Chapter 1

Fundamental Concepts and Basic Results

1.1 The Königsberg bridge problem

In an old city of Eastern Prussia, named *Königsberg*, there was a river, called River Pregel, flowing through its centre. In the 18^{th} century, there were seven bridges over the river connecting the two islands (B and D) and two opposite banks (A and C) as shown in Figure 1.1.

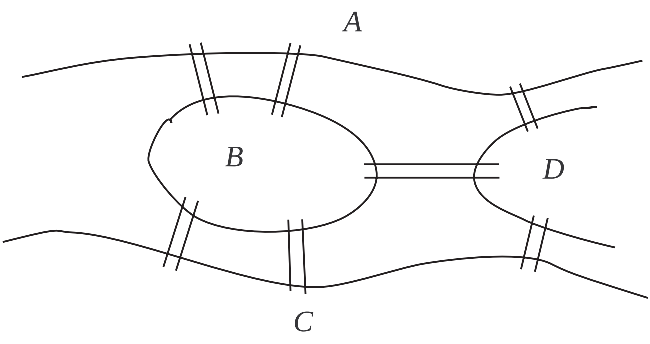


Figure 1.1

It was said that the people in the city had always amused themselves with the following problem:

> *Starting with any one of the four places A, B, C or D as shown in Figure 1.1, is it possible to have a walk which passes through each of the seven bridges once and only once, and return to where you started?*

No one could find such a walk; and after a number of tries, people believed that it is simply not possible, but no one could prove it either.

Leonhard Euler, the greatest mathematician that Switzerland has ever produced, was told of the problem. He noticed that the problem was very much different in nature from the problems in traditional geometry, and instead of considering the original problem, he studied its much more general version which encompassed any number of islands or banks, and any number of bridges connecting them. His finding was contained in the article [E] (the English translation of its title is: *The solution of a problem to the geometry of position*) published in 1736. As a direct consequence of his finding, he deduced the impossibility of having such a walk in the Königsberg bridge problem. This was historically the first time a proof was given from the mathematical point of view.

How did Euler generalize the Königsberg bridge problem? How did he solve his more general problem? What is his finding?

1.2 Multigraphs and graphs

Euler observed that the Königsberg bridge problem had nothing to do with traditional geometry where the measurements of lengths and angles, and relative locations of vertices count. How large the islands and banks are, how long the bridges are, and whether an island is at the south or north of a bank are immaterial. The key ingredients are whether the islands or banks are connected by a bridge, and by how many bridges.

Euler's idea was essentially as follows: represent the islands or banks by '**dots**', one for each island or bank, and two dots are joined by k '**lines**' (not necessarily straight), where $k \geq 0$, when and only when the respective islands or banks represented by the dots are connected by k bridges. Thus the situation for the Königsberg bridge problem is represented by the diagram in Figure 1.2.

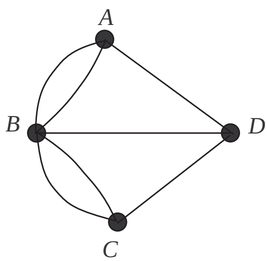


Figure 1.2

The diagram in Figure 1.2 is now known as a **multigraph**. *Intuitively*, a **multigraph** is a diagram consisting of 'dots' and 'lines', where each line joins some pair of dots, and two dots may be joined by no lines or any number of lines. More formally, we call a 'dot' a **vertex** (plural, vertices) and call a 'line' an **edge**.

For instance, in the multigraph of Figure 1.2, there are four vertices and seven edges, where each edge joins some pair of vertices; vertices A and C are not joined by any edges, A and D are joined by one edge, and B and C are joined by two edges, etc.

Note that the sizes and the relative locations of dots (vertices), and the lengths of the lines (edges) are immaterial. Only the 'linking relations' among the vertices and the number of edges that join two vertices count. Thus, the situation for the Königsberg bridge problem can equally well be represented by the multigraph of Figure 1.3.

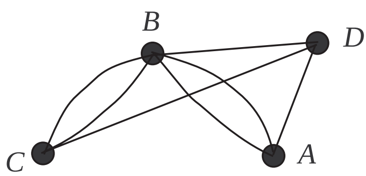


Figure 1.3

Let us give more examples of multigraph which represent certain situations in different nature.

Example 1.2.1. *There were six people: A, B, C, D, E and F in a party and several handshakes among them took place. Suppose that*

A shook hands with B, C, D, E and F,
B, in addition, shook hands with C and F,
C, in addition, shook hands with D and E,
D, in addition, shook hand with E,
E, in addition, shook hand with F.

This situation can be clearly shown by the multigraph in Figure 1.4, where people are represented by vertices and two vertices are joined by an edge whenever the corresponding persons shook hands.

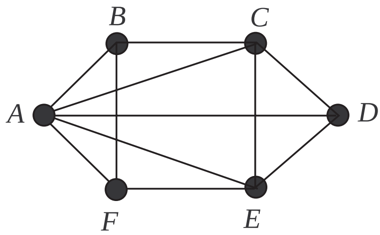


Figure 1.4

Example 1.2.2. *The diagram in Figure 1.5 is a multigraph which shows the availability of flights operated by an airline company between a number of cities. The vertices represent the cities, and two vertices are joined by an edge if there is a flight available between the two corresponding cities.*

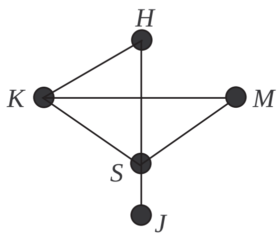


Figure 1.5

Example 1.2.3. *The diagram in Figure 1.6 is a multigraph which models a job-application situation. The vertices are divided into two parts:* X *and* Y*, where the vertices in* X *represent the applicants, while those in* Y *represent the jobs available. A vertex in* X *is joined to a vertex in* Y *by an edge if the corresponding applicant applies for the corresponding job.*

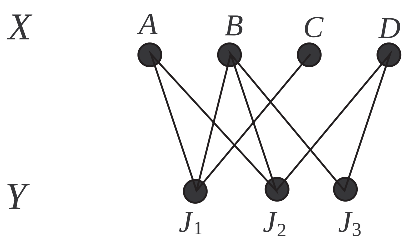


Figure 1.6

Question 1.2.1. *Give three examples from our everyday life where the situations can be modeled by multigraphs.*

It is noted that in the three multigraphs shown in Figures 1.4 - 1.6, every two vertices are joined by at most one edge (that is, either no edges or exactly one edge). These situations are different from the multigraph in Figure 1.2 (or Figure 1.3) where there are vertices joined by more than one edge. To distinguish them, we call the diagrams in Figures 1.4 - 1.6 **simple graphs**, or simply, **graphs**. Thus the diagram in Figure 1.2 (or Figure 1.3) is a multigraph, but not a (simple) graph.

Let us consider another example.

