *The local Lie group structure of G'_y canonically induces a local Lie group structure on G''_y, and the C^∞ local action of G'_y on X'_y canonically induces a C^∞ effective isometric local action of G''_y on X'_y.*

Proof. Let $\widetilde{K}_y \subset \widetilde{G}'$ be the closed normal local Lie subgroup of elements in $\widetilde{G}'$ that fix all points of X'_y. Right translations obviously define a local action of $\widetilde{K}_y$ on G'_y whose orbit space will be denoted by $G'_y/\widetilde{K}_y$. Since $\widetilde{K}_y$ is normal in $\widetilde{G}'$, the local group structure of G'_y induces a local group structure on $G'_y/\widetilde{K}_y$, and the local action of G'_y on X'_y induces a local action of $G'_y/\widetilde{K}_y$ on X'_y.

The pseudogroup on G'_y generated by the local action of $\widetilde{K}_y$ preserves the parallelism defined by any frame of right invariant vector fields, and thus is Riemannian. This pseudogroup is closed because $\widetilde{K}_y$ is closed in $\widetilde{G}'$. Moreover it is obviously complete. So $G'_y/\widetilde{K}_y$ is a manifold and the quotient map $G'_y \to G'_y/\widetilde{K}_y$ is a C^∞ submersion. Therefore the result follows by showing that $G'_y/\widetilde{K}_y = G''_y$.

To show this assertion, observe that, if the product $g'a'$ is defined in G'_y for some $g' \in G'_y$ and some $a' \in \widetilde{K}_y$, then g' and $g'a'$ have the same restriction to X'_y, and thus are equivalent. Hence it only remains to show that two equivalent elements g', h' of G'_y are in the same $\widetilde{K}_y$-orbit. But for such g', h', the inverse g'^{-1} is also an element of G'_y, and thus the product

$a' = g'^{-1}h'$ is defined in $\widetilde{G}'$. Moreover a' fixes every point of X'_y because g' and h' are equivalent; *i.e.*, $a' \in \widetilde{K}_y$. So g' and $h' = g'a'$ are in the same $\widetilde{K}_y$-orbit as desired. □

Lemma 11.9 *There is a homomorphism of local Lie groups, $F_y : G_y \to G''_y$, such that the restriction $\phi : X_y \to X'_y$ is F_y-equivariant.*

Proof. For each $g \in G_y$, there is some $g' \in G'_y$ so that $g' \circ \phi = \phi \circ g$ on X_y. Any other element of G'_y satisfying the same property defines the same element in G''_y; thus the notation $F_y(g) \in G''_y$ makes sense for the element defined by g'. This defines a map $F_y : G_y \to G''_y$, which is a homomorphism of local groups by Corollary 11.3. Moreover $\phi : X_y \to X'_y$ clearly is F_y-equivariant. On the other hand, Corollary 11.1 implies that F_y is also continuous, and thus analytic too [4, Theorem 1, page 225]. □

Corollary 11.10 *The map $\phi : X_y \to X'_y$ is C^∞.*

Proof. This is a consequence of Lemma 11.9, since small enough open sets of X_y and X'_y are images of surjective submersions defined on open subsets of G_y and G''_yso that ϕ is locally induced by F_y. □

Since X_y and X'_y are open in the orbit closures of $\mathcal{H}$ and $\mathcal{H}'$, Corollary 11.10 means that ϕ is C^∞ along the orbit closures, showing Theorem 9.1-(iii) when the orbits are dense. More steps are needed in the general case.

12 The strong plaquewise topology

Let X and Y be foliated spaces with foliated structures $\mathcal{F}$ and $\mathcal{G}$. Fix the following data

- Any foliated map $f : \mathcal{F} \to \mathcal{G}$.
- Any locally finite collection $\mathcal{U} = \{U_i\}$ of simple open sets of X.
- A family $\mathcal{V} = \{V_i\}$ of simple open sets of Y, with the same index set. Let $q_i : V_i \to Q_i$ be the projection of each V_i to its quotient space of plaques.
- A family $\mathcal{K} = \{K_i\}$ of compact subsets of X, with the same index set, such that $K_i \subset U_i$ and $f(K_i) \subset V_i$ for all i.

Let $\mathcal{N}(f, \mathcal{U}, \mathcal{V}, \mathcal{K})$ be the set of foliated maps $g : \mathcal{F} \to \mathcal{G}$ such that $g(K_i) \subset V_i$ and $q_i \circ g(x) = q_i \circ f(x)$ for each i and every $x \in K_i$. Such sets $\mathcal{N}(f, \mathcal{U}, \mathcal{V}, \mathcal{K})$ form a base of a topology on $C(\mathcal{F}, \mathcal{G})$, called the *strong plaquewise topology*, and the corresponding space is denoted by $C_{SP}(\mathcal{F}, \mathcal{G})$. A weak version of this topology can be defined by taking finite families, and both topologies are equal if X is compact. If two foliated maps are close enough with respect to the strong plaquewise topology, then they are leafwisely homotopic and induce the same holonomy morphism (a leafwise homotopy can be defined along the plaques in a standard way).

