Description of nuclear octupole and quadrupole deformation close to the axial symmetry and phase transitions in the octupole mode

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The dynamics of nuclear collective motion is investigated in the case of reflection-asymmetric deformation, with the purpose of describing the critical point of phase transitions between different shapes. The model is based on the Bohr hydrodynamical approach and employs a new parametrization of the octupole and quadrupole degrees of freedom, valid for nuclei close to the axial symmetry. Three particular cases are discussed in some detail: octupole critical point in nuclei which already possess a permanent quadrupole deformation (^{226}Th); octupole vibrations in nuclei at the X(5) critical point of quadrupole mode (^{150}Nd, ^{152}Sm); and critical point in both quadrupole and octupole modes (^{224}Ra, ^{224}Th). Results are compared with experimental data.

Keywords: Nuclear collective models; Octupole excitations; Phase transitions

1. Introduction

My talk will concern the description of quadrupole and octupole degrees of freedom, in the frame of Bohr hydrodynamical model, and in conditions close to the axial symmetry. The main motivation of this work was to provide a theoretical frame to investigate the possible phase transitions in nuclear shapes involving the octupole degrees of freedom, in a way analogous to the one used to describe the phase transitions in the quadrupole shapes. The subject of quantum phase transitions has been introduced by Prof. Zamfir at the very beginning of this School, and phase transitions in the quadrupole modes have been extensively reviewed yesterday by Prof. Bonatsos.

In my first lecture, I will discuss the basis of the model and introduce the formalism that will be used to treat, in the second lecture, a few particular cases: the phase transition from octupole vibrations to stable octupole deformation in nuclei with permanent quadrupole deformation; the octupole vibrations in nuclei at the X(5) critical point; and finally the phase transition from quadrupole–octupole vibrations to stable quadrupole–octupole deformation (*i.e.*, the reflection asymmetric rotor).

But, as a first point, I will remind you a few points concerning the basic assumptions and limitations of the original Bohr model, which will obviously concern also its extension to the octupole deformation.

The distinctive feature of our approach is the use of an intrinsic frame referred to the principal axes of the tensor of inertia of the deformed nucleus. In our opinion, properties of phase transition can be better described in this scheme than with one of the many models that have been introduced in the past to describe reflection-asymmetric deformation. Here, I will only remind some of them shortly.

In principle, algebraic models (like the spdf Interacting Boson Model [1–4], or the Extended Coherent State Model [5] discussed in this School by Prof. A.Raduta, or various forms of Cluster models [6,7]) are perfectly able to describe the entire region of transitional nuclei between the two extreme simple cases, including the critical point of the phase transition. But just because all of them are described on the same footing, it will be difficult to identify the particular features of the phase transition point. This is the reason why Iachello himself uses a geometrical–model basis to discuss the phase transitions between the IBM limiting symmetries U(5) and O(6) or U(5) and SU(3).

In the geometrical approach, the most complete treatment of the octupole degrees of freedom has been proposed by Donner and Greiner [8]. Like the original Bohr model, it defines a non–inertial intrinsic frame, which however is not referred to the principal axes of the overall tensor of inertia, but to the principal axes of the quadrupole alone [8–10]. This approach would therefore coincide with ours only when the octupole amplitudes are very small compared to quadrupole. A number of alternative models choose to work in a reduced space, usually limited to axial quadrupole and octupole deformations [9,11–17]. In this case, part of the dynamical variables (those concerning non axial degrees of freedom) are frozen to zero from the very beginning. As we shall see, such a choice has consequences also on the differential equations describing the allowed axial modes. Finally, a parametrization of the octupole mode alone in its own intrinsic frame (referred to the principal axes of its tensor of inertia) has been reported by Wexler and Dussel [18]. We shall return to this point in the Sec. 4.

2. The basis of the Bohr model

The hydrodynamical model of nuclear collective motion was introduced by A. Bohr in a famous paper of 1952 [19] concerning the coupling of collective motion to the single particle degrees of freedom. Here we are interested in the first part of the paper, where the collective motion is described as the irrotational motion of a drop of *homogeneous* liquid, induced by *small* deformations of the surface. An implicit assumption is that the typical excitation energies of the collective modes be small compared to those of single-particle excitations. Therefore, phenomena concerning a relatively large excitation energy and/or angular momentum, as band crossing / backbending, band termination, isovector excitations (Giant Dipole Resonance) cannot be described in this simple frame. Instead, the model has proved to be very useful to describe the new symmetries at the critical point of phase transitions, and we can expect that it will be so also for its extension to the octupole modes. Phenomena involving higher energies and angular momenta, as the "beat pattern" in the parity straggling of alternate parity bands remain outside our possibilities. Other lectures of this School do treat them by means of more suitable models.

Our starting point is the equation of the nuclear surface, that is given in spherical coordinates as

$$R(\theta,\phi) = R_0\Big[1+\sum_\lambda \sum_{\mu=-\lambda,\lambda} \alpha_\mu^{(\lambda)} Y^*_{\lambda,\mu}(\theta,\phi)\Big] , \qquad \alpha_\mu^{(\lambda)\,*} = (-)^\mu \alpha_{-\mu}^{(\lambda)} .$$

With this definition, the $\alpha_\mu^{(\lambda)}$ are components of an irreducible tensor of rank λ. The amplitudes $\alpha_\mu^{(\lambda)}$ are assumed to be small. Why? Consider, *e.g.*, the nuclear volume V

$$V = \int \mathrm{d}\Omega \int_0^{R(\theta,\phi)} r^2 \,\mathrm{d}r = R_0^3 \left[\frac{4\pi}{3} + \sqrt{4\pi}\alpha_0^{(0)} + \sum_{\lambda,\mu} \left|\alpha_\mu^{(\lambda)}\right|^2 + ...\right] .$$

Higher order terms are neglected, and this already means that the $\alpha_\mu^{(\lambda)}$ are assumed to be small. Up to the second order, to keep the volume constant one should put $\alpha_0^{(0)} = -(4\pi)^{-1/2} \sum_{\lambda,\mu} |\alpha_\mu^{(\lambda)}|^2$. If, and only if, all the amplitudes $\alpha_\mu^{(\lambda)}$ are small enough to neglect their modulus squared, we can assume $\alpha_0^{(0)} = 0$ with the nuclear volume *approximately* conserved.

In a similar way it is possible to show that, if even and odd values of λ are present in the sum, then to keep fixed the center-of-mass position it is necessary to assume a well definite form for the amplitudes $\alpha_\mu^{(1)}$ as bilinear combinations of amplitudes with $\lambda \geq 2$. Again, if all these amplitudes are small enough, we can assume $\alpha_\mu^{(1)} \approx 0$.