Example 1.2.4. *In the diagram shown in Figure 1.7, there are*

- *four vertices:* u, v, w *and* z*, and*
- *eight edges:* f_1 *and* f_2 *joining* u *and* v*;* e_1, e_2 *and* e_3 *joining* w *and* z*;* h_1 *joining* v *and* w*;* h_2 *joining* u *and* w*;* h_3 *joining* v *to itself.*

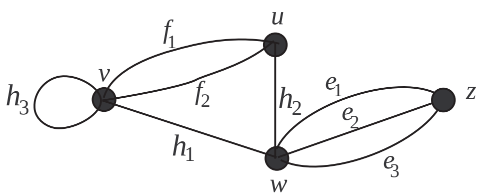


Figure 1.7

Two or more edges joining the same pair of vertices are called **parallel edges**. Thus, in Figure 1.7, f_1 and f_2 are parallel edges; e_1, e_2 and e_3 are parallel edges.

Any edge joining a vertex to itself is called a **loop**. Thus, in Figure 1.7, h_3 is a loop.

Remarks. (1) In this book, we shall not consider 'loops' in any diagram of vertices and edges unless otherwise stated. A diagram with the existence of parallel edges is **not** a **(simple) graph**. Another example of a multigraph which is **not** a (simple) graph is shown in Figure 1.8.

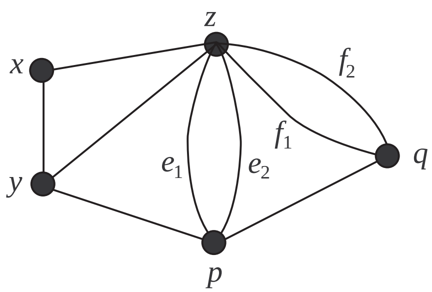


Figure 1.8

(2) Bear in mind that a 'graph' or a 'multigraph' in Graph Theory is not a geometrical figure. Thus we do not consider

- the size of a 'dot',
- the location of a vertex, and
- the shape of an edge.

(3) When there is only one edge joining a pair of vertices, say a and b, we may denote this edge by ab. For example, the edge h_1 in Figure 1.7 can also be denoted by vw.

We now give formal definitions of a 'graph' and 'multigraph'.

A **multigraph** G consists of a non-empty finite set $V(G)$ of vertices together with a finite set $E(G)$ (possibly empty) of edges such that

(1) each edge joins two distinct vertices in $V(G)$ and

(2) any two distinct vertices in $V(G)$ are joined by a finite number (including zero) of edges.

The sets $V(G)$ and $E(G)$ are called the **vertex set** and the **edge set** of G respectively.

The number of vertices in G, denoted by $v(G)$, is called the **order** of G (thus $v(G) = |V(G)|$). The number of edges in G, denoted by $e(G)$, is called the **size** of G (thus $e(G) = |E(G)|$).

A multigraph G is called a (simple) **graph** if any two vertices in $V(G)$ are joined by at most one edge (that is, either they are not joined by an edge or joined by exactly one edge).

(a) It follows from the above definitions that

(i) every graph is a multigraph but not vice versa and

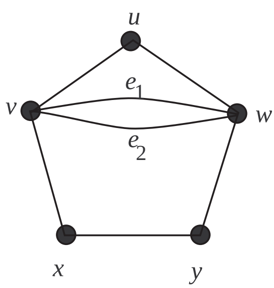
(ii) no loops are allowed in any multigraph.

When a concept is defined or a statement is made for multigraphs, they are also valid, in particular, for graphs.

(b) If e is the only edge joining two vertices u and v, then we may write $e = uv$ or $e = vu$. The ordering of u and v in the expression is immaterial.

Example 1.2.5. *Let G be the multigraph shown in Figure 1.8. Then*
$V(G) = \{x, y, z, p, q\}$,
$E(G) = \{xy, xz, yz, yp, e_1, e_2, f_1, f_2, pq\}$,
$v(G) = 5$ *and* $e(G) = 9$.
Let H be the graph shown in Figure 1.4. Then
$V(H) = \{A, B, C, D, E, F\}$,
$E(H) = \{AB, AC, AD, AE, AF, BC, BF, CD, CE, DE, EF\}$,
$v(H) = 6$ *and* $e(H) = 11$.

Question 1.2.2. *Let G be the multigraph shown below. Find $V(G), E(G), v(G)$ and $e(G)$.*



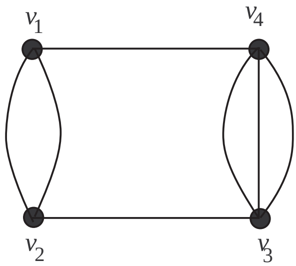
Question 1.2.3. *Let H be the graph with $V(H) = \{a, b, c, x, y, z\}$ and $E(H) = \{ab, ay, bx, by, cx, cz, xz, yz\}$. Find $v(H)$ and $e(H)$, and draw a*

diagram of H.

Matrices and multigraphs

As discussed earlier, a multigraph G can be represented by a diagram consisting of 'dots' and 'lines', and can be defined in terms of its vertex set $V(G)$ and edge set $E(G)$. Multigraphs can also be represented by matrices in various ways. In what follows, we introduce one of them.

Example 1.2.6. *Let G be the multigraph shown below, where its four vertices are named as v_1, v_2, v_3 and v_4.*



Consider also the following 4×4 matrix A:

$$A = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}$$

Can you find any relation between G and A?

What is the value of the $(1,2)$-entry in A? It is '2'. How many edges in G joining v_1 and v_2? There are '2' also.

How many edges in G joining v_3 and v_4? There are '3'. What is the value of the $(3,4)$-entry in A? It is '3' also.

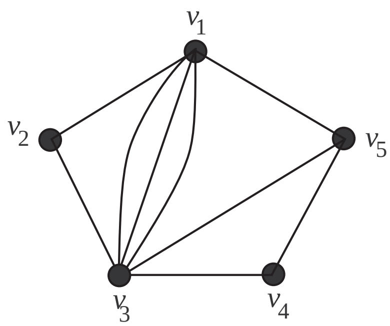
Indeed, it is observed that the value of the (i,j)-entry in A is the number of edges in G joining v_i and v_j, where $i, j \in \{1,2,3,4\}$. Note that the value of each (i,i)-entry (that is, a diagonal entry) in A is '0' as there is no edge in G joining v_i to itself. We call A the **adjacency matrix** of G. Two vertices are adjacent if they are joined by an edge. Evidently, the matrix is dependent on the labelling of the vertices.

Let G be a multigraph of order n with $V(G) = \{v_1, v_2, \cdots, v_n\}$. The **adjacency matrix** of G is the $n \times n$ matrix

$$A(G) = (a_{i,j})_{n\times n},$$

where $a_{i,j}$, the (i,j)-entry in $A(G)$, is the number of edges joining v_i and v_j for all $i, j \in \{1, 2, \cdots, n\}$.

