

# HYPERGEOMETRIC SUMMATION REVISITED

S. A. ABRAMOV\*

*Dorodnicyn Computing Centre, Russian Academy of Sciences,  
Vavilova 40, Moscow GSP-1, 119991, Russia  
E-mail: sabramov@ccas.ru*

M. PETKOVŠEK†

*Faculty of Mathematics and Physics, University of Ljubljana,  
Jadranska 19, SI-1000 Ljubljana, Slovenia  
E-mail: marko.petkovsek@fmf.uni-lj.si*

We consider hypergeometric sequences, i.e., the sequences which satisfy linear first-order homogeneous recurrence equations with relatively prime polynomial coefficients. Some results related to necessary and sufficient conditions are discussed for validity of discrete Newton-Leibniz formula  $\sum_{k=v}^w t(k) = u(w+1) - u(v)$  when  $u(k) = R(k)t(k)$  and  $R(k)$  is a rational solution of Gosper's equation.

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## 1. Introduction

Let  $K$  be a field of characteristic zero ( $K = \mathbb{C}$  in all examples). If  $t(k) \in K(k)$  then the *telescoping equation*

$$u(k+1) - u(k) = t(k) \tag{1}$$

may or may not have a rational solution  $u(k)$ , depending on the type of  $t(k)$ . Here the telescoping equation is considered as an equality in the rational-function field, regardless of the possible integer poles that  $u(k)$  and/or  $t(k)$  might have.

An algorithm for finding rational  $u(k)$  was proposed in 1971 (see Ref. 1). It follows from that algorithm that if  $t(k)$  has no integer poles, then a

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rational  $u(k)$  satisfying (1), if it exists, has no integer poles either, and the *discrete Newton-Leibniz formula*

$$\sum_{k=v}^w t(k) = u(w+1) - u(v) \quad (2)$$

is valid for any integer bounds  $v \leq w$ . Working with polynomial and rational functions we will write  $f(k) \perp g(k)$  for  $f(k), g(k) \in K[k]$  to indicate that  $f(k)$  and  $g(k)$  are coprime; if  $R(k) \in K(k)$ , then  $\text{den}(R(k))$  is the monic polynomial from  $K[k]$  such that  $R(k) = \frac{f(k)}{\text{den}(R(k))}$  for some  $f(k) \in K[k], f(k) \perp \text{den}(R(k))$ .

The problem of solving equation (1) can be considered for sequences. If  $t(k)$  is a sequence, we use the symbol  $E$  for the shift operator w.r. to  $k$ , so that  $Et(k) = t(k+1)$ . In the rest of the paper we assume that the sequences under consideration are defined on an infinite interval  $I$  of integers and either  $I = \mathbb{Z}$ , or

$$I = \mathbb{Z}_{\geq l} = \{k \in \mathbb{Z} \mid k \geq l\}, \quad l \in \mathbb{Z}.$$

If a sequence  $t(k)$  defined on  $I$  is given, and a sequence  $u(k)$ , which is also defined on  $I$  and satisfies (1) for all  $k \in I$ , is found (any such sequence is a *primitive* of  $t(k)$ ), then we can use formula (2) for any  $v \leq w$  with  $v, w \in I$ .

Gosper's algorithm,<sup>6</sup> which we denote hereafter by  $\mathcal{GA}$ , discovered in 1978, focuses on the case where a given  $t(k)$  and an unknown  $u(k)$  are hypergeometric sequences.

**Definition 1.1.** A sequence  $y(k)$  defined on an infinite interval  $I$  is hypergeometric if it satisfies the equation  $Ly(k) = 0$  for all  $k \in I$ , with

$$L = a_1(k)E + a_0(k) \in K[k, E], \quad a_1(k) \perp a_0(k). \quad (3)$$

$\mathcal{GA}$  starts by constructing the operator  $L$  for a given concrete hypergeometric sequence  $t(k)$ , and this step is not formalized. On the next steps  $\mathcal{GA}$  works with  $L$  only, while the sequence  $t(k)$  itself is ignored (more precisely, in the case of  $L = a_1(k)E + a_0(k)$ ,  $\mathcal{GA}$  works with the *certificate* of  $t(k)$ , i.e., with the rational function  $-\frac{a_0(k)}{a_1(k)}$ , but this is not essential). The algorithm tries to construct a rational function  $R(k)$ , which is a solution in  $K(k)$  of *Gosper's equation*