3. The Bohr model with Quadrupole and Octupole

Our next step is the description of quadrupole and octupole deformation in a proper intrinsic frame, referred to the principal axes of the overall tensor of inertia. The equation of the nuclear surface is now:

$$r(\theta,\phi) = R_0\Big[1 + \sum_{\lambda=2,3} \sum_{\mu=-\lambda,\lambda} \alpha_\mu^{(\lambda)} Y^*_{\lambda,\mu}(\theta,\phi)\Big] .$$

In the Bohr hydrodynamical model, the classical expression of the kinetic energy is

$$T = \frac{1}{2}B_2 \sum_{\mu=-2,2} \left|\dot{\alpha}_\mu^{(2)}\right|^2 + \frac{1}{2}B_3 \sum_{\mu=-3,3} \left|\dot{\alpha}_\mu^{(3)}\right|^2$$

with constant B_2, B_3. For a classical drop, such an expression is obtained assuming irrotational flow. We now express $\alpha_\mu^{(\lambda)}$ in terms of the amplitudes $a_\nu^{(\lambda)}$ in a (non-inertial) intrinsic frame:

$$\sqrt{B_\lambda}\; \alpha_\mu^{(\lambda)} = \sum_\nu a_\nu^{(\lambda)} D_{\mu\nu}^{(\lambda)*}(\theta_i)$$

where $D^{(\lambda)}(\theta_i)$ are the Wigner matrices and θ_i the Euler angles. In order to simplify the notation in the following, we include the inertia coefficient B_λ in the definition of the collective variables $a_\mu^{(\lambda)}$. The classical expression of the kinetic energy in terms of time derivatives of the intrinsic variables $a_\mu^{(\lambda)}$ and of the intrinsic components q_i of the angular velocity, is now the sum of a *vibrational* term T_{vib}, a *rotational* term T_{rot} and a *coupling* term T_{coup}, which is not present in the case of pure quadrupole deformation:

$$T_{\text{vib}} = \frac{1}{2}\sum_{\lambda,\mu} \left|\dot{a}_\mu^{(\lambda)}\right|^2 \qquad T_{\text{rot}} = \frac{1}{2}\sum_{k,k'} q_k q_{k'} \mathcal{J}_{kk'}$$

$$T_{\text{coup}} = i\sum_\lambda \sqrt{3(2\lambda+1)}\left[q^{(1)} \otimes \left[a^{(\lambda)} \otimes \dot{a}^{(\lambda)}\right]^{(1)}\right]_0^{(0)} .$$

The diagonal and non–diagonal components of the tensor of inertia are

$$\mathcal{J}_{\frac{1}{2}} = \sum_\lambda \Big\{ C_0(\lambda)\left[a^{(\lambda)} \otimes a^{(\lambda)}\right]_0^{(0)} + \frac{1}{\sqrt{6}} C_2(\lambda)\left[a^{(\lambda)} \otimes a^{(\lambda)}\right]_0^{(2)}$$
$$\mp C_2(\lambda)\mathrm{Re}\left[a^{(\lambda)} \otimes a^{(\lambda)}\right]_2^{(2)} \Big\}$$

$$\mathcal{J}_3 = \sum_\lambda \Big\{ C_0(\lambda)\left[a^{(\lambda)} \otimes a^{(\lambda)}\right]_0^{(0)} - \sqrt{\frac{2}{3}}\, C_2(\lambda)\left[a^{(\lambda)} \otimes a^{(\lambda)}\right]_0^{(2)} \Big\}$$

$$\mathcal{J}_{13}+i\ \mathcal{J}_{23} = \sum_\lambda C_2(\lambda)\ \left[a^{(\lambda)}\otimes a^{(\lambda)}\right]^{(2)}_1$$

$$\mathcal{J}_{12} = \sum_\lambda C_2(\lambda)\ \mathrm{Im}\left[a^{(\lambda)}\otimes a^{(\lambda)}\right]^{(2)}_2$$

with
$$C_0(\lambda) = (-1)^\lambda\ \lambda(\lambda+1)\sqrt{2\lambda+1}\ /3$$
$$C_2(\lambda) = (-1)^{\lambda+1}\ \sqrt{\lambda(\lambda+1)(2\lambda+3)(4\lambda^2-1)/30}\ .$$

We do not give here the proof of these relations, that can be found, *e.g.* in the book by Eisenberg and Greiner [10].

4. Intrinsic amplitudes for quadrupole and octupole

In the spirit of the original Bohr paper, we decide to use as intrinsic frame the one referred to the principal axes of the overall tensor of inertia, and – preferably – define a parametrization that automatically implies vanishing of the three products of inertia $\mathcal{J}_{12}$, $\mathcal{J}_{13}$, and $\mathcal{J}_{23}$.

We have seen that in the case of a pure quadrupole deformation, this result is obtained with the Bohr parametrization

$$\begin{aligned} a_0^{(2)} &= \beta\cos\gamma \\ a_1^{(2)} = a_{-1}^{(2)} &= 0 \\ a_2^{(2)} = a_{-2}^{(2)} &= (1/\sqrt{2})\beta\sin\gamma\ . \end{aligned}$$

In the case of a pure octupole deformation, without any contribution of quadrupole, this is also possible, with the parametrization proposed by Wexler and Dussel [18]. We can introduce a very similar one:

$$\begin{aligned} a_0^{(3)} &= \beta_3\ \cos\gamma_3 \\ a_1^{(3)} &= -(5/2)\ (X+iY)\ \sin\gamma_3 \\ a_2^{(3)} &= \sqrt{1/2}\,\beta_3\ \sin\gamma_3 \\ a_3^{(3)} &= X\left[\cos\gamma_3 + (\sqrt{15}/2)\ \sin\gamma_3\right] + i\ Y\left[\cos\gamma_3 - (\sqrt{15}/2)\ \sin\gamma_3\right]\ , \end{aligned}$$

with $a_{-\mu}^{(3)} = (-)^\mu a_\mu^{(3)*}$. Also with this choice, the tensor of inertia turns out to be diagonal for a pure octupole deformation. To the intrinsic parameters (2 for the pure quadrupole, 4 for the pure octupole) one must add the three Euler angles to obtain the right number of parameters (5 or 7) needed to describe the nuclear deformation. We observe that in both cases the amplitudes with $\mu = \pm 2$ are real and those with $\mu = \pm 1$ are either zero (for pure quadrupole) or small of the second order, if we consider small of the first order other amplitudes with $\mu \neq 0$.