Question 1.2.4. *Let G be the multigraph shown below.*



(i) *Find $A(G)$.*

(ii) *Is $A(G)$ symmetric (i.e., (i,j)-entry $=$ (j,i)-entry)?*

(iii) *What is the sum of the values of the entries in each row (respectively, column)?*

(iv) *What is your interpretation of the 'sum' obtained in (iii)?*

Question 1.2.5. *The adjacency matrix of a multigraph G is given below:*

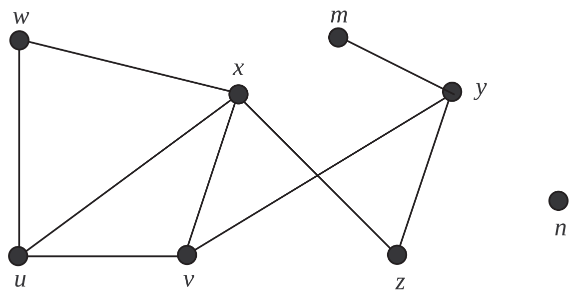
$$A = \begin{pmatrix} 0 & 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \end{pmatrix}$$

Draw a diagram of G.

Remark. There are many ways of storing multigraphs in computers. The use of the adjacency matrices is, perhaps, one of the most common and convenient ways.

Exercise 1.2

(1) Let G be the multigraph representing the following diagram. Determine $V(G)$, $E(G)$, $v(G)$ and $e(G)$. Is G a simple graph?



(2) Draw the graph G modeling the flight connectivity between twelve capital cities with the following vertex set $V(G)$ and edge set $E(G)$.

$V(G) =$ {Asuncion, Beijing, Canberra, Dili, Havana, Kuala Lumpur, London, Nairobi, Phnom Penh, Singapore, Wellington, Zagreb}.

$E(G) =$ {Asuncion-London, Asuncion-Havana, Beijing-Canberra, Beijing-Kuala Lumpur, Beijing-London, Beijing-Singapore, Beijing-Phnom Penh, Dili-Kuala Lumpur, Dili-Singapore, Dili-Canberra, Havana-London, London-Wellington, Kuala Lumpur-London, Kuala Lumpur-Phnom Penh, Kuala Lumpur-Singapore, Kuala Lumpur-Wellington, London-Nairobi, Phnom Penh-Singapore, London-Singapore, London-Zagreb, Singapore-Wellington, Havana-Nairobi}.

(Note that your may use A to represent Asuncion, B to represent Beijing, C to represent Canberra, etc.)

(3) Define a graph G such that $V(G) = \{2, 3, 4, 5, 11, 12, 13, 14\}$ and two vertices s and t are adjacent if and only if $gcd\{s, t\} = 1$. Draw a diagram of G and find its size $e(G)$.

(4) The diagram in page 12 is a map of the road system in a town. Draw a multigraph to model the road system, using a vertex to represent a junction and an edge to represent a road joining two junctions.

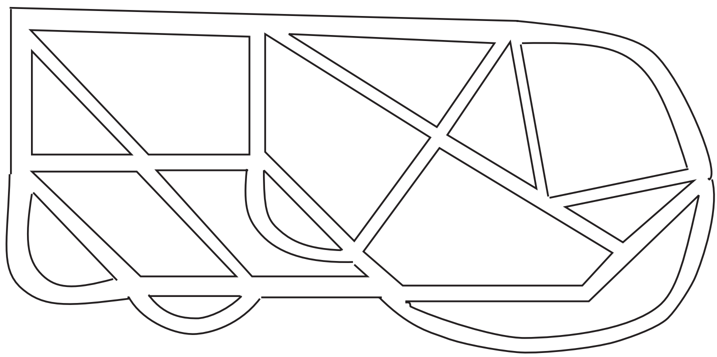


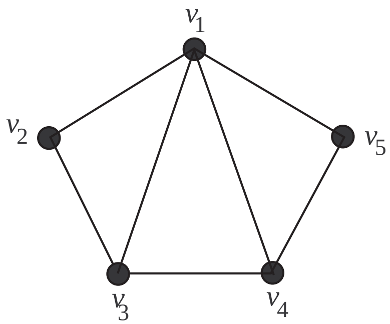
Diagram for Problem 4

(5) Let G be a graph with $V(G) = \{1, 2, \cdots, 10\}$, such that two numbers i and j in $V(G)$ are adjacent if and only if $|i - j| \leq 3$. Draw the graph G and determine $e(G)$.

(6) Let G be a graph with $V(G) = \{1, 2, \cdots, 10\}$, such that two numbers i and j in $V(G)$ are adjacent if and only if $i + j$ is a multiple of 4. Draw the graph G and determine $e(G)$.

(7) Let G be a graph with $V(G) = \{1, 2, \cdots, 10\}$, such that two numbers i and j in $V(G)$ are adjacent if and only if $i \times j$ is a multiple of 10. Draw the graph G and determine $e(G)$.

(8) Find the adjacency matrix of the following graph G.



(9) The adjacency matrix of a multigraph G is shown below:

$$\begin{pmatrix} 0 & 1 & 0 & 2 & 3 \\ 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 & 0 \end{pmatrix}$$

Draw a diagram of G.

(10) Four teams of three specialist soldiers each (a scout, a signaler and a sniper) are to be sent into enemy territory. However, some of the soldiers cannot work well with some others. The following table shows the soldiers, their specializations and who they cannot work with.

Soldier	Specialization	Cannot cooperate with
1	Scout	$5, 7, 10$
2	Scout	–
3	Scout	$5, 6, 8, 9, 11$
4	Scout	$8, 12$
5	Signaler	$1, 3, 9$
6	Signaler	$3, 10, 11$
7	Signaler	$1, 9, 12$
8	Signaler	$3, 4, 9, 10$
9	Sniper	$3, 5, 7, 8$
10	Sniper	$1, 6, 8$
11	Sniper	$3, 6$
12	Sniper	$4, 7$

(i) Draw a multigraph to model the situation so that we may see how to form 3-man teams such that each specialization is represented and every member of the team can work with every other. State clearly what the vertices represent and under what condition(s) two vertices are joined by an edge.

(ii) Can you form four 3-man teams such that each specialization is represented and all members of the team can work with one another?

1.3 Vertex degrees

Let G be a multigraph.

Two vertices u and v in G are said to be **adjacent** if they are joined by an edge, say, e in G. In the case when e is the only edge joining u and v, we also write $e = uv$, and we say that
(1) u is a **neighbour** of v and vice versa,
(2) the edge e is **incident with** the vertex u (and v) and
(3) u and v are the two **ends** of e.