13 Smooth approximation for Riemannian foliations

Let $\mathcal{F}$ and $\mathcal{G}$ be transversely complete Riemannian foliations on manifolds M and N. Consider a transversely complete bundle-like metric on N. Fix the following data:

- Any $f \in C(\mathcal{F}, \mathcal{G})$.
- Any locally finite family $\mathcal{Q} = \{Q_a\}$ of saturated closed subsets of M with compact projection to $M/\overline{\mathcal{F}}$.
- A family $\mathcal{E} = \{\epsilon_a\}$ of positive numbers, with the same index set.
- A neighborhood $\mathcal{N}$ of f in $C_{SP}(\mathcal{F}, \mathcal{G})$.

Let $\mathcal{M}(f, \mathcal{Q}, \mathcal{E}, \mathcal{N})$ be the set of continuous foliated maps $g : \mathcal{F} \to \mathcal{G}$ such that there is some sequence $(f'_0, f'_1, \ldots)$ in $C(\mathcal{F}, \mathcal{G})$ such that $f'_0 \in \mathcal{N}$; $f'_k = g$ on each Q_a for all but finitely many $k \in \mathbb{N}$; and, for all $x \in M$ and $k \in \mathbb{Z}^+$, there is a geodesic arc $c_{x,k}$ orthogonal to the leaves between $f'_{k-1}(x)$ and $f'_k(x)$, so that $\sum_{k=1}^{\infty} \operatorname{length}(c_{x,k}) < \epsilon_a$ whenever $x \in Q_a$. The family of such sets $\mathcal{M}(f, \mathcal{Q}, \mathcal{E}, \mathcal{N})$ form a base of a topology, called the *strong adapted topology*, and the corresponding space is denoted by $C_{SA}(\mathcal{F}, \mathcal{G})$. A weak version of this topology can be defined by considering finite families and neighborhoods in $C_{WP}(\mathcal{F}, \mathcal{G})$; both of these topologies are equal when $\mathcal{F}$ has compact leave closures and $M/\overline{\mathcal{F}}$ is compact. If $\mathcal{F}$ has compact leaf closures, then the strong adapted topology equals the strong compact-open topology on $C(\mathcal{F}, \mathcal{G})$.

Theorem 13.1 *With the above notation, $C^\infty(\mathcal{F},\mathcal{G})$ is dense in $C_{SA}(\mathcal{F},\mathcal{F}')$.*

This result follows from Theorem 9.1-(iii): for any foliated map $f : \mathcal{F} \to \mathcal{G}$, since the differentiability of $\operatorname{Hol}(f)$ along the orbit closures is granted, we only have to approximate f improving differentiability along the leaves and transversely to the leaf closures, which can be done with standard arguments.

Theorem 13.2 *With the above notation, if two foliated maps are close enough in $C_{SA}(\mathcal{F}, \mathcal{G})$, then there exists a foliated homotopy between them.*

Proof. For any $f \in C(\mathcal{F}, \mathcal{G})$, consider a neighborhood $\mathcal{M} = \mathcal{M}(f, \mathcal{Q}, \mathcal{E}, \mathcal{N})$ of the above type. For any $g \in \mathcal{M}$, take f'_k and $c_{x,k}$ as above. Suppose for simplicity that there is some $m \in \mathbb{Z}^+$ so that $f'_k = g$ for all $k \geq m$. Since the metric of N is bundle-like and transversely complete, the numbers ϵ_a can be chosen so small that the mapping $(x,t) \mapsto c_{x,k}(t)$ is a foliated homotopy between each f'_{k-1} and f'_k. On the other hand, there is a leafwise homotopy between f and f'_0 when $\mathcal{N}$ is small enough. □

Corollary 13.3 *Any foliated map $\mathcal{F} \to \mathcal{G}$ is foliatedly homotopic to a C^∞ foliated map. Moreover, if two C^∞ foliated maps $\mathcal{F} \to \mathcal{G}$ are foliatedly homotopic, then there is a C^∞ foliated homotopy between them.*

14 The spectral sequence of a C^∞ foliation

Let $\mathcal{F}$ be a C^∞ foliation of dimension p and codimension q on a C^∞ manifold M. Consider the decreasing filtration of the de Rham differential algebra $(\Omega = \Omega(M), d)$ by the differential ideals

$$\Omega = F^0\Omega \supset F^1\Omega \supset \ldots \supset F^q\Omega \supset F^{q+1}\Omega = 0 ,$$

where and r-form is in $F^k\Omega$ if it vanishes when $r-k+1$ vectors are tangent to the leaves; intuitively, this means that its "transverse degree" is $\geq k$. The induced spectral sequence $(E_i = E_i(\mathcal{F}), d_i)$ is a differentiable invariant of $\mathcal{F}$. The terms $E_1^{0,\bullet}$ and $E_2^{\bullet,0}$ are respectively called *leafwise cohomology* and *basic cohomology*, and $E_2^{\bullet,p}$ is isomorphic to the *transverse cohomology* [9] (also called *Haefliger cohomology*).