Instead, in the presence of both quadrupole and octupole deformations, also the amplitudes with $\mu = \pm 1$ and the imaginary part of those with $\mu = \pm 2$ must be considered. They are, however, not independent of one another, due to the requirement that

$$\mathcal{J}_{12} = \mathcal{J}_{13} = \mathcal{J}_{23} = 0 \tag{1}$$

The Eqs. 1 are non linear, but if we assume that non–axial amplitudes are small (of the first order) compared to the axial ones and we neglect the 2nd order terms, they reduce to the linear equations

$$\begin{aligned} \mathcal{J}_{13} + i\mathcal{J}_{23} &= \sqrt{6}\left(\beta_2 a_1^{(2)} + \sqrt{2}\,\beta_3 a_1^{(3)}\right) = 0 \\ \mathcal{J}_{12} &= -2\sqrt{6}\left(\beta_2 \mathrm{Im}\, a_2^{(2)} + \sqrt{5}\,\beta_3 \mathrm{Im}\, a_2^{(3)}\right) = 0 \end{aligned} \tag{2}$$

which are authomatically verified if one defines

$$\begin{aligned} a_1^{(2)} &= -c_1\,\sqrt{2}\,\beta_3\,(\eta_c + i\zeta_c) \qquad & a_1^{(3)} &= c_1\,\beta_2\,(\eta_c + i\zeta_c) \\ \mathrm{Im}\, a_2^{(2)} &= -c_2\,\sqrt{5}\,\beta_3\,\xi_c & \mathrm{Im}\, a_2^{(3)} &= c_2\,\beta_2\,\xi_c \end{aligned} \tag{3}$$

where c_1 and c_2 are arbitrary functions of β_2 and β_3.

With this choice, at the first order in the "small" quantities $\gamma_2,\ \gamma_3, X,\ Y,\ \eta_c,\ \zeta_c,\ \xi_c$, the intrinsic amplitudes of quadrupole and octupole deformation are

$$\begin{aligned} a_0^{(2)} &= \beta_2 \cos\gamma_2 \approx \beta_2\left[1 - (1/2)\,\gamma_2^2\right] \\ a_1^{(2)} &= -c_1\,\sqrt{2}\,\beta_3\,(\eta_c + i\zeta_c) \\ a_2^{(2)} &= \sqrt{1/2}\,\beta_2\,\sin\gamma_2\, -ic_2\,\sqrt{5}\,\beta_3\,\xi_c \approx \sqrt{1/2}\,\beta_2\,\gamma_2\, -ic_2\,\sqrt{5}\,\beta_3\,\xi_c \\ a_0^{(3)} &= \beta_3\,\cos\gamma_3 \approx \beta_3\left[1 - (1/2)\,\gamma_3^2\right] \\ a_1^{(3)} &= -(5/2)\,[X + iY]\,\sin\gamma_3 + c_1\,\beta_2\,(\eta_c + i\zeta_c) \approx c_1\,\beta_2\,(\eta_c + i\zeta_c) \\ a_2^{(3)} &= \sqrt{1/2}\,\beta_3 \sin\gamma_3\, +ic_2\,\beta_2\,\xi_c \approx \sqrt{1/2}\,\beta_3\,\gamma_3\, +ic_2\,\beta_2\,\xi_c \\ a_3^{(3)} &= X\left[\cos\gamma_3 + \sqrt{15/4}\,\sin\gamma_3\right] + iY\left[\cos\gamma_3 - \sqrt{15/4}\,\sin\gamma_3\right] \approx X + iY \end{aligned} \tag{4}$$

With this choice, the three products of inertia $\mathcal{J}_{\kappa\kappa'}$, and also the diagonal term $\mathcal{J}_{33}$, are small of the second order in the "small" variables $\gamma_2,\ \gamma_3,\ X,\ Y,\ \xi_c,\ \eta_c,\ \zeta_c$.

In the following, we are going to consider the equation of motion for β_2, β_3 assuming that it is effectively decoupled from that of all other independent variables (wich will not necessarily coincide with those defined in the Eq. 4). The identification of a set of variables for which this condition holds is a crucial point in our work, as it determines the form of the kinetic energy operator also in the sector involving β_2 and β_3.

5. Quantization of the quadrupole – octupole Hamiltonian

We must now pass from the classical expression T of the kinetic enrgy to the quantum kinetic energy operator $\hat{T}$. This can be done with the Pauli procedure [20] for quantization in a non Cartesian reference.

The classical expression of the kinetic energy has the form

$$T = \frac{1}{2}B_2 \sum_{\mu=-2,2} \left|\dot{\alpha}_\mu^{(2)}\right|^2 + \frac{1}{2}B_3 \sum_{\mu=-3,3} \left|\dot{\alpha}_\mu^{(3)}\right|^2 \Rightarrow \frac{1}{2}\sum_{\mu\nu} \mathcal{G}_{\mu\nu} v_\mu v_\nu \qquad (5)$$

where v_ν are the components of the generalized velocity vector $\boldsymbol{v}$. Here, $\boldsymbol{v} = \{\dot{\beta}_2, \dot{\gamma}_2, \dot{\beta}_3, \dot{\gamma}_3, \dot{X}, \dot{Y}, \dot{\xi}_c, \dot{\eta}_c, \dot{\zeta}_c, q_1, q_2, q_3\}$, and q_μ are the components of the angular velocity $\vec{q}$ along the axes of the intrinsic frame.

According to the Pauli recipe, the kinetic energy operator is

$$\hat{T} = -\sum \frac{\hbar^2}{2} G^{-1/2} \frac{\partial}{\partial Q_\mu} [G^{1/2} (\mathcal{G}^{-1})_{\mu\nu} \frac{\partial}{\partial Q_\nu}] \qquad (6)$$

where the Q_ν are the dynamical variables and G is the Determinant of the matrix $\mathcal{G}$ defined in the Eq. 5.

If one makes use of the dynamical variables defined in the Eq. 4, the result obviously depends on the choice of the *arbitrary functions* $c_1(\beta_2, \beta_3)$ and $c_2(\beta_2, \beta_3)$. *E.g.*, with the simplest possible choice $c_1 = c_2 = 1$, we would obtain a determinant $G = \mathrm{Det}\mathcal{G} = 1152\beta_2^2\beta_3^2 (\beta_2^2 + 2\beta_3^2)^4 (\beta_2^2\gamma_2^2 + \beta_3^2\gamma_3^2)^2$. In this case, at the limit for $\beta_3 \to 0$, we obtain $G \propto \beta_2^{14}$, while at the same limit the Bohr model gives $G \propto \beta_2^8\ \gamma_2^2$ and in the case of small–amplitude vibrations around a well deformed shape ($\beta \approx \bar{\beta}$ constant) the Frankfurt model [10] gives $G \propto \bar{\beta}_2^4$. Therefore, the choice $c_1 = c_2 = 1$ seems not to be adequate.

Apparently, a better choice could be $c_1 = (\beta_2^2 + 2\beta_3^2)^{1/2}$, $c_2 = (\beta_2^2 + 5\beta_3^2)^{1/2}$, which gives the $\mathcal{G}$ matrix shown in the Table 1. All the non diagonal matrix elements, apart from those of the last line and column, are small at least of the first order in comparison with the corresponding diagonal elements, and it is possible to show [21] that they can be neglected. Those of the last line or column are still small of the first order, but they must be compared with the diagonal element $\mathcal{J}_3$, which is *small of the second order*. Their effect will be discussed in the next Sec. 6.