The set of all neighbours of v in G is denoted by $N(v)$; that is,

$$N(v) = \{x | x \text{ is a neighbour of } v\}.$$

Example 1.3.1. *Let G be the multigraph shown in Figure 1.9.*

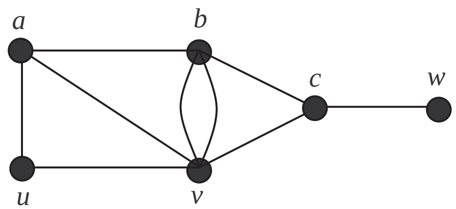


Figure 1.9

Then
(1) the vertices a and b are adjacent, so are b and v, but not a and c;
(2) the vertices a and u are the two ends of the edge au;
(3) the edge av is incident with the vertices a and v;
(4) the vertex a has three neighbours, namely, b, u and v; and
(5) $N(a) = \{b, u, v\}$, $N(b) = \{a, v, c\}$, $N(w) = \{c\}$, etc.

Let G be a multigraph. We now introduce a very useful and important number associated with each vertex in G.

> Given a vertex v in G, the **degree** of v in G, denoted by $d_G(v)$, is defined as the number of edges incident with v.

For simplicity, we shall replace $d_G(v)$ simply by $d(v)$ if there is no danger of confusion.

Question 1.3.1.

(i) Find the degree of each vertex in G of Figure 1.9.

(ii) Find $N(x)$ for each vertex x in G of Figure 1.9.

(iii) By definition, is it true that $d(v) = |N(v)|$?

Example 1.3.2. *Let G be the multigraph of Figure 1.10.*

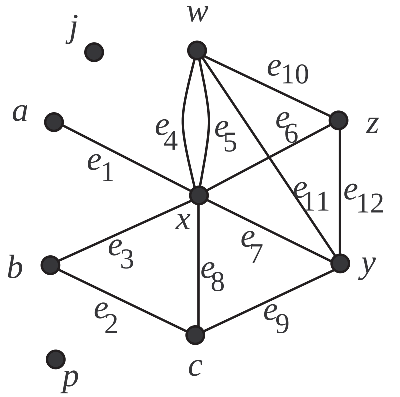


Figure 1.10

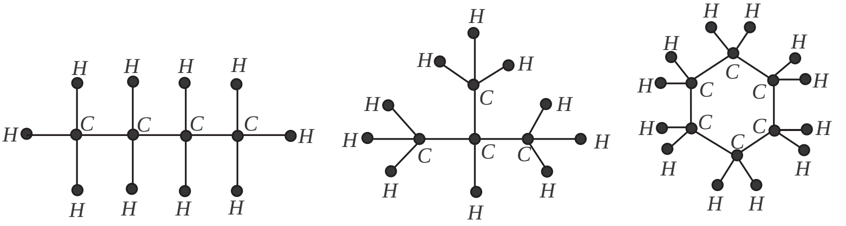
Observe that there are seven edges incident with the vertex x. Thus, $d(x) = 7$. There are no edges incident with the vertex j. Thus, $d(j) = 0$. The degrees of the vertices in G are shown in Table 1.11.

Vertex	*a*	*b*	*c*	*j*	*p*	*w*	*x*	*y*	*z*
Degree	1	2	3	0	0	4	7	4	3

Table 1.11

The degree of a vertex is also called the **valency** of the vertex as it is related to the valency of an atom in chemical compounds as shown in

Figure 1.12.



Butane and Isobutane, C_4H_{10} Cyclohexane, C_6H_{12}

Figure 1.12

Two types of vertices having smallest degrees have special names.

A vertex v is called an **isolated-vertex** if $d(v) = 0$; it is called an **end-vertex** if $d(v) = 1$.

Thus, in the multigraph of Figure 1.10, the vertices j and p are isolated-vertices and the vertex a is an end-vertex.

Remark. An *end-vertex* and an *end* of an edge (see page 14) are two different concepts. While an *end-vertex* is a vertex of degree one, an *end* of an edge has nothing to do with its degree.

In the multigraph G of Figure 1.10, the total sum of the degrees of its vertices, as can be seen from Table 1.11, is 24. What is the size of G? The answer is: $e(G) = 12$. Observe that the total sum 24 is double the size 12 of G. Is this a coincidence?

Question 1.3.2. *Consider the multigraph G of Figure 1.9. Find $e(G)$ and the sum of the degrees of the six vertices. Is the sum twice of $e(G)$?*

In general, is the sum of the degrees of the vertices in a multigraph always double its size?

There is a Chinese saying: *whenever you drink water, think of its source.* Where do the *degrees* of the vertices come from? The answer is: the *existence* of '*edges*'. No edges implies no degrees. How many degrees can each edge contribute? The answer is '2' as an edge is incident with its two ends. *To compute the sum of degrees of vertices, we count each edge twice, once for each end of the edge.* Thus, we have the following result due to Euler [E]:

Theorem 1.1 Let G be a multigraph with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Then

$$\sum_{i=1}^{n} d(v_i) = 2e(G).$$

Remarks. (1) If the vertices of G in Theorem 1.1 are not named as shown, the result can also be expressed as

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

(2) A group of $n(\geq 2)$ persons were together and various handshakes took place among them (two persons might even shake hands more than once). Each person recorded the number of handshakes he or she had shaken. The sum of these n numbers would be double the number of handshakes that took place in the gathering. Theorem 1.1 is, therefore, also known as the **Handshaking Lemma**.

An important way to classify the vertices of a multigraph G is by means of the **parity** (i.e., being even or odd) of their degrees.

> A vertex w in G is said to be **even** if $d(w)$ is even; and said to be **odd** if $d(w)$ is odd.

Thus, in the multigraph G of Figure 1.10, there are five even vertices: b, j, p, w and y; and four odd vertices: a, c, x and z.

Question 1.3.3. *(1) How many odd vertices are there in each of the multigraphs shown in the previous examples?*
(2) Can you construct a multigraph containing (i) exactly one odd vertex? (ii) exactly three odd vertices?

Instead of merely considering the multigraph of Figure 1.2, which represents the Königsberg bridge problem, Euler [E] studied a much more general problem: *Let G be a multigraph. Suppose that one starts with an arbitrary vertex in G, and finds that it is possible to have a walk which passes through each edge exactly once and then be able to end at the starting vertex. What can be said about such a multigraph?*

Evidently, in order to have such a walk one must be able to enter and exit a vertex; hence each vertex must be even. Is the converse true? We will discuss this in Chapter 6.

In order to study this problem and its related issues, Euler introduced the notion of 'odd vertices' and determined the parity of the number of 'odd vertices'. To get to this, Euler first established Theorem 1.1, and then deduced from it the following consequence:

Corollary 1.2 The number of odd vertices in any multigraph is even.

Proof. Let G be a multigraph. Let A be the set of odd vertices in G, and B be the set of even vertices in G. Our aim is to show that $|A|$ is even. Indeed, as $V(G) = A \cup B$ and by Theorem 1.1, we have

$$\sum_{v \in A} d(v) + \sum_{v \in B} d(v) = \sum_{v \in V(G)} d(v) = 2e(G).$$

Since $\sum_{v \in B} d(v)$ and $2e(G)$ are even, $\sum_{v \in A} d(v)$ is also even. As $d(v)$ is odd for each v in A, it follows that $|A|$ must be even, as required. □

Let us proceed to introduce two useful quantities pertaining to the degrees of vertices of a multigraph.

