RECONSTRUCTION OF SOUND PRESSURE FIELD BY IFEM

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This talk discusses an inverse problem of acoustic. The aim is to reconstruct the sound pressure field of a cavity based on a small number of measurements. In the calculation, arbitrary admittance boundary conditions are considered. Therefore, the inverse formulation requires to include the boundary admittance as a coefficient of the Robin boundary condition for the Helmholtz differential equation. In order to support a minimization of the necessary number of measurements, the new approach is based on an inverse formulation of the finite element method for the acoustical boundary value problem, of which its facility to extract a modal solution can be advantageous.

1 Introduction

This contribution reports about the progress on a FEM-based approach to solve the inverse acoustic problem of an internal space considering admittance boundary condition, called IFEM. In spaces bounded by structures with complex geometry it is difficult to measure directly the boundary admittance, that is an essential parameter for acoustic simulations.

The authors have not found any methods in literature to globally estimate the boundary admittance of arbitrarily shaped cavities by using inverse methods. Apparently, in inverse acoustics two major types of algorithms are developed usually to detect sources. There is the near-field acoustic holography (NAH) [1,2] and the inverse frequency response function (IFRF), also called inverse boundary element method (IBEM) [3–8]. A so-called hybrid NAH [9] was developed to combine the advantages of NAH and IBEM. Although this method is applicable to arbitrarily shaped surfaces, it does still not consider or calculate admittance boundary conditions.

One major motivation for our rather exceptional FEM-based investigations in inverse acoustics is based upon the capability of FEM to extract modal information. This feature shall be used to decrease the experimental expenses. Owing to its properties, an orthogonal modal basis might be better suited for sound field reconstruction than other basis functions. Further, it is shown in principal that boundary admittance can be explicitly evaluated based on the surface sound pressure [10]. The surface sound pressure itself my be caculated from pressure measurements in the interior domain by solving a Dirichlet problem and computing an ill-conditioned inversion in a second step, but without quantifying the admittance boundary condition [11].

Hence, an algorithm is investigated to firstly calculate the surface sound pressure based on sound pressure measurements in the interior domain using a FEM formulation. The governing equations of the damped FEM acoustics as well as its well known forward solution are refereed to now, before we face the actual inverse problem.

The boundary value problem

$$\begin{aligned} \Delta p(\vec{x}) + k^2 p(\vec{x}) &= 0\,, \quad \vec{x} \in \Omega \subset \mathbb{R}^d \\ p_{,n}(\vec{x}) &= s\,k[v_s(\vec{x}) + Y(\vec{x})\,p(\vec{x})]\,, \quad \vec{x} \in \Gamma\,, \; s = \mathrm{i}\,\rho_0\,c\,, \end{aligned} \tag{1}$$

i.e. the Helmholtz differential equation together with the Robin boundary condition, describes the sound pressure field $p(\vec{x})$ at wavenumber k of a one-way structure-fluid interaction model for a cavity. The fluid properties are given by the density ρ_0 and the speed of sound c. The boundary condition incorporates the surface velocity v_s as well as damping, elasticity and mass influence of the boundary Γ through the complex coefficient Y, the boundary admittance.

The discretisation of the acoustic boundary value problem by means of FEM results in

$$(\boldsymbol{K} - k^2\,\boldsymbol{M} - \mathrm{i}\,k\,\boldsymbol{D})\,\boldsymbol{p} = \boldsymbol{b}\,, \tag{2}$$

with the stiffness, mass and damping matrices $\boldsymbol{K}$, $\boldsymbol{M}$ and $\boldsymbol{D}$, respectively. The matrices are of size

NxN where N is the number of nodes. The excitation appears in $\boldsymbol{b} = s\,k\,\boldsymbol{F}\,\boldsymbol{v}_s$, where $\boldsymbol{F}$ denotes the boundary mass matrix. The damping matrix can be formally written as $\boldsymbol{D} = \rho_0\,c\,Y\,\boldsymbol{F}$, whereas $\boldsymbol{D}$ is actually a superposition of element boundary mass matrices with Y being constant on each boundary face. Herein, we assume the admittance to be constant over frequency to enable a modal solution.

The modal forward solution of Eq. (2)

$$\boldsymbol{p} = -\sum_{i=1}^{2N-N_{red}} \frac{\boldsymbol{v}_i \cdot \boldsymbol{b}}{\alpha_i + \mathrm{i}\,k\,\beta_i}\,\boldsymbol{v}_i\,. \tag{3}$$

is obtained by a superposition of eigenvectors $\boldsymbol{v}_i$ that are computationally produced via the solution of the general linear eigenvalue problem of the state space transform of Eq. (2). N_{red} defines the depth of modal reduction; α_i and β_i are the eigenpairs that are connected to the eigenvalues λ_i through $\lambda_i = \alpha_i/\beta_i$ ($\beta_i \neq 0$).