Now the determinant G has a more reasonable behaviour:

$$G = \mathrm{Det}\mathcal{G} = 1152\beta_2^2\beta_3^2 (\beta_2^2 + 2\beta_3^2)^2 (\beta_2^2 + 5\beta_3^2)^{-1} (\beta_2^2\gamma_2 + \sqrt{5}\ \beta_3^2\gamma_3)^2 \qquad (7)$$

and, when $\beta_3 \to 0$, $G \propto \beta_2^8$ as in the original Bohr model. We can consider

Table 1. The matrix of inertia $\mathcal{G}$: leading terms and relevant first-order terms. Other first-order terms are indicated with the symbol ≈ 0.

	$\dot\beta_2$	$\dot\gamma_2$	$\dot\beta_3$	$\dot\gamma_3$	$\dot X$	$\dot Y$	$\dot\xi$	$\dot\eta$	$\dot\zeta$	q_1	q_2	q_3
$\dot\beta_2$	1	0	0	0	0	0	0	0	0	≈ 0	≈ 0	0
$\dot\gamma_2$	0	β_2^2	0	0	0	0	0	0	0	≈ 0	≈ 0	$\frac{-\sqrt{40}\beta_2\beta_3\xi}{\sqrt{\beta_2^2+5\beta_3^2}}$
$\dot\beta_3$	0	0	1	0	0	0	0	0	0	≈ 0	≈ 0	0
$\dot\gamma_3$	0	0	0	β_3^2	≈ 0	≈ 0	0	≈ 0	≈ 0	≈ 0	≈ 0	$\frac{\sqrt{8}\beta_2\beta_3\xi}{\sqrt{\beta_2^2+5\beta_3^2}}$
$\dot X$	0	0	0	≈ 0	2	0	0	≈ 0	0	≈ 0	≈ 0	$6Y$
$\dot Y$	0	0	0	≈ 0	0	2	0	0	≈ 0	≈ 0	≈ 0	$-6X$
$\dot\xi$	0	0	0	0	0	0	2	0	0	≈ 0	≈ 0	$\frac{\sqrt{8}\beta_2\beta_3\gamma}{\sqrt{\beta_2^2+5\beta_3^2}}$
$\dot\eta$	0	0	0	≈ 0	≈ 0	0	0	2	0	≈ 0	≈ 0	2ζ
$\dot\zeta$	0	0	0	≈ 0	0	≈ 0	0	0	2	≈ 0	≈ 0	-2η
q_1	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	$\mathcal{J}_1$	0	0
q_2	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	0	$\mathcal{J}_2$	0
q_3	0	[..]	0	[..]	$6Y$	$-6X$	[..]	2ζ	-2η	0	0	$\mathcal{J}_3$

Note: Here $\gamma = \sqrt{5}\gamma_2 - \gamma_3$, $\mathcal{J}_1 = 3(\beta_2^2 + 2\beta_3^2) + 2\sqrt{3}(\beta_2^2\gamma_2 + \sqrt{5}\beta_3^2\gamma_3)$; $\mathcal{J}_2 = 3(\beta_2^2 + 2\beta_3^2) - 2\sqrt{3}(\beta_2^2\gamma_2 + \sqrt{5}\beta_3^2\gamma_3)$; and $\mathcal{J}_3 = 4(\beta_2^2\gamma_2^2 + \beta_3^2\gamma_3^2) + 18(X^2 + Y^2) + 2(\eta^2 + \zeta^2) + 8\xi^2$.

therefore the present choice of c_1, c_2 and the results given in the Table 1 as a good starting point for our future work.

6. The third component of the angular momentum

Surface vibrations that conserve the axial symmetry correspond to an angular momentum component $L_3 = 0$ along the intrinsic symmetry axis. It is interesting to investigate the relation between the non–axial modes of vibration, described by the dynamical variables γ_2, γ_3, X, Y, ξ, η, ζ and other eigenvalues of L_3

To this purpose, I will follow here a semiclassical approach, simpler and more transparent than the completely consistent one, which can be found in the Ref. [21]. The classical expression for the intrinsic components of the angular momentum is

$$L_\kappa = \partial T / \partial q_\kappa \tag{8}$$

While, at the leading order, the components L_1, L_2 have the usual form $L_\kappa = \mathcal{J}_\kappa q_\kappa$, $\kappa = 1,\ 2$, the third component has a very complicated expression, due to the effect of non diagonal terms of the last line and column:

$$L_3 = \mathcal{J}_3 q_3 + \Big[\frac{\sqrt{8}\beta_2\beta_3}{\sqrt{\beta_2^2 + 5\beta_3^2}}(\gamma\dot\xi - \xi\dot\gamma) + 6(Y\dot X - X\dot Y) + 2(\zeta\dot\eta - \eta\dot\zeta) \Big] \tag{9}$$

where we have put

$$\gamma = \sqrt{5}\gamma_2 - \gamma_3 \ . \tag{10}$$

At this point, it will be convenient to express the variables γ_2 and γ_3 as linear combinations of two new variables, one of which is, obviously, $\gamma = \sqrt{5}\gamma_2 - \gamma_3$. The other one, γ_0, can be chosen proportional to the linear combination which enters in the expression of the determinant G,

$$\gamma_0 = c_0 \left(\beta_2^2\gamma_2 + \sqrt{5}\ \beta_3^2\gamma_3\right) . \tag{11}$$

where, again, the factor c_0 will be considered as an arbitrary function of β_2, β_3, with the only condition that $c_0 \to \beta_2^2$ when $\beta_3 \to 0$.

The expression of L_3 can be substantially simplified with the introduction of a new set of variables v, ϑ, u, φ, w, χ, u_0 and the corresponding conjugate moments p_v, p_ϑ, p_u, p_φ, p_w, p_χ, p_{u_0}. These new variables are related to the old ones by the expressions

$$\begin{aligned}
X &= w \sin\vartheta \\
Y &= w \cos\vartheta \\
\eta &= v \sin\varphi \\
\zeta &= v \cos\varphi \\
\xi &= u \sin\chi \quad . \\
\gamma &\equiv \sqrt{5}\gamma_2 - \gamma_3 = \sqrt{2}\ \left(\sqrt{\beta_2^2 + 5\beta_3^2}\ /\ \beta_2\beta_3\right) u\cos\chi \\
\frac{\gamma_0}{c_0} &\equiv \beta_2^2\gamma_2 + \sqrt{5}\ \beta_3^2\gamma_3 = \sqrt{\beta_2^2 + 5\beta_3^2}\ u_0
\end{aligned} \tag{12}$$