$$a_0(k)R(k+1) + a_1(k)R(k) = -a_1(k) \quad (4)$$

(such  $R(k)$ , when it exists, can also be found by general algorithms from Refs. 2,3). If such  $R(k)$  exists then

$$R(k+1)t(k+1) - R(k)t(k) = t(k)$$

is valid for *almost* all integers  $k$ . The fact is that even when  $t(k)$  is defined everywhere on  $I$ , it can happen that  $R(k)$  has some poles belonging to  $I$ , and  $u(k) = R(k)t(k)$  cannot be defined in such a way as to make (1) valid for all integers from  $I$ . One can encounter the situation where formula (2) is not valid even when all of

$$t(v), t(v + 1), \dots, t(w), \quad u(v), u(w + 1)$$

are well-defined. The reason is that (1) may fail to hold at certain points  $k$  of the summation interval. However, sometimes it is possible to define the values of  $u(k) = R(k)t(k)$  appropriately for all integers  $k$ , even though  $R(k)$  has some integer poles. In such well-behaved cases (2) can be used to compute  $\sum_{k=v}^w t(k)$  for any  $v \leq w, v, w \in I$ .

**Example 1.1.**

Gosper’s equation, corresponding to  $L = kE - (k + 1)^2$ , has a solution  $R = \frac{1}{k}$ . The sequences

$$t_1(k) = \begin{cases} 0, & \text{if } k < 0, \\ k \cdot k!, & \text{if } k \geq 0 \end{cases}$$

and

$$t_2(k) = \begin{cases} \frac{(-1)^k k}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \geq 0 \end{cases}$$

both satisfy  $Ly = 0$  on  $I = \mathbb{Z}$ .

Generally speaking, (2) is not applicable to  $t_1(k)$ , but is applicable to  $t_2(k)$ . We can illustrate this as follows. Applying (2) to  $t_1(k)$  with  $v = -1, w = 1$ , we have

$$t_1(-1) + t_1(0) + t_1(1) = \frac{1}{k}t_1(k) \Big|_{k=2} - \frac{1}{k}t_1(k) \Big|_{k=-1} = \frac{1}{2} \cdot 4 - 0 = 2$$

which is wrong, because  $t_1(-1) + t_1(0) + t_1(1) = 0 + 0 + 1 = 1$ . Applying (2) to  $t_2$  with the same  $v, w$ , we have

$$t_2(-1) + t_2(0) + t_2(1) = \frac{1}{k}t_2(k) \Big|_{k=2} - \frac{1}{k}t_2(k) \Big|_{k=-1} = 0 - (-1) = 1$$

which is correct, because  $t_2(-1) + t_2(0) + t_2(1) = 1 + 0 + 0 = 1$ .

In this paper we discuss some results related to necessary and sufficient conditions for validity of formula (2) when  $u(k) = R(k)t(k)$ , and  $R(k)$  is a rational solution of corresponding Gosper’s equation. If such  $R(k)$  exists, then we describe the linear space of all hypergeometric sequences  $t(k)$  that are defined on  $I$  and such that formula (2) is valid for  $u = Rt$  and any

integer bounds  $v \leq w$  such that  $v, w \in I$ . The dimension of this space is always positive (it can be even bigger than 1). We will denote

- by  $\mathcal{H}_I$  the set of all hypergeometric sequences defined on  $I$ ;
- by  $\mathcal{L}$  the set of all operators of type (3);
- by  $V_I(L)$ , where  $L \in \mathcal{L}$ , the  $K$ -linear space of all sequences  $t(k)$  defined on  $I$  for which  $Lt(k) = 0$  for all  $k \in I$ ;
- by  $W_I(R(k), L)$ , where  $L \in \mathcal{L}$  and  $R(k) \in K(k)$  is a solution of the corresponding Gosper's equation, the  $K$ -linear space of all  $t(k) \in V_I(L)$  such that (2) with  $u(k) = R(k)t(k)$  is valid for all  $v \leq w$  with  $v, w \in I$ .