Let $\mathcal{G}$ be another C^∞ foliation on a manifold N. For each $f \in C^\infty(\mathcal{F},\mathcal{G})$, the corresponding homomorphism $f^* : \Omega(N) \to \Omega(M)$ preserves the filtrations and induces a spectral sequence homomorphism $E_i(f) : (E_i(\mathcal{G}), d_i) \to (E_i(\mathcal{F}), d_i)$. Since the operator on differential forms defined by a C^∞ foliated homotopy decreases the filtration degrees at most by one, we get the following.

Proposition 14.1 *If there is a C^∞ foliated homotopy between C^∞ foliated maps f and g, then $E_i(f) = E_i(g)$ for $i \geq 2$.*

15 Foliated homotopy invariance of the spectral sequence

Let $\mathcal{F}$ and $\mathcal{G}$ be transversely complete Riemannian foliations. According to Corollary 13.3, any $f \in C(\mathcal{F},\mathcal{G})$ is foliatedly homotopic to some $g \in C^\infty(\mathcal{F},\mathcal{G})$. Moreover, by Corollary 13.3 and Proposition 14.1, the homomorphism $E_i(g)$ is independent of the choice of g for $i \geq 2$, and can be denoted by $E_i(f)$. The following is also a consequence of Corollary 13.3 and Proposition 14.1.

Corollary 15.1 *If $f, g \in C(\mathcal{F}, \mathcal{G})$ are foliatedly homotopic, then $E_i(f) = E_i(g)$ for $i \geq 2$. If $f \in C(\mathcal{F}, \mathcal{G})$ is a foliated homotopy equivalence, then $E_i(f)$ is an isomorphism for $i \geq 2$.*

Corollary 15.1 is a generalization of the topological invariance of the basic cohomology obtained by El Kacimi-Alaoui and Nicolau [8].

If the leaves are dense, then we can similarly get the invariance of E_1 by leafwise homotopy equivalences; in particular, this holds for the leafwise cohomology.

There is a version of Corollary 13.3 for complete pseudogroups of local isometries. This implies that the basic and transverse cohomology of transversely complete Riemannian foliations is a homotopy invariant of their holonomy pseudogroup in the sense of Section 2.

16 Examples

Theorem 9.1-(i) supplies a large class of complete morphisms. Another source of complete morphisms is the following one. Any pseudogroup generated by a group action is complete, and any equivariant map generates a complete morphism.

For $\lambda > 1$, the mapping $x \mapsto \lambda x$ generates a complete pseudogroup $\mathcal{H}$, whose restriction to $U = (-1, 1)$ is not complete. Since U cuts every $\mathcal{H}$-orbit, completeness is not invariant by pseudogroup equivalences, unless we are considering only pseudogroups of local isometries on Riemannian manifolds. The identity morphism of $\mathcal{H}|_U$ is not complete.

Consider Arnold's example of a diffeomorphism of the circle which is topologically conjugated but not C^1 conjugated to a rotation [3]. By suspension, we get examples of homeomorphic C^∞ foliations with non-isomorphic basic cohomology [8]; so these foliations can not be diffeomorphic. Hence Theorem 13.1 fails in this example, and thus the morphism generated by such topological conjugation does not satisfy Theorem 9.1-(iii); the hypothesis of this theorem is not satisfied because Arnold's diffeomorphism is not isometric for any Riemannian metric. But it generates an equicontinuous action, showing that Theorem 9.1-(iii) cannot be generalized to equicontinuous pseudogroups [7, Section 5], [1].

17 Open problems

We may ask whether results like the Hurewicz isomorphism theorem [16] or the Van Kampen theorem [13] can be generalized to pseudogroups in the sense of Section 2.

It seems that morphisms of pseudogroups induce morphisms between the corresponding C^*-algebras and their K-theory. What can be said about them?

A version of Theorem 9.1 for "measurable morphisms" seems to be possible.

Even though Theorem 9.1-(iii) cannot be generalized to equicontinuous pseudogroups, what about its assertions (i) and (ii)?

A *pseudomonoid* can be defined like a pseudogroup with arbitrary continuous maps and without using inversion. By the lack of inversion, it makes sense to consider α- and ω-*orbits*, referring to the backwards and forwards direction. We similarly have α- and ω-*morphisms*, and α- and ω-*equivalences*. The *holonomy pseudomonoid* of a foliated space can be defined as the family of maps between local quotients induced by inclusions of its simple open subsets. Its α-class gives the holonomy pseudogroup of all open subsets. This may be useful to deal with invariants like the *transverse LS-category*, whose definition involves non-saturated open sets [6]. For instance, to address the problem of knowing to what extent is it a transverse invariant.

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