# Preface

The laws which govern the small vibrations of elastic plates are contained in the three-dimensional equations of the linear theory of elasticity. There is no essential difficulty in obtaining solutions of the equations for infinite plates, at least as far as the establishment of the secular equations. When the faces of the plate are free of traction, the solutions reveal the existence of an infinite number of modes of vibration for each wave-length in the plane of the plate. In a finite plate, with free or clamped edges, each of these modes (or its overtones) couples with all the others (or their overtones), leading to an extraordinarily complex spectrum of frequencies whose details are buried in an infinite system of transcendental equations.

Inasmuch as the governing differential equations are linear, each of the modes of an infinite plate has overtones in a finite plate, but no subtones. Accordingly, the lower modes of an infinite plate influence the high-frequency spectrum of a finite plate but the high modes of an infinite plate do not greatly affect the spectrum of a finite plate at lower frequencies. This circumstance permits the formulation of approximate equations with high-frequency cut-offs at various levels. The best known examples are the classical, two-dimensional equations of vibration of thin plates. The Germain-Lagrange-Cauchy flexural equations and the Poisson-Cauchy extensional equations contain only the lowest modes of an infinite plate and describe adequately the frequency spectra of finite plates as long as the frequency is well below that of the lowest absent mode.

The extension of the two-dimensional equations to accommodate higher frequencies involves the incorporation of the next higher modes of an infinite plate. The first steps in this direction have led to applications (see the Appendix) of interest in the field of frequency control. As a result, an advisory committee, composed of E. A. Gerber, R. A. Sykes and K. S. Van Dyke, recommended to the Office of the Chief Signal Officer that a monograph be prepared in which the derivation of the new equations might be explained in detail.

It turned out that a reasonably full exposition of the foundations of the theory of vibrations of plates required preliminary studies in areas hitherto unexplored. These investigations have proved so fruitful and so necessary to an understanding of the theory that they comprise the major part of the monograph. The equations which formed the basis of the previous applications now appear as almost trivial special cases of more general equations covering a much wider field of applications.

Two-dimensional equations are, of course, approximations. The first question that arises in attempting to explain the development of the equations is "What is it that is being approximated?" The point of view adopted here is that the relation between frequency and wave-length is the element of prime importance. Accordingly, it is the aim to produce equations which will yield the appropriate relation, at least in a limited range. Now, the appropriate relation is the one given by the three-dimensional equations and, if these could be solved for all shapes and boundary conditions, there would be no need for approximations. Fortunately, it is only essential to have available the solution for an infinite plate. For the isotropic material this was given by Rayleigh in 1889.

Most of Chapter 2 is devoted to the study of Rayleigh's problem. Of particular interest is the method adopted to lead up to the final solution via the simpler problem of mixed boundary conditions. This enables the prediction of some of the major features of the complicated, coupled spectrum without detailed analysis or computations. Rayleigh's final transcendental equations are deceivingly simple in appearance. After more than sixty years during which the roots in various regions had been studied extensively (notably by Lamb in 1917 and Holden in 1951) there still remained unexplored the part of the spectrum of greatest importance for the development of two-dimensional equations. This part (the high-frequency, long wave-length region) is analyzed in detail in Section 2.11.

Analogous to Rayleigh's solution, there is one by Ekstein (in 1945) for the monoclinic crystal, which has important applications to the rotated Y-cuts of quartz. In view of the length of time it has taken to gain an understanding of Rayleigh's problem it is not surprising that Ekstein's solution of a much more complicated problem has not yet been explored fully. Inasmuch as a large portion of the monograph is devoted to crystal plates, it would have been appropriate to describe Ekstein's problem in as great detail as Rayleigh's. However, there was not enough time available to complete the work. Consequently, a test of the first-order approximation for crystals, analogous to the test of the equations for isotropic plates at the end of Chapter 5, had to be postponed.

As early as 1828, Poisson and Cauchy had already deduced the two-dimensional equations of low-frequency flexural and extensional vibrations of plates from the three-dimensional equations. They started with full expansions in infinite series of powers of the thickness-coordinate and then exercised great ingenuity in discarding higher powers and combining what was left so as to reach the desired equations. They had before them the ingredients of the higher order equations, but there was no interest in high-frequency vibrations at that time. Not long afterwards, Kirchhoff introduced energy considerations and integral theorems into the theory of plates; but he, too, was interested only in low frequencies and included just enough terms of the series for his immediate purpose. He did settle, though, the difficult question of boundary conditions.

In recent years there has been much interest in higher order theories of plates. Some of the developments have been based on analogy with the Bress-Timoshenko theory of high-frequency, flexural vibrations of bars, while others have been based on more fundamental energetic considerations. The earliest of the latter is E. Reissner's theory of flexure which appeared in 1945. Since that time there have been produced about as many ways of deriving higher order theories of bars, plates and shells as there have been authors to write about them. My own preference is to return to Poisson, Cauchy and Kirchhoff and pick up where they left off.

In Chapter 3 the three-dimensional equations of elasticity are converted to an infinite series of two-dimensional equations. This is done by expanding the displacement, in the variational equation of motion, in an infinite series of powers of the thickness-coordinate of the plate and integrating through the thickness. It seems to me that this is what Cauchy and Poisson might have done if they had had available what Kirchhoff was to develop later. The infinite series of equations are, of course, no easier to handle than the original equations; but the series-expressions of displacement, strain, stress, energy and equations of motion are of aid in deciding what to include in various orders of approximation and in understanding the implications of what is discarded and what retained.

The remaining three chapters deal with a variety of truncations of the series-expressions. In each case an explanation is given of how and why the truncation is performed. Comparisons are made between the resulting predictions of frequency vs. wave-length in an infinite plate and the corresponding relations obtained from the three-dimensional theory, where the latter are available. These comparisons serve to delineate the ranges of usefulness of the approximate equations. In many cases the estimates have been confirmed by comparison of particular solutions with experiments. The details are described in the papers listed in the Appendix.

The display of the infinite-series expansion, in Chapter 3, could lead the reader to expect that the subsequent process of truncation might be executed with some semblance of mathematical rigor. This is not the case. The series-expressions serve only to exhibit the branches which must be pruned. Where the cuts are made depends on physical considerations and on intuition based, in turn, on an understanding of the solution of the three-dimensional equations for the case of the infinite plate. However, when a truncation is completed, the kinetic energy-density and the strain-energy-density are formulated and a theorem of uniqueness of solutions of the approximate equations is established. Thus, each order of approximation is finally expressed as a self-sufficient mathematical system.

December 1955

R. D. Mindlin