The expression of the kinetic energy, in terms of the new variables and of their time derivatives, is

$$\begin{aligned}
T = \frac{1}{2}\Big\{ &\dot\beta_2^2 + \dot\beta_3^2 + \dot u_0^2 + 2(\dot v^2 + v^2\dot\varphi^2) + 2(\dot u^2 + u^2\dot\chi^2) + 2(\dot w^2 + w^2\dot\vartheta^2) \\
&+ 2q_3\left[2v^2\dot\varphi + 4u^2\dot\chi + 6w^2\dot\vartheta\right] + \mathcal{J}_1 q_1^2 + \mathcal{J}_2 q_2^2 + \mathcal{J}_3 q_3^2 \ \Big\}
\end{aligned} \tag{13}$$

with $\mathcal{J}_1 \approx \mathcal{J}_2 \approx 3(\beta_2^2 + 2\beta_3^2)$, $\mathcal{J}_3 = 4u_0^2 + 2v^2 + 8u^2 + 18w^2$. The expression of L_3 takes the simpler form

$$\begin{aligned}
L_3 &= \mathcal{J}_3 q_3 + \left[2v^2\dot\varphi + 4u^2\dot\chi + 6w^2\dot\vartheta\right] \\
&= 2v^2\left(q_3 + \dot\varphi\right) + 4u^2\left(2q_3 + \dot\chi\right) + 6w^2\left(3q_3 + \dot\vartheta\right) + 4u_0^2 q_3 \ .
\end{aligned} \tag{14}$$

We also evaluate the *conjugate moments* of the new variables φ, χ, ϑ and invert this system of equations to obtain their time derivatives in terms of the conjugate moments and L_3:

$$\begin{aligned}
p_\varphi &= 2v^2\left(\dot\varphi + q_3\right) & \dot\varphi &= \frac{p_\varphi}{2v^2} - \frac{1}{u_0^2}\left(L_3 - p_\varphi - 2p_\chi - 3p_\vartheta\right) \\
p_\chi &= 2u^2\left(\dot\chi + 2q_3\right) & \dot\chi &= \frac{p_\chi}{2u^2} - \frac{2}{u_0^2}\left(L_3 - p_\varphi - 2p_\chi - 3p_\vartheta\right) \\
p_\vartheta &= 2w^2(\dot\vartheta + 3q_3) & \dot\vartheta &= \frac{p_\vartheta}{2\chi_0^2} - \frac{3}{u_0^2}\left(L_3 - p_\varphi - 2p_\chi - 3p_\vartheta\right).
\end{aligned} \tag{15}$$

and

$$q_3 = \frac{1}{u_0}\left(L_3 - p_\varphi - 2p_\chi - 3p_\vartheta\right) \tag{16}$$

This last relation shows that, when $u_0 \to 0$, then $q_3 \to \infty$ unless $L_3 = \Omega \equiv p_\varphi - 2p_\chi - 3p_\vartheta$. This relation has a very simple meaning if the potential does not depend on the variables φ, χ or ϑ. In such a case (a sort of model φ-χ-ϑ–instable, in the sense of the γ–instable model by Wilets and Jean [22]) the conjugate moments of these three angular variables are constants of the motion, with integer eigenvalues n_φ,n_χ and n_ϑ (in units of $\hbar$), and the operator L_3 is diagonal, with eigenvalues $K = n_\varphi + 2n_\chi + 3n_\vartheta$. Therefore, the three degrees of freedom corresponding to the pairs of variables v, φ, u, χ and w, ϑ can be associated with non-axial excitation modes with K =1, 2 and 3, respectively. By using their definitions (Eq. 12), it is easy to verify that they also carry *negative parity.*

It remains to discuss the role of u_0. The variable u_0 measures the triaxiality of the overall tensor of inertia. In this sense, it plays a role similar to that of a_2 in the pure–quadrupole case and, like a_2 in the case of small triaxial deformation, it can be eventually replaced by an angular variable times a proper combinations of β_2 and β_3. If we assume that the differential equation for u_0 can be decoupled from the others, this equation is

$$\left\{\frac{1}{u_0}\frac{\partial}{\partial u_0}\left[u_0\frac{\partial}{\partial u_0}\right] + \frac{2}{\hbar^2}\left[E_{u_0} - U(u_0)\right] - \frac{1}{u_0^2}\left[\frac{K_0}{2}\right]^2\right\}\phi(u_0) = 0 \tag{17}$$

where we have put $K_0 = L_3 - \Omega$. The variable u_0 can take positive as well as negative values. The condition of continuity for the wavefunction $\phi(u_0)$ and its derivative at $u_0 = 0$ imposes that $K_0 = 2n_{u_0}$, with n_{u_0} integer. We can conclude that the degree of freedom associated to the variable u_0 carries two units of angular momentum along the intrinsic axis 3, and it is possible to show that it also carries positive parity.

With this choice of dynamical variables, the determinant G is

$$G \equiv \mathrm{Det}\mathcal{G} = 2304\, u_0^2\, v^2 u^2 w^2 \left(\beta_2^2 + 2\beta_3^2\right)^2 \tag{18}$$

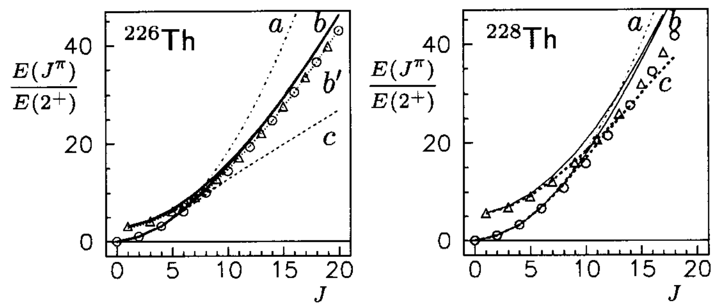

Fig. 1. (From Ref. [21]). Excitation energies for states of positive parity (circles) and negative parity (triangles), in units of $E(2^+)$, for the ground-state band of ^{226}Th and ^{228}Th. Theoretical curves: a – rigid rotor; b (b') – present model with critical potential, fitted on the 1^- state (on high-spin states); c – present model with harmonic potential.

and the inverse of the matrix $\mathcal{G}$ turns out to be diagonal (at the relevant order) in the space of momenta conjugate to the variables defined in the Eq. 15 and of the angular momentum components L_1, L_2 and $K_0 = L_3 - \Omega$. A more formal derivation of these results, involving the derivatives of the Euler angles, can be found in the Appendix C of Ref. [21].

7. Axial octupole mode with stable quadrupole deformation

This is the simplest case in which the properties of octupole excitation can be followed from the limit of harmonic oscillations around the reflection symmetric core to the opposite limit of stable octupole deformation. A detailed discussion of this subject can be found in the Ref. [21]. Here, we only summarize these results.