Let G be a multigraph. The **maximum degree** of G, denoted by $\Delta(G)$, is defined as the **maximum number** among all vertex degrees in G.

Likewise, the **minimum degree** of G, denoted by $\delta(G)$, is defined as the **minimum number** among all vertex degrees in G.

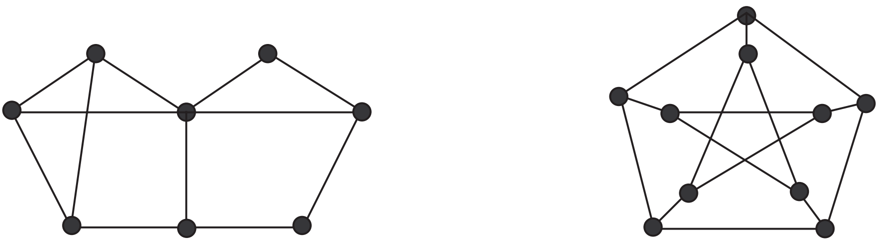
That is,

$$\Delta(G) = \max\{d(v) | v \in V(G)\} \text{ and}$$

$$\delta(G) = \min\{d(v) | v \in V(G)\}.$$

Thus, in the multigraph G of Figure 1.10, $\Delta(G) = 7$ and $\delta(G) = 0$.

Example 1.3.3. *Consider the graphs G and H shown in Figure 1.13 (a) and (b) respectively.*



(a) G (b) H

Figure 1.13

It can be checked that $\Delta(G) = 5$ while $\delta(G) = 2$, and $\Delta(H) = \delta(H) = 3$.

Remark. The graph H shown in Figure 1.13(b) is a famous graph, known as the **Petersen graph**. It was named after Julius Petersen (1839-1910), a Danish mathematician, who discussed the graph in the paper [P].

Notice that in the Petersen graph H, every vertex has the same degree, namely '3'. There are many graphs in which every vertex has the same degree. We now single out this special family of graphs by giving these graphs a name.

> A graph G is said to be **regular** if every vertex in G has the same degree. More precisely, G is said to be k-**regular** if $d(v) = k$ for each vertex v in G, where $k \geq 0$.

Thus, a graph G is k-regular if and only if $\Delta(G) = \delta(G) = k$. Note that the Petersen graph is 3-regular. For $k = 0, 1, 2, 3$ and 4, a k-regular graph is shown in Figure 1.14.

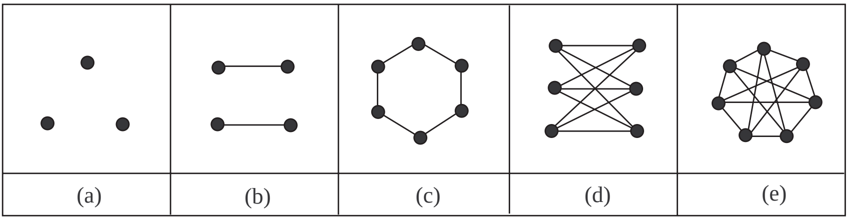


Figure 1.14

Remark. A 3-regular graph is also called a **cubic graph**.

Question 1.3.4. *Construct a* 5*-regular graph of order* 10*. What is its size?*

To end this section, we introduce three important families of regular graphs: the '*null graphs*', '*complete graphs*' and '*cycles*'.

(1) **The null graphs**

By the definition of a graph G, the vertex set $V(G)$ is never empty, but its edge set $E(G)$ may be empty. The graph (a) in Figure 1.14 is an example.

> A graph G is called a **null graph** (or **empty graph**) if $E(G)$ is empty. A null graph of order n is denoted by N_n.

Clearly, each N_n is a 0-regular graph and $e(N_n) = 0$. The five smallest null graphs are shown in Figure 1.15.

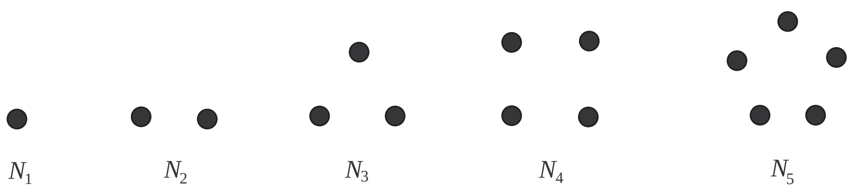


Figure 1.15

Among the (simple) graphs G of a fixed order n, at one extreme, the null graph N_n contains no edges (the least possible size). At the other extreme, we may ask:

Question 1.3.5. *What is the largest possible size that G can have? Which graph has its size attaining this largest possible number?*

(2) **The complete graphs**

While a null graph is one in which no two vertices are adjacent, a graph is called a **complete graph** if every two of its vertices are adjacent. A complete graph of order n is denoted by K_n.

Clearly, each K_n is $(n-1)$-regular and $e(K_n) = \binom{n}{2}$. The first five smallest complete graphs are shown in Figure 1.16.

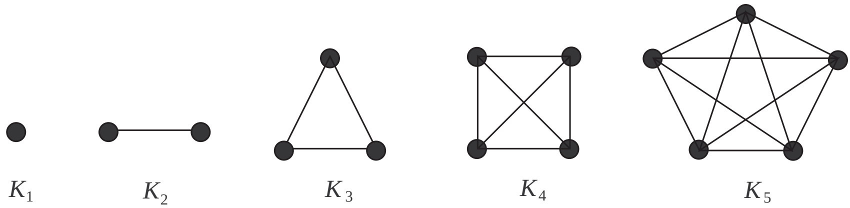


Figure 1.16

(3) **The cycles**

The 2-regular graph shown in Figure 1.14(c) is called a **cycle**.

A graph G of order $n \geq 3$ is called a **cycle** if its n vertices can be named as $v_1, v_2, \cdots, v_n$ such that v_1 is adjacent to v_2, v_2 is adjacent to v_3, $\cdots$, v_{n-1} is adjacent to v_n, v_n is adjacent to v_1, and no other adjacency exists; that is,

$$V(G) = \{v_1, v_2, \cdots, v_n\} \text{ and}$$
$$E(G) = \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n, v_nv_1\}.$$

A cycle of order n is denoted by C_n. We call C_n an n-**cycle**, and C_3 a **triangle**.

Clearly, every cycle is 2-regular and $e(C_n) = v(C_n) = n$. Three more examples of cycles are shown in Figure 1.17.

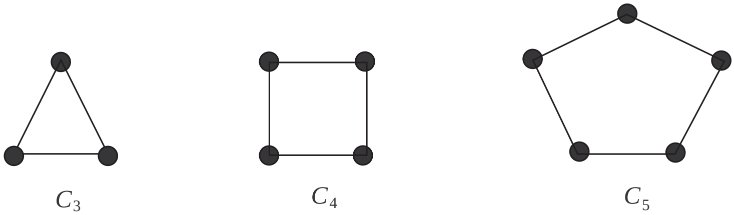


Figure 1.17

Remark. The graph C_n is defined for $n \geq 3$. For $n = 2$, as shown in Figure 1.18, C_2 is also called a cycle, but it is not a graph (it is a multigraph).