2 Inverse Problem

The objective of this section is the inverse formulation of Eq. (2) which is to be used to reconstruct the whole sound pressure field from pressure measurements $\boldsymbol{p}_m$ located at internal grid points. The pressure at the remaining internal nodes $\boldsymbol{p}_f$ and at the boundary $\boldsymbol{p}_b$ are to be estimated without knowledge about the boundary admittance. Eq. (2) is rearranged in terms of the type of the nodes

$$\begin{bmatrix} \boldsymbol{G}_{bb} & \boldsymbol{G}_{bf} & \boldsymbol{G}_{bm} \\ \boldsymbol{G}_{fb} & \boldsymbol{G}_{ff} & \boldsymbol{G}_{fm} \\ \boldsymbol{G}_{mb} & \boldsymbol{G}_{mf} & \boldsymbol{G}_{mm} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_b \\ \boldsymbol{p}_f \\ \boldsymbol{p}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_b + \mathrm{i}\,k\,\boldsymbol{D}_{bb}\,\boldsymbol{p}_b \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{4}$$

with submatrices $\boldsymbol{G}_{ij} = \boldsymbol{K}_{ij} - k^2\,\boldsymbol{M}_{ij}\,,\ \{i,j\} = \{b,f,m\}$. Motivated by the observations the Dirichlet problem in [11] we extract only the lower row of submatrices and have to solve the incomplete but linear system of equations

$$\underbrace{\begin{bmatrix} \boldsymbol{G}_{ff} & \boldsymbol{G}_{fm} \\ \boldsymbol{G}_{mf} & \boldsymbol{G}_{mm} \end{bmatrix}}_{\boldsymbol{G}_D} \begin{bmatrix} \boldsymbol{p}_f \\ \boldsymbol{p}_m \end{bmatrix} = -\begin{bmatrix} \boldsymbol{G}_{fb} \\ \boldsymbol{G}_{mb} \end{bmatrix} \boldsymbol{p}_b\,. \tag{5}$$

An eigenvalue analysis of the symmetric system matrix $\boldsymbol{G}_D$ of the Dirichlet problem provides us with a set of global and orthogonal basis functions that are used to express $\boldsymbol{G}_D^{-1}$. After some rearrangement we end up at the linear equation

$$(\boldsymbol{A} - k^2\,\boldsymbol{B})\,\boldsymbol{p}_u = \boldsymbol{q} \tag{6}$$

for the unknown pressure values $\boldsymbol{p}_u^T = [\boldsymbol{p}_b^T\,,\,\boldsymbol{p}_f^T]$.

3 Solution Techniques

$\boldsymbol{A}$ and $\boldsymbol{B}$ are static matrices so as to allow for a modal solution. But they are, depending on the over- or under-determination of the problem, rectangular and as most inverse problems heavily ill-conditioned, impeding sensible results of a modal superposition.

Instead, a Tikhonov regularization

$$\|\boldsymbol{Q}\,\boldsymbol{p}_u - \boldsymbol{q}\|^2 + \alpha^2\,\|\boldsymbol{p}_u\|^2 \quad \rightarrow \text{min!} \tag{7}$$

at fixed wavenumbers k shall get the ill-posedness of the system matrix $\boldsymbol{Q} = \boldsymbol{A} - k^2\,\boldsymbol{B}$ under control [12, 13]. Here, the sums of the errors of the residual and the system variable are minimized by imposing a weighting upon them with the regularization parameter α. To solve Eq. (7), a singular value decomposition (SVD) does reveal the behavior of the ill-posed system to help finding an optimal regularization parameter α_{opt}, cf. [14]. With the eigenvalues λ_j and eigenvectors $\boldsymbol{v}^j$, $\boldsymbol{u}^j$ of the matrix products $\boldsymbol{Q}^T\,\boldsymbol{Q}$, $\boldsymbol{Q}\,\boldsymbol{Q}^T$ the solution of Tikhonov regularization may be written as superposition of these modes

$$\boldsymbol{p}_u = \sum_{j=1}^{r} f_j \frac{\boldsymbol{u}^j \cdot \boldsymbol{q}}{\sigma_j}\,\boldsymbol{v}^j\,. \tag{8}$$

Here r stands for the rank of matrix $\boldsymbol{Q}$. Note that all eigenvalues are positive. Thus, the singular values can be defined as $\sigma_j = \sqrt{\lambda_j}$, cf. [15], $f_j = \sigma_j^2/(\sigma_j^2 + \alpha^2)$ denotes the jth filter factor. For any of the investigated problems in [16] the condition number, that is defined by the ratio of the highest and the lowest singular value, turns out to be $\mathrm{cond}(\boldsymbol{Q}) \gg 1$, indicating ill-posedness.

An optimal regularization parameter α_{opt} truncates the high frequency components (at high indices j) by means of regularization and is capable of producing a result with minimized error inflicted by the solution technique. The standard L-curve criterion, well explained in [12], is chosen for finding α_{opt}.

Incorporating realistic noise impaired inputs mean a significant hurdle for finding a sensible solution of the ill-posed inverse acoustic problem, let alone the aim to minimize the measurement expenses. However, noise was neglected during the tests in [16]. Instead, the forward solution (3) was used providing reference values for p_b, p_f and simulated data for p_m in order to check the derived algorithm with virtually undisturbed input values as a first step.

4 Conclusion

The method that has been outlined is based on an inverse finite element formulation using the modal basis of the Dirichlet problem and Tikhonov regularization.

The tests of the linear approach on two-dimensional examples as for instance a passenger compartment of a car revealed that it works for the very special case of Dirichlet boundary conditions, i.e. very high values for the boundary admittance. The numerical errors could be minimized by Tikhonov regularization. Still, most cases showed a lack of accuracy in the reconstructed sound pressure field aspecially near the boundary. This fact can be explained by the missing evaluation of the information of Eq. (4) that connects the nodes at and near the boundary. However, since not only over-determined but also under-determined cases behaved in the same way of featuring a good reconstruction of the sound pressure away from the boundary, it is the belief of the authors to be able to decrease the experimental expenses by limiting the number of measurements and applying more appropriate basis functions. Thus, the focus will be on under-determined systems.

Hence, it might be reasonable to further search for an approach that utilizes the remaining system equations and a modal bases adjusted to the actual distribution of the boundary admittance. It might be one possibility to estimate the boundary admittance and then adjust the sound pressure field. In any case, optimization techniques with different basis functions will be focused. It remains the aim of the authors to decrease the experimental expenses and, thus, focus on the under-determined systems.

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