The paper is a summary of the results that have been published in Refs. 4,5. In addition we consider the case where Gosper's equation has non-unique rational solution (Section 3.2). In Section 2 we consider individual hypergeometric sequences while in Section 3 we concentrate on spaces of the type  $W_I(R(k), L)$ .

## 2. Validity conditions of the discrete Newton-Leibniz formula

### 2.1. A criterion

**Theorem 2.1.**<sup>4,5</sup> *Let  $L \in \mathcal{L}$ ,  $t(k) \in V_I(L)$ , and let Gosper's equation corresponding to  $L$  have a solution  $R(k) \in K(k)$ , with  $\text{den}(R) = g(k)$ . Then  $t(k) \in W_I(R(k), L)$  iff there exists a  $\bar{t}(k) \in \mathcal{H}_I$  such that  $t(k) = g(k)\bar{t}(k)$  for all  $k \in I$ .*

**Example 2.1.** Consider again the sequences  $t_1(k), t_2(k)$  on  $I = \mathbb{Z}$  from Example 1.1. We have  $t_2(k) = k\bar{t}_2(k)$ , where

$$\bar{t}_2(k) = \begin{cases} \frac{(-1)^k}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \geq 0 \end{cases}$$

is a hypergeometric sequence defined everywhere:

$$E\bar{t}_2(k) - (k+1)\bar{t}_2(k) = 0.$$

On the other hand, if  $t_1(k) = k\bar{t}_1(k)$  for some sequence  $\bar{t}_1(k)$ , then

$$\bar{t}_1(k) = \begin{cases} 0, & \text{if } k < 0, \\ \zeta, & \text{if } k = 0, \\ k!, & \text{if } k > 0 \end{cases}$$

where  $\zeta \in \mathbb{C}$ . Notice that the sequence  $\bar{t}_1(k)$  is not hypergeometric on  $\mathbb{Z}$ , for any  $\zeta \in \mathbb{C}$ .

**2.2. Summation of proper hypergeometric sequences**

**Definition 2.1.** Following conventional notation, the *rising factorial power*  $(\alpha)_k$  and its *reciprocal*  $1/(\beta)_k$  are defined for  $\alpha, \beta \in K$  and  $k \in \mathbb{Z}$  by

$$(\alpha)_k = \begin{cases} \prod_{m=0}^{k-1} (\alpha + m), & k \geq 0; \\ \prod_{m=1}^{|k|} \frac{1}{\alpha - m}, & k < 0, \alpha \neq 1, 2, \dots, |k|; \\ \text{undefined,} & \text{otherwise;} \end{cases}$$

$$\frac{1}{(\beta)_k} = \begin{cases} \prod_{m=0}^{k-1} \frac{1}{\beta + m}, & k \geq 0, \beta \neq 0, -1, \dots, 1 - k; \\ \prod_{m=1}^{|k|} (\beta - m), & k < 0; \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Note that if  $(\alpha)_k$  resp.  $1/(\beta)_k$  is defined for some  $k \in \mathbb{Z}$ , then  $(\alpha)_{k+1}$  resp.  $1/(\beta)_{k-1}$  is defined for that  $k$  as well. Thus  $(\alpha)_k$  and  $1/(\beta)_k$  are hypergeometric sequences which satisfy

$$(\alpha)_{k+1} = (\alpha + k)(\alpha)_k, \quad (\beta + k)/(\beta)_{k+1} = 1/(\beta)_k \tag{5}$$

whenever  $(\alpha)_k$  and  $1/(\beta)_{k+1}$  are defined.

**Example 2.2.** Let  $t(k) = (k - 2)(-1/2)_k/(4k!)$ . This hypergeometric sequence is defined for all  $k \in \mathbb{Z}$  (note that  $t(k) = 0$  for  $k < 0$ ) and satisfies  $Lt(k) = 0$  for all  $n \in \mathbb{Z}$  where  $L = a_1(k)E + a_0(k)$  with  $a_0(k) = -(k - 1)(2k - 1)$  and  $a_1(k) = 2(k - 2)(k + 1)$ . Gosper’s equation, corresponding to  $L$ , has a rational solution