C_2:

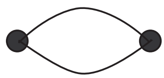
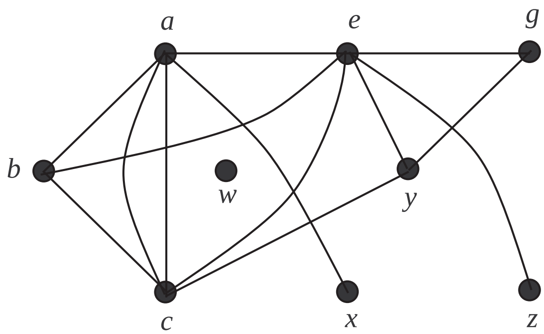


Figure 1.18

Exercise 1.3

(1) In the following multigraph G, find

- (i) the size of G,
- (ii) the degree of each vertex,
- (iii) the sum $\sum\{d(v)|v \in V(G)\}$,
- (iv) the number of odd vertices,
- (v) $\Delta(G)$, and
- (vi) $\delta(G)$.



Is your answer for (iii) double your answer for (i)? Is your answer for (iv) an even number?

(2) Construct a multigraph of order 6 and size 7 in which every vertex is odd.

(3) Let G be a multigraph with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Prove that the sum of all the entries in the ith row of the adjacency matrix $A(G)$ is the degree of the vertex v_i for each $i = 1, 2, \cdots, n$.

(4) Let G be a graph of order 8 and size 15 in which each vertex is of degree 3 or 5. How many vertices of degree 5 does G have? Construct one such graph G.

(5) Let H be a graph of order 10 such that $3 \leq d(v) \leq 5$ for each vertex v in H. Not every vertex is even. No two odd vertices are of the same degree. What is the size of H?

(6) Let G be a graph of order 14 and size 30 in which every vertex is of degree 4 or 5. How many vertices of degree 5 does G have? Construct

one such graph G.

(7) Does there exist a multigraph G of order 8 such that $\delta(G) = 0$ while $\Delta(G) = 7$? What if 'multigraph G' is replaced by 'graph G'?

(8) Characterize the 1-regular graphs.

(9) Draw all regular graphs of order n, where $2 \leq n \leq 6$.

(10) (i) Does there exist a graph G of order 5 such that $\delta(G) = 1$ and $\Delta(G) = 4$?
(ii) Does there exist a graph G of order 5 which has two vertices of degree 4 and $\delta(G) = 1$?

(11) Let H be a graph of order 8 and size 13 with $\delta(H) = 2$ and $\Delta(H) = 4$. Denote by n_i the number of vertices in H of degree i, where $i = 2, 3, 4$. Assume that $n_3 \geq 1$. Find all possible answers for (n_2, n_3, n_4). For each of your answers, construct a corresponding graph.

(12) Suppose G is a k-regular graph of order n and size m, where $k \geq 0$, $m \geq 0$ and $n \geq 1$. Find a relation linking k, n and m. Justify your answer.

(13) Does there exist a 3-regular graph with eight vertices? Does there exist a 3-regular graph with nine vertices?

(14) Construct a cubic graph of order 12. What is its size? Does there exist a cubic graph of order 11? Why?

(15) Let H be a k-regular graph of order n. If $e(H) = 10$, find all possible values for k and n; and for each case, construct one such graph H.

(16) (+) Let G be a 3-regular graph with $e(G) = 2v(G) - 3$. Determine the values of $v(G)$ and $e(G)$. Construct all such graphs G.

(17) Find all integers n such that $100 \leq e(K_n) \leq 200$.

(18) (+) Let G be a multigraph of order 13 in which each vertex is of degree 7 or 8. Show that G contains **at least eight** vertices of degree 7 or **at least seven vertices** of degree 8.

(19) (+) Let G be a graph of order n in which there exist **no** three vertices u, v and w such that uv, vw and wu are all edges in G. Show that $n \geq \delta(G) + \Delta(G)$.

(20) (+) There were n (≥ 2) persons at a party and, as usually happens, some shake hands with others. No one shook hands with the same

person more than once. Show that there are at least two persons in the party who had the same number of handshakes.

(21) The preceding problem says that in any graph of order $n \geq 2$, there exist two vertices having the same degree. Is the result still valid for multigraphs?

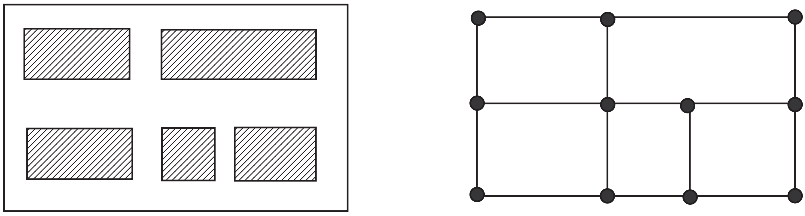
(22) (+) Mr. and Mrs. Samy attended an exclusive party where in addition to themselves, there were only another 3 couples. As usually happens, some shake hands with others. No one shook hands with the same person more than once and no one shook hands with his/her spouse. After all the handshakes had been done, Mr. Samy asked each person, including his wife, how many hands he/she had shaken. To everyone's amusement, each one gave a different answer. How many hands did Mrs. Samy shake?

(23) (+) In the preceding problem, there were four couples altogether in a party. Solve the general problem where 'four couples' is replaced by '$n(\geq 2)$ couples'.

(24) ($*$) There are $n \geq 2$ distinct points in the plane such that the distance between any 2 points is at least one. Prove that there are at most $3n$ pairs of these points at distance exactly one.

1.4 Paths, cycles and connectedness

Figure 1.19(a) shows a section of the street system of a town. It can be modeled as a graph as shown in Figure 1.19(b), where a vertex represents a junction and two vertices are joined by an edge if and only if the corresponding junctions are linked by a street. For certain purposes, we may have to traverse the street system by passing through some junctions and streets. In order to show more precisely and succinctly the way we traverse, in this section, we shall introduce some basic terms in general multigraphs which serve the purpose.



(a) (b)

Figure 1.19

Consider the multigraph H of Figure 1.20. If we start at vertex a, then we can reach vertex x via the edge f_1, and from x to y via the edge f_7. We can further proceed to reach z via f_{12}. This process can be conveniently expressed by the following alternating sequence of vertices and edges:

$$a\ f_1\ x\ f_7\ y\ f_{12}\ z.$$

Such a sequence is called a **walk** or, more precisely, an $a - z$ walk as a and z are, respectively, the **initial** and **terminal** vertices of the walk.