A preliminary comment is in order. The properties of the quadrupole vibrations around an axially deformed core are better described [10] with respect to the intrinsic parameters $a_2^{(2)}$, $\delta a_0^{(2)} = a_0^{(2)} - \bar{a}_0^{(2)}$ than in terms of the Bohr parameters β_2 and γ_2. We have seen that the parameter u_0 defined in the Eq. 12 plays, in our treatment, a role analogous to that of $a_2^{(2)}$ in the pure quadrupole Hamiltonian. It appears reasonable, therefore, to use it as a dynamical variable instead of defining an angle variable similar to the γ_2 of the quadrupole case. Therefore, we can use the expression of G given in Eq. 18, to derive with the Pauli recipe the differential equation for β_3 (in doing this, we assume decoupling of the β_3 motion from the small-amplitude oscillations in all other degrees of freedom). One obtains

$$\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{2x}{1+x^2}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x} + \left[\epsilon - \frac{J(J+1)}{6(1+x^2)} - v(x)\right]\psi(x) = 0 \qquad (19)$$

where $x = \sqrt{2}\ \beta_3/\bar{\beta}_2$, while $v(x)$, ϵ are the potential energy and the energy eigenvalue in a proper energy unit, and $\psi(-x) = (-)^J\psi(x)$. As for the potential $v(x)$, we have considered two simple cases: a quadratic expression $v = \frac{1}{2}cx^2$ or a critical (square-well) potential, as in the X(5) model: $v(x) = 0$ for $|x| < b$ and $= +\infty$ for $|x| > b$. In both cases, the model has one free parameter (c or b) to be adjusted to fit the experimental data. In the Fig. 1 the energies of positive and negative parity levels of the ground-state band of ^{226}Th and ^{228}Th are compared with different model predictions. The former turns out to be close to the results we obtain for a critical-point potential, while the latter is closer to those obtained with a quadratic potential. Relative values of experimental transition strengths, B(E1) and B(E2) have also been calculated and compared with existing experimental data. The results can be found in the Ref. [21]. The agreement is satisfactory, within the (admittedly large) experimental errors.

8. Going close to the quadrupole critical point

If we want to consider the case where the dependence of the potential energy on β_2 is that of a square well extending from $\beta_2 = 0$ to some finite limit β_2^{w}, we need that the results of our model converge to those of the Bohr model in the limit of small octupole deformation. This is not the case for the Hamiltonian we have used in the previous Section, where u_0 has been used as independent dynamical variable. Since in the Bohr model the variable γ_2 is used in the place of $a_2^{(2)}$, it is now necessary to replace u_0 with a proper adimensional variable γ_0 which would reduce to γ_2 for $\beta_3 \to 0$. A necessary condition we have to fulfill by means of this substitution is that the limit of the determinant G for $\beta_3 \to 0$ assume the correct dependence on β_2^8, as in the Bohr model. However, this is not enough to ensure that, at this limit, the model Hamiltonian converge to that of Bohr.

In fact, if we assume that the complete differential equation obtained with the Pauli quantization rule can be effectively separated in one part depending only on β_2, β_3 and another containing all other dynamical variables, for the former we obtain

$$\left\{\frac{1}{g}\frac{\partial}{\partial\beta_2}\left[g\frac{\partial}{\partial\beta_2}\right]+\frac{1}{g}\frac{\partial}{\partial\beta_3}\left[g\frac{\partial}{\partial\beta_3}\right]+\epsilon-V-\frac{A_J}{\beta^2+2\beta_3^2}\right\}\Psi(\beta_2,\beta_3)=0 \quad (20)$$

where $g \propto G^{1/2}$, $\epsilon = 2E/\hbar^2$, $V = V(\beta_2, \beta_3)$ and $A_J = J(J+1)/3$. It is convenient to eliminate in the Eq. 20 the first–derivative terms, with the substitution $\Psi(\beta_2, \beta_3) = g^{-1/2}\ \Psi_0(\beta_2, \beta_3)$, to obtain

$$\left\{ \frac{\partial^2}{\partial \beta_2^2} + \frac{\partial^2}{\partial \beta_3^2} + \epsilon - V(\beta_2, \beta_3) + V_g(\beta_2, \beta_3) - \frac{A_J}{\beta^2 + 2\beta_3^2} \right\} \Psi_0 = 0 \quad (21)$$

where

$$V_g(\beta_2, \beta_3) = \frac{1}{4g^2} \left[\left(\frac{\partial g}{\partial \beta_2} \right)^2 + \left(\frac{\partial g}{\partial \beta_3} \right)^2 \right] - \frac{1}{2g} \left[\frac{\partial^2 g}{\partial \beta_2^2} + \frac{\partial^2 g}{\partial \beta_3^2} \right] . \quad (22)$$

We need, therefore, that also V_g takes the correct value at the limit $\beta_3 \ll \beta_2$. To this purpose, it is sufficient [23] that the first and second derivatives of G with respect to β_3 vanish when $\beta_3 \to 0$. A possible (perhaps, non unique) choice leading to this result corresponds to assuming $c_0 = 1/\sqrt{(\beta_2^2 + \beta_3^2)(\beta_2^2 + 2\beta_3^2)}$ in the definition of γ_0 (Eq. 11). In this case one obtains

$$G \propto \frac{(\beta_2^2 + \beta_3^2)^2 (\beta_2^2 + 2\beta_3^2)^4}{(\beta_2^2 + 5\beta_3^2)^2} \Rightarrow \beta_2^8 \left[1 + 16\, (\beta_3/\ \beta_2)^4 + ... \right]$$

and the first and second derivative of g with respect to β_3 vanish for $\beta_3 \to 0$.

9. Specific models for quadrupole–octupole oscillations

We now consider the case of simultaneous quadrupole–octupole oscillations, and in particular the quadrupole motion corresponding to the critical point of phase transition described by the X(5) model. We can start from what seems to be the simplest case: β_3 oscillations of very small amplitude compared to those in β_2, and, therefore, large excitation energy of the negative-parity levels. Experimentally, this happens in ^{150}Nd and ^{152}Sm.