$$R(k) = \frac{2k(k + 1)}{k - 2}. \tag{6}$$

Equation (1) indeed fails at  $k = 1$  and  $k = 2$  because  $u(k) = R(k)t(k)$  is undefined at  $k = 2$ . But if we cancel the factor  $k - 2$  and replace  $u(k)$  by the sequence

$$\bar{u}(k) = k(k + 1) \frac{(-1/2)_k}{2k!},$$

then equation

$$\bar{u}(k+1) - \bar{u}(k) = t(k) \quad (7)$$

holds for all  $k \in \mathbb{Z}$ , and

$$\sum_{k=v}^w t(k) = \bar{u}(w+1) - \bar{u}(v). \quad (8)$$

The sequence  $t(k)$  from Example 2.2 is an instance of a *proper hypergeometric* sequence which we are going to define now. As it turns out, there are no restrictions on the validity of the discrete Newton-Leibniz formula for proper sequences (Theorem 2.2).

**Definition 2.2.** A hypergeometric sequence  $t(k)$  defined on an infinite interval  $I$  of integers is *proper* if there are

- a constant  $z \in K$ ,
- a polynomial  $p(k) \in K[k]$ ,
- nonnegative integers  $q, r$ ,
- constants  $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_r \in K$

such that

$$t(k) = p(k)z^k \frac{\prod_{i=1}^q (\alpha_i)_k}{\prod_{j=1}^r (\beta_j)_k} \quad (9)$$

for all  $k \in I$ .

**Theorem 2.2.**<sup>4</sup> Let  $t(k)$  be a proper hypergeometric sequence defined on  $I$  and given by (9). Denote  $a(k) = z \prod_{i=1}^q (k + \alpha_i)$  and  $b(k) = \prod_{j=1}^r (k + \beta_j)$ . If a polynomial  $y(k) \in K[k]$  satisfies

$$a(k)y(k+1) - b(k-1)y(k) = p(k) \quad (10)$$

and if

$$\bar{u}(k) = y(k)z^k \frac{\prod_{i=1}^q (\alpha_i)_k}{\prod_{j=1}^r (\beta_j)_{k-1}}$$

for all  $k \in I$ , then equation (7) holds for all  $k \in I$ , and the discrete Newton-Leibniz formula (8) is valid for all  $v \leq w$ , when  $v, w \in I$ .

Notice that (10) has a solution in  $K[k]$  iff Gosper's equation, corresponding to the operator from  $\mathcal{L}$ , annihilating  $t(k)$ , has a solution in  $K(k)$ .

**Example 2.3.** The hypergeometric sequence

$$t(k) = \frac{\binom{2k-3}{k}}{4^k}, \tag{11}$$

which is defined for all  $k \in \mathbb{Z}$  can be written as

$$t(k) = \begin{cases} 2s(k), & k < 2, \\ s(k), & k \geq 2, \end{cases}$$

where

$$s(k) = (2 - k) \frac{(-1/2)_k}{4(1)_k}$$

is the proper sequence from Example 2.2. For  $w \geq 1$ , one should first split summation range in two

$$\sum_{k=0}^w t(k) = \frac{3}{4} + \sum_{k=2}^w s(k),$$

then the discrete Newton-Leibniz formula can be safely used to evaluate the sum on the right. However, applying directly (2) to (11) with (6) we obtain

$$\sum_{k=0}^w t(k) \stackrel{(?)}{=} u(w+1) - u(0) = \frac{(w+1)(w+2)\binom{2w-1}{w+1}}{2(w-1)4^w}. \tag{12}$$

If we assume that the value of  $\binom{2k-3}{k}$  is 1 when  $k = 0$  and  $-1$  when  $k = 1$  (that is natural from combinatorial point of view) then the expression on the right gives the true value of the sum only at  $w = 0$ .