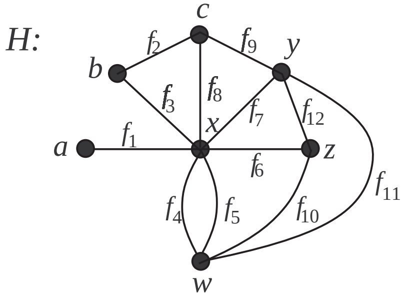


Figure 1.20

A **walk** in a multigraph G is an alternating sequence of vertices and edges beginning and ending at vertices:

$$(\#) \qquad v_0 e_0 v_1 e_1 v_2 \cdots v_{k-1} e_{k-1} v_k,$$

where $k \geq 1$ and e_i is incident with v_i and v_{i+1}, for each $i = 0, 1, \cdots, k-1$. The walk (#) is also called a $v_0 - v_k$ **walk** with its **initial vertex** v_0 and **terminal vertex** v_k. The **length** of the walk (#) is defined as 'k', which is the number of occurrences of edges in the sequence.

Remark. The vertices v_i's or edges e_i's in (#) need not be distinct.

Question 1.4.1. *Is the following sequence:*

$$cf_9yf_{11}wf_1b$$

a walk in H of Figure 1.20? Why?

Example 1.4.1. *Some walks in H of Figure 1.20 and their respective lengths are shown in Table 1.21.*

	sequence	walk	length
(1)	$bf_3xf_4wf_5xf_4wf_{10}zf_{10}w$	$b-w$	6
(2)	$bf_3xf_4wf_5xf_7y$	$b-y$	4
(3)	$bf_3xf_4wf_{11}yf_9c$	$b-c$	4
(4)	$bf_2cf_8xf_4wf_5xf_3b$	$b-b$	5
(5)	$bf_2cf_9yf_{12}zf_6xf_3b$	$b-b$	5

Table 1.21

The definition of a *walk* is quite general. In certain circumstances, we will need some special types of walks. A highway inspector may not want to inspect a road twice. And a traveller may not want to visit a city more than once.

A walk is called a **trail** if no *edge* in it is traversed more than once.
A walk is called a **path** if no *vertex* in it is visited more than once.

Question 1.4.2. *Is every trail a path? Is every path a trail?*

In Example 1.4.1,
walk (1) is neither a trail nor a path (why?);
walk (2) is a trail but not a path (why?);
walk (3) is both a trail and a path (why?);
walks (4) and (5) are both trails but not paths (why?).

Remark. It is now clear that if some edge is traversed more than once, then at least one of its two ends is visited more than once. Thus, **every path must be a trail**. Its converse is, however, not true.

The walk (1) in Example 1.4.1 is a $b-w$ walk, but not a $b-w$ path as the vertices x and w are visited more than once. However, it can be cut short, say to bf_3xf_4w, to become a $b-w$ path. Likewise, the $b-y$ walk is not a $b-y$ path, but it can be cut short to bf_3xf_7y to become a $b-y$ path.

Question 1.4.3. *Is it true that every $u-v$ walk always contains a $u-v$ path?*

> A $u-v$ walk is said to be **closed** if $u=v$, that is, its initial and terminal vertices are the same; and **open** otherwise.

Thus, in Example 1.4.1, the walks (1), (2) and (3) are open while the walks (4) and (5) are closed.

> A closed walk of length at least two in which no edge is repeated is called a **circuit**.

Thus, in Example 1.4.1, the closed walks (4) and (5) are circuits. Note that vertices are allowed to be repeated in a circuit.

Question 1.4.4. *In the multigraph H of Figure 1.20, find a circuit of length* 2 *and a circuit of length* 8.

A circuit is called a **cycle** if no vertex is repeated (except the initial and terminal vertices).

Thus, in Example 1.4.1, while the circuit (4) is not a cycle (why?), the circuit (5) is a cycle.

Question 1.4.5. *For each $k = 3, 4, 5, 6$, find a cycle of length k which passes through the vertex z in the multigraph of Figure 1.20.*

Question 1.4.6. *The circuit (4) in Example 1.4.1 is not a cycle, but it can be cut short to $bf_2cf_8xf_3b$ to become a cycle. Is it true that every circuit always contains a cycle?*

Remark. At the end of Section 1.3, a cycle is introduced as a graph. In the above discussion, however, a cycle is regarded as a special closed walk in a multigraph. What a cycle means should be clear from the context when it is used.

Remark. We express a walk in a multigraph as an alternating sequence of vertices and edges. It is necessary to name the edges since two vertices may be joined by more than one edge and we want to know which edge is traversed to visit these two vertices. However, when we confine ourselves to graphs, as two adjacent vertices are joined by a unique edge, such an expression can be simplified by dropping the names of edges. Thus, in the graph G of Figure 1.22, the $u - w$ walk $ue_1ve_7ye_4ze_3w$ can simply be denoted by $uvyzw$ without any ambiguity.

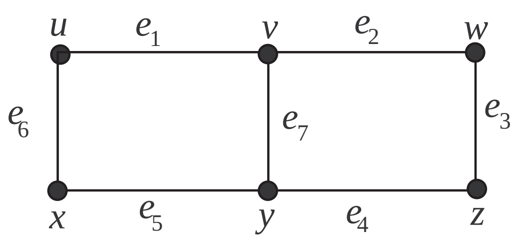
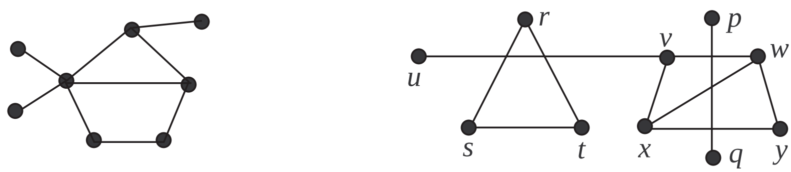


Figure 1.22

The notion of walks or paths enables us to introduce a very important class of multigraphs, called **connected** multigraphs.

A multigraph G is said to be **connected** if every two vertices in G are joined by a path.

Example 1.4.2. *There are two graphs G and H in Figure 1.23. It can easily be checked that every two vertices in G are joined by a path. Thus G is a connected graph. However, the graph H is not connected since, for instance, the vertices r and u in H are not joined by any path.*



(a) G (b) H

Figure 1.23

A multigraph is said to be **disconnected** if it is not connected.

Thus the graph H in Figure 1.23(b) is disconnected.

Question 1.4.7. *Consider the disconnected graph H in Figure 1.23(b).*
(1) Which vertices are reachable from the vertex r via a path?
(2) Which vertices are reachable from the vertex u via a path?
(3) Which vertices are reachable from the vertex p via a path?

The answers to (1), (2) and (3) of the above question are, respectively, shown in Figure 1.24. Note that each of them is a connected 'piece', and is called a **connected component** of H. From now on, we simply call a connected component a **component**. Thus the disconnected graph H has three components.