The situation is very much simplified in this case, as far as $\beta_3 \ll \beta_2$ and we can neglect β_3^2 in comparison to β_2^2. One could express the quadrupole and octupole amplitudes in terms of two new variables β and δ,

$$\beta_2 = \beta \cos \delta \qquad \beta_3 = \beta \sin \delta \quad (23)$$

and confine δ to very small values by a proper potential term. It is clear that also the amplitudes v, u, w must be small compared to β, as well as γ_0 compared to 1. At the moment, however, we will forget their presence and only discuss the Schrödinger equation involving the variables β, δ and the Euler angles. With this *ansatz*, and assuming that the potential energy has the form $(\hbar^2/2)\ V(\beta, \delta)$, for $K = 0$ we obtain

$$\left\{ \frac{1}{g} \frac{\partial}{\partial \beta} \left[g \frac{\partial}{\partial \beta} \right] + \frac{1}{g} \frac{\partial}{\partial \delta} \left[\frac{1}{\beta^2} g \frac{\partial}{\partial \delta} \right] + \epsilon - V - \frac{A_J}{\beta^2 (1 + \sin^2 \delta)} \right\} \Psi(\beta, \delta) = 0 \quad (24)$$

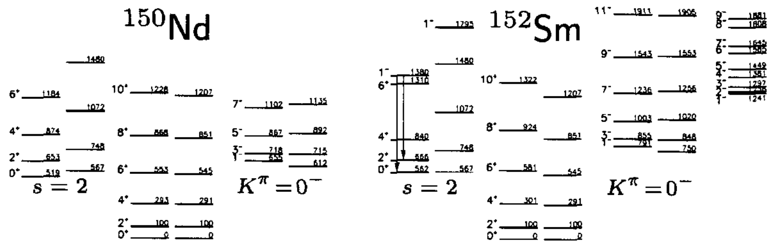

Fig. 2. The experimental (left) and theoretical energies (normalized to that of the first excited state) of the $K = 0$ bands in ^{150}Nd and ^{152}Sm. In the latter, the experimental values for the $K^\pi = 1^-$ band are also shown. Theoretical values for even J and parity are those of the X(5) model, for odd J and parity are obtained with $\Delta_1 = 15$ for ^{150}Nd and with $\Delta_1 = 20$ for ^{152}Sm.

where $g \propto G^{1/2} \propto \beta^5\ [(1+\sin^2\delta)^2/\ (1+4\sin^2\delta)]\ \gamma_0\ v\ u\ w$, $\epsilon = 2E/\hbar^2$, and $A_J = J(J+1)/3$. Again, it is convenient to eliminate in the Eq. 24 the first–derivative terms, with the substitution $\Psi(\beta,\delta) = g^{-1/2}\ \Psi_0(\beta,\delta)$ giving, at the limit $\delta \ll 1$,

$$\left\{\frac{\partial^2}{\partial\beta^2} + \frac{1}{\beta^2}\frac{\partial^2}{\partial\delta^2} + \epsilon - V(\beta,\delta) - \frac{A_J - 7/4}{\beta^2}\right\}\Psi_0 = 0\ . \qquad (25)$$

The Eq. 25 has a structure very similar to that of the Bohr equation for pure quadrupole motion at the limit close to the axial symmetry, with our parameter δ in the place of γ_2. We could assume, also here, that the potential $V(\beta,\delta)$ is the sum of two independent terms, $V = V_\beta(\beta) + V_\delta(\delta)$, and try – as in the X(5) model [24] – an approximate separation of the variables, substituting the factor $1/\beta^2$ with a proper average value in the differential equation for δ. The result would be a level spacing in the excited band very similar to the ground–state band, and, at least in ^{152}Sm and ^{150}Nd, this is the case for the γ band but not for the negative-parity ones [25].

Now we want to explore some alternative procedure which could better account for the experimental data. If, *e.g.* one assumes for δ a square-well potential $V_\delta(\delta) = 0$ for $\delta < \delta_{\rm w}$ and $= +\infty$ elsewhere, the equation becomes exactly separable, with $\Psi_0 = \psi(\beta)\phi(\delta)$:

$$\frac{{\rm d}^2\phi_{\rm k}(\delta)}{{\rm d}\delta^2} + \left[A'_k - V_\delta(\delta)\right]\phi_k(\delta) = 0 \qquad (26)$$

$$\frac{{\rm d}^2\psi_{\rm skJ}(\beta)}{{\rm d}\beta^2} + \left[\epsilon_{\rm skJ} - \tilde{V}_\beta(\beta) - \frac{1}{\beta^2}\left(A_J + 2 + \Delta_k\right)\right]\psi_{\rm skJ}(\beta) = 0$$

where $\tilde{V}_\beta = V_\beta(\beta) + (1/\beta^2)[A'_0 - 1/4]$ and $\Delta_k = A'_k - A'_0$. We now consider in particular the case where $\tilde{V}_\beta$ is a critical–point potential, $\tilde{V}_\beta = 0$ when

Table 2. Experimental and calculates B(E1) strengths in ^{152}Sm

Transition	E_i [keV]	$E\gamma$ [keV]	B(E1) [10^{-2}W.u.] experimental	calculated
$1^- \to 0^+$	963	963	0.42 (4)	0.42 (norm.)
$1^- \to 2^+$	963	841	0.77 (7)	0.97
$3^- \to 2^+$	1041	919	0.81 (16)	0.62
$3^- \to 4^+$	1041	675	0.82 (16)	0.99
$1^- \to 0^+_2$	963	279	weak	0.37
$1^- \to 2^+_2$	963	153	0.013 (4)	0.61

Note: Calculated strengths are normalized to the experimental one for the $1^- \to 0^+$ transition.

β is in the interval $0 - \beta_{\rm w}$ and $= +\infty$ elsewhere*. In this case, for $k = 0$ one obtains the X(5) solution. For the first excited band of negative parity ($k = 1$), the term $\Delta_1 = A'_1 - A'_0$ will be considered as an adjustable parameter. In this case, the spectrum of eigenvalues is given by

$$\epsilon(s, J, k) - \epsilon_0 = C\ [x_\nu(s, J, k)]^2 \tag{27}$$

with C constant, $x_\nu(s, J, k)$ the $s^{\rm th}$ zero of the Bessel function $J_\nu(x)$ and $\nu = \sqrt{J(J+1)/3 + 9/4 + \Delta_k}$. The Fig. 2 shows a partial level scheme of ^{152}Sm and ^{150}Nd, normalized to the energy of the first excited state, and compared with the values derived from Eq. 27 (which, for positive–parity states, coincide with those of the X(5) model). The agreement is fairly good in both cases. In ^{152}Sm, the comparison can be extended to the lowest negative–parity state of the $s = 2$ band, if the level reported as $1^{(-)}$ really belongs to this band as it would be suggested by its decay. If it is so, its energy is significantly lower than the model prediction, but this happens also for all the $s = 2$ excited states of even parity and spin.

The B(E1) strength of a few transitions is known, but the theoretical analysis of this information is not easy. In fact, if we remain strictly in the frame of the hydrodynamical model (with a fluid of constant charge density) all E1 transitions are predicted to vanish (at least, in the long wavelength limit). An extension of the model, assuming a constant charge polarizability of the fluid, gives an Electric Dipole operator in the form [26]

$$D^{(1)}_\mu = c\ \beta_2\beta_3 Y^{(1)}_\mu \tag{28}$$

*With a separation constant $A'_0 = 1/4$ one would obtain exactly $\tilde{V}_\beta \equiv V_\beta$. Actually, it is probably unnecessary to assume that this relation holds. In fact, if a phenomenological potential is used for the active variables, this potential should already include the zero–point energies of all other degrees of freedom not explicitly taken into account.