**2.3. When the interval  $I$  contains no leading integer singularity of  $L$**

**Definition 2.3.** For a linear difference operator (3) we call  $M = \max(\{k \in \mathbb{Z}; a_1(k-1) = 0\} \cup \{-\infty\})$  the *maximal leading integer singularity* of  $L$ ,

**Proposition 2.1.**<sup>4</sup> *Let  $R(k)$  be a rational solution of (4). Then  $R(k)$  has no poles larger than  $M - 1$ .*

**Theorem 2.3.**<sup>4</sup> *Let  $L \in \mathcal{L}$ ,  $M$  be the maximal integer singularity of  $L$ ,  $l \geq M$ ,  $I = \mathbb{Z}_{\geq l}$  and  $t(k) \in V_I(L)$ . Let Gosper's equation, corresponding to  $L$ , have a solution  $R(k)$  in  $K(k)$ . Then  $t(k) \in W_I(R(k), L)$ .*

**Example 2.4.** For the sequence (11) we have  $a_0(k) = -(2k-1)(k-1)$ ,  $a_1(k) = 2(k+1)(k-2)$ ,  $R(k) = 2k(k+1)/(k-2)$ , and  $u(k) = 2k(k+1) \binom{2k-3}{k} / ((k-2)4^k)$ . Thus  $M = 3$ , and the only pole of  $R(k)$  is  $k = 2$ . As predicted by Theorem 2.3, the discrete Newton-Leibniz formula is valid when, e.g.,  $3 \leq v \leq w$ .

### 3. The spaces $V_I(L)$ and $W_I(R(k), L)$

#### 3.1. The structure of $W_I(R(k), L)$

**Theorem 3.1.**<sup>5</sup> *Let  $L \in \mathcal{L}$  and Gosper's equation, corresponding to  $L$ , have a solution  $R(k) \in K(k)$ ,  $\text{den}(R) = g(k)$ . Then*

$$W_I(R(k), L) = g(k) \cdot V_I(\text{pp}(L \circ g(k))),$$

where the operator  $\text{pp}(L \circ g(k))$  is computed by removing from  $L \circ g$  the greatest common polynomial factor of its coefficients.

In addition, if  $R = \frac{f(k)}{g(k)}$ ,  $f(k) \perp g(k)$ , then the space of the corresponding primitives of the elements of  $W_I(R(k), L)$  can be described as  $f(k) \cdot V_I(\text{pp}(L \circ g(k)))$ .

We will denote by  $\bar{L}$  the operator  $\text{pp}(L \circ g(k))$ .

**Example 3.1.** Consider again the operator  $L = kE - (k+1)^2$  from Example 1.1 with  $I = \mathbb{Z}$ . We have  $R = \frac{1}{k}$ , and

$$L \circ k = kE \circ k - (k+1)^2 k = k(k+1)E - (k+1)^2 k = k(k+1)(E - k - 1),$$

$$\bar{L} = E - (k+1).$$

The space  $W_I(R(k), \bar{L})$  is generated by  $\bar{t}_2$ , and, resp., the space  $k \cdot W_I(R(k), \bar{L})$  is generated by  $k\bar{t}_2$ . In accordance with Theorem 3.1 the space  $W_I(R(k), L)$  coincides with  $k \cdot V_I(\bar{L})$ .

It is possible to give examples showing that in some cases  $\dim W_I(R(k), L) > 1$ .

#### Example 3.2.

Let  $L = 2(k^2 - 4)(k - 9)E - (2k - 3)(k - 1)(k - 8)$ ,  $I = \mathbb{Z}$ . Then Gosper's equation, corresponding to  $L$ , has the rational solution

$$R(k) = -\frac{2(k-3)(k+1)}{k-9}.$$

Here  $g(k) = k - 9$  and  $\bar{L} = 2(k^2 - 4)E - (2k - 3)(k - 1)$ . Any sequence  $\bar{t}$  which satisfies the equation  $\bar{L}\bar{t} = 0$  has  $\bar{t}(k) = 0$  for  $k = 2$  or  $k \leq -2$ . The

values of  $\bar{t}(1)$  and  $\bar{t}(3)$  can be chosen arbitrarily, and all the other values are determined uniquely by the recurrence  $2(k^2 - 4)\bar{t}(k+1) = (2k - 3)(k - 1)\bar{t}(k)$ . Hence  $\dim V_I(\bar{L}) = 2$ .

At the same time,  $\dim V_I(L) = 3$ . Indeed, if  $Lt = 0$ , then  $t(-2) = t(2) = t(9) = 0$ . The value  $t(k) = 0$  from  $k = -2$  propagates to all  $k \leq -2$ , but on each of the integer intervals  $[-1, 0, 1]$ ,  $[3, 4, 5, 6, 7, 8]$  and  $[10, 11, \dots]$  we can choose one value arbitrarily, and the remaining values on that interval are then determined uniquely. A sequence  $t(k) \in V_I(L)$  belongs to  $W_I(R(k), L)$  iff  $22t(10) - 13t(8) = 0$ . So  $\dim W_I(R(k), L) = 2$ .