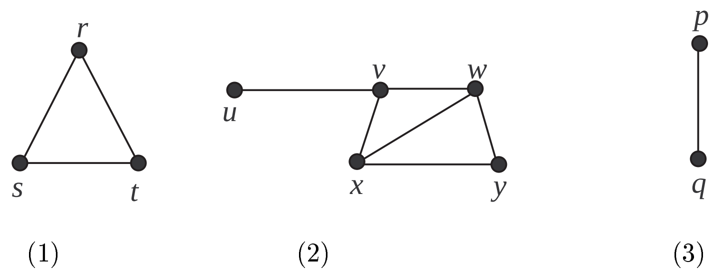


Figure 1.24

Question 1.4.8. *Consider the graph of order* 12 *and size* 9 *in Figure 1.25. Is the graph connected? If not, how many components does it have?*

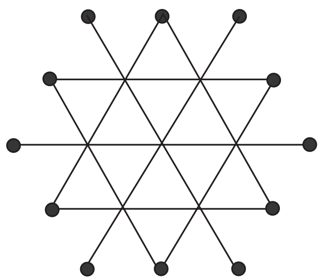
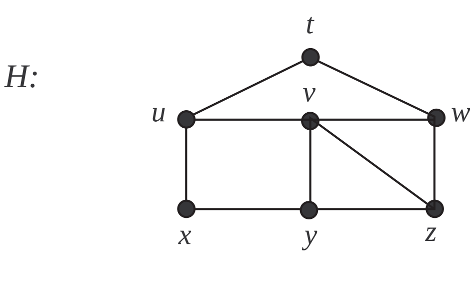


Figure 1.25

To end this chapter, we introduce in what follows an important quantity associated with a pair of vertices in a connected multigraph.

The graph H shown in Figure 1.26(a) is connected, and any two vertices in H are joined by at least one path. For instance, some paths joining the vertices x and w and their respective lengths are shown in Figure 1.26(b).



(a)

$x - w$ path	length
$xyzvutw$	6
$xuvyzw$	5
$xyzvw$	4
$xutw$	3
$xyzw$	3

(b)
Figure 1.26

We notice that '3' is the smallest length in the list in Figure 1.26(b). We ask: is there any $x - w$ path of length less than '3' in H? The answer is 'no'. That is, among all paths joining x and w in H, the smallest length is '3'. In this case, we say that the **distance from x to w is '3'**, and we write $d(x, w) = 3$.

> Let G be a connected multigraph, and u, v be any two vertices in G. The **distance from u to v**, denoted by $d(u, v)$, is defined as the *smallest length* of all $u - v$ paths in G.

Question 1.4.9. *In the graph H of Figure 1.26(a), find $d(u, u), d(x, y)$, $d(y, x)$, $d(x, z)$, $d(z, x), d(u, z)$, and $d(t, y)$.*

Let G be a connected multigraph, and x, y, z be any vertices in G. Some facts on distances are stated below.

(1) $d(x, x) = 0$,
(2) $d(x, y) > 0$ if $x \neq y$,
(3) $d(x, y) = d(y, x)$,
(4) $d(x, y) + d(y, z) \geq d(x, z)$.

Question 1.4.10. *(1) If $d(x, y) = 1$, what is the relation between x and y?*
(2) If $d(x, y) > 1$, what is the relation between x and y?
(3) Is it possible that $d(x, y) + d(y, z) = d(x, z)$?
(4) Is it possible that $d(x, y) + d(y, z) > d(x, z)$?

(∗)

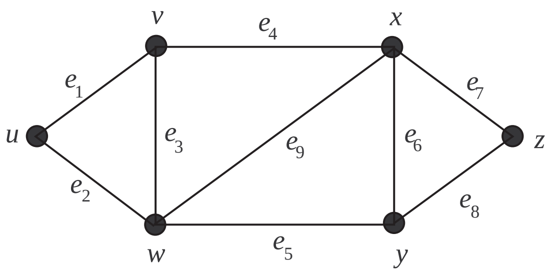
The greatest distance between any two vertices in a graph G (i.e. $\max\{d(u,v)|u,v \in V(G)\}$) is called the **diameter** of G. For instance, the diameter of the graph of Figure 1.26 (a) is '3'. Just as the diameter of a circle is the greatest distance between any two points on the circle, the diameter of a graph is an indication of how 'far apart' vertices in a graph are.

Suppose we represent each person on earth by a vertex and put an edge between two vertices if the two persons are acquainted. What will be the diameter of this 'acquaintance' graph? In 1990, John Guare wrote a play called 'Six Degrees of Separation' in which it is claimed that any two persons can be connected by a chain of at most 6 intermediaries, i.e. the diameter of the acquaintance graph is 6. This claim was 'tested' by Stanley Milgram. He gave some volunteers in Nebraska and Kansas (both Midwestern states of the USA) packages with the name of an individual in Massachusetts on the east coast of the USA. Each volunteer was asked to pass the package on to an acquaintance who he thought could get the package 'nearer' to the intended recipient, with the instruction to pass it on to another acquaintance, and so on, till it finally reaches the intended recipient. The average number of intermediaries for the packages that finally made it to the recipient was 5.5!

Does this prove Guare's claim? Certainly not, since the packages that did not reach their destination could not be counted, and also those who did may not have taken the shortest 'path'. In addition, the experiment was only within the USA and there are many parts of the world that are clearly more isolated than Nebraska or Kansas. Still, the idea of a 'small world' seems plausible and the diameter of the acquaintance graph may indeed be a small number.

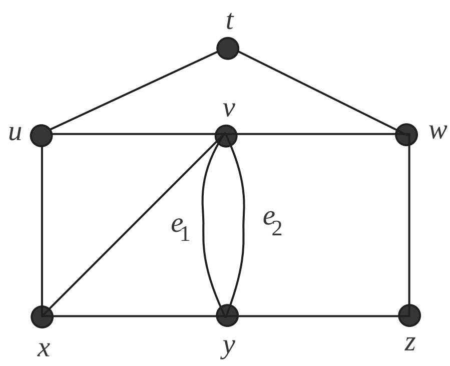
Exercise 1.4

(1) Consider the following graph H.



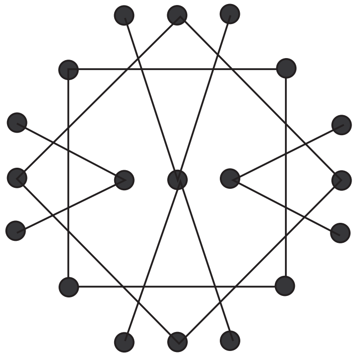
(a) Which of the following sequences represents a $u - z$ walk in H?
(i) $ue_2we_5xe_7z$
(ii) $ue_1ve_5ye_8z$
(iii) $ue_1ve_3we_3ve_4xe_7z$
(b) Find a $u - z$ trail in H that is not a path.
(c) Find all $u - z$ paths in H which pass through e_9.

(2) Consider the following multigraph G:



(a) Find $d(t, v)$, $d(t, y)$, $d(x, w)$ and $d(u, z)$.
(b) For $k = 2, 