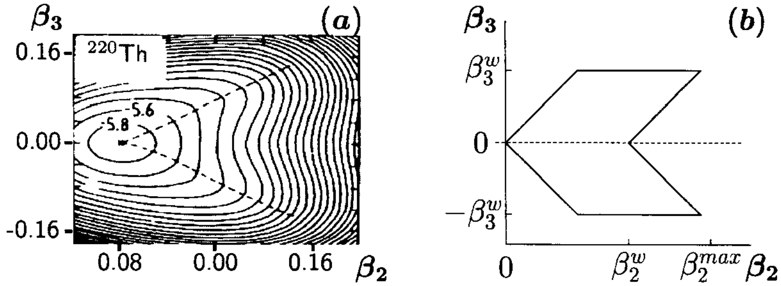

Fig. 3. Part *a*: Potential energy $V(\beta_2, \beta_3)$ calculated by Nazarwicz et al. [29] for ^{220}Th. From this situation of *correlated* quadrupole–octupole vibrations, moving towards a stable quadrupole–octupole deformation, one should cross a critical point approximately described by the potential shown in the part *b*, with $V = 0$ inside the marked area and $= +\infty$ outside. The dotted lines inside this area show the shape of the lattice used for the numerical integration.

with c dependent on the nuclear polarizability.

It would be important to check the validity of Eq. 28 in real nuclei by means of microscopic calculations. Results obtained with a Skyrme Hartree–Foch approach for well deformed nuclei of the Ra – Th region [27], show that the nuclear polarizability c is almost constant in a given nucleus, but changes drastically (even in sign) from one isotope to the next. It is not clear what can happen outside the region of stable quadrupole deformation.

In the Table 2, the values of B(E1) calculated [28] according to Eq. 28 are compared with experimental data. The strengths of E1 transitions inside the $s = 1$ sector of the level scheme are reasonably reproduced, but the experimental values of B(E1) for transitions to the $s = 2$ band are much smaller than the calculated values. Apparently, a selection rule exists, that the model is not able to reproduce.

10. Critical-point behaviour for ^{224}Ra and ^{224}Th ?

When the amplitude of the octupole vibrations is not negligible compared with the quadrupole, *i.e.* when the excitation energy of negative-parity levels is comparable with that of positive-parity ones, the above approximations are no longer valid and, at the moment, the only practicable way seems to be a numerical approach. The phase transition could take place, in this case, between a situation of quadrupole–octupole vibration around a spherical shape (with a potential like in Fig. 3*a*) directly to a permanent quadrupole–octupole deformation. The flat potential at the critical point has been schematized as shown in Fig. 3*b*. With the substitution

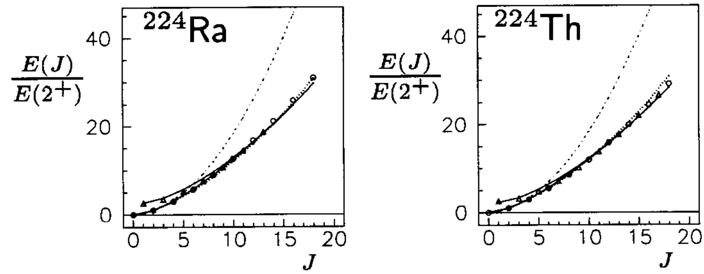

Fig. 4. Experimental excitation energies of the positive parity levels (circles) and of the negative parity ones (triangles) for ^{224}Ra and ^{224}Th, compared with the results of the present model (full line) with the follwing values of the parameters: for ^{224}Ra, $\beta_3^{\rm w}/\beta_2^{\rm w} = 0.915$, $\beta_2^{\rm max}/\beta_2^{\rm w} = 1.333$; for ^{224}Th, $\beta_3^{\rm w}/\beta_2^{\rm w} = 0.800$, $\beta_2^{\rm max}/\beta_2^{\rm w} = 1.360$ (note that the assumed values are not the result of a real optimization, still to be done). The predictions of the X(5) model (dotted lines) and of a rigid-rotor model (dashed-dotted) are shown for comparison.

$\Psi = g^{-1/2}\Psi_0$, the differential equation to be solved takes the form

$$\left[\frac{\partial^2}{\partial\beta_2^2} + \frac{\partial^2}{\partial\beta_3^2} + \epsilon - V(\beta_2,\beta_3) - V_{\rm g}(\beta_2,\beta_3) - \right] \Psi_0(\beta_2,\beta_3) = 0 \qquad (29)$$

with $V_{\rm g}$ given in the Eq. 22 and $\Psi_0 = 0$ on the contour of Fig. 3*b*.

The numerical integration has been performed with the finite difference method. Namely, the space is discretized on a rectangular lattice and values of Ψ_0 at the lattice nodes are taken as independent variables. In the place of second derivatives, the ratios of finite differences are used: *e.g.*,

$$\left(\frac{\partial^2\Psi_0}{\partial\beta_2^2}\right)_{x,y} \Rightarrow \frac{\Psi_0(x+\Delta_x,y) - 2\ \Psi_0(x,y) + \Psi_0(x-\Delta_x,y)}{\Delta_x^2}$$

As $\Psi_0(\beta_2,\beta_3) = \pm\Psi_0(\beta_2,-\beta_3)$ for J even/odd, it is enough to consider only the region $\beta_3 > 0$. The lattice centers are chosen as

$$\beta_3 = (k_3 + 1/2)\Delta_y \qquad \beta_2 = (k_2 + k_3 + 1/2)\Delta_x \qquad (30)$$

with $k_2 = 1...n_2$, $k_3 = 1...n_3$, and $\Delta_x = \beta_2^w/(n_2+1)$, $\Delta_y = 2\beta_3^w/(2n_3+1)$. The ratio Δ_y/Δ_x determines the slope of the lateral borders. With this choice, all borders contain a line of nodes, on which we assume $\Psi_0 = 0$. The number of nodes internal to the integration region – and therefore the number of variables – is now $N = n_2 \cdot n_3$, and we obtain a finite dimensional $N \times N$ Hamiltonian matrix. This Hamiltonian has been diagonalized with the Implicitly Restarted Arnoldi – Lanczos method, using the ARPACK package [30]. This kind of analysis is still in progress. As an example, we give in Fig. 4 the results for two particular values of the parameters $\beta_2^{\max}/\beta_2^{\mathrm{w}}$ and $\beta_3^{\mathrm{w}}/\beta_2^{\mathrm{w}}$, which satisfactorily reproduce the experimental values of $E(J^{\pi})/E(2^{+})$ for ^{224}Ra ^{224}Th. We note that in both cases the positive part of the band is very close to the X(5) predictions, while the negative part fits satisfactorily the experimental results. This fact suggests that ^{224}Ra and ^{224}Th really lye close to the critical point of a phase transition between spherical shape and reflection–asymmetric deformation.

Obviously, this conclusion should be substantiated by further calculations, in particular for the transition amplitudes. This work is in program.

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