**3.2. When a rational solution of Gosper’s equation is not unique**

We give an example showing that if  $L \in \mathcal{L}$  and Gosper’s equation, corresponding to  $L$ , has different solutions  $R_1(k), R_2(k) \in K(k)$ , then it is possible that  $W_I(R_1(k), L) \neq W_I(R_2(k), L)$ . Moreover, these two spaces can have different dimensions.

**Example 3.3.** If  $L = kE - (k + 1)$ , then Gosper’s equation, corresponding to  $L$ , is

$$-(k + 1)R(k + 1) + kR(k) = -k,$$

and its general rational solution is

$$\frac{k - 1}{2} + \frac{c}{k} = \frac{k^2 - k + 2c}{2k}.$$

Consider the solutions

$$R_1(k) = \frac{k - 1}{2} \quad (g_1(k) = 1), \quad \text{and} \quad R_2(k) = \frac{k^2 - k + 2}{2k} \quad (g_2(k) = k).$$

We have  $L \circ g_1(k) = L$ , and  $W_I(R_1(k), L) = V_I(L)$ . This space has a basis that consists of two linearly independent sequences:

$$t_1(k) = \begin{cases} k, & \text{if } k \leq 0, \\ 0, & \text{if } k > 0 \end{cases}$$

and

$$t_2(k) = \begin{cases} 0, & \text{if } k \leq 0, \\ k, & \text{if } k > 0. \end{cases}$$

So this space contains, e.g., the sequence  $t(k) = |k|$ .

We have  $L \circ g_2(k) = k(k + 1)(E - 1)$ , therefore  $W_I(R_2(k), L)$  is generated by the sequence  $t(k) = k$ .

If Gosper's equation, corresponding to  $L \in \mathcal{L}$ , has non-unique solution in  $K(k)$ , then the equation  $Ly = 0$  has a non-zero solution in  $K(k)$ .

**3.3. If Gosper's equation has a rational solution  $R(k)$  then  $W_I(R, L) \neq 0$**

**Theorem 3.2.**<sup>5</sup> *Let  $L \in \mathcal{L}$  and let Gosper's equation, corresponding to  $L$ , have a solution  $R(k) \in K(k)$ . Then  $W_I(R(k), L) \neq 0$  (i.e.,  $\dim W_I(R(k), L) \geq 1$ ).*

**Example 3.4.**

Let  $L = (k+2)E - k$ . The rational function  $\frac{1}{k(k+1)}$  is a solution in  $K(k)$  of the equation  $Ly = 0$ . Here  $R(k) = -k - 1$ , and  $-1/k$  is a solution of the corresponding telescoping equation:

$$-\frac{1}{k+1} + \frac{1}{k} = \frac{1}{k(k+1)}.$$

The rational functions

$$\frac{1}{k(k+1)} \quad \text{and} \quad -\frac{1}{k}$$

have integer poles. Nevertheless, by Theorem 3.2 it has to be  $W_I(R(k), L) \neq 0$  even when  $I = \mathbb{Z}$ . The space  $W_I(R(k), L)$  is generated by the sequence

$$t(k) = \begin{cases} 1, & \text{if } k = -1, \\ -1, & \text{if } k = 0, \\ 0, & \text{otherwise,} \end{cases}$$

while the primitive of  $t(k)$  is

$$(-k-1)t(k) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

If  $I = \mathbb{Z}_{\geq 1}$ , then  $W_I(R(k), L)$  is generated by the sequence  $t'(k) = \frac{1}{k(k+1)}$ .

By Theorem 2.3, if  $M$  is the maximal integer singularity of  $L$ ,  $l \geq M$ ,  $I = \mathbb{Z}_{\geq l}$ , and Gosper's equation, corresponding to  $L$ , has a solution  $R(k)$  in  $K(k)$ , then  $V_I(L) = W_I(R(k), L)$ . As a consequence,  $\dim V_I(L) = \dim W_I(R(k), L) = 1$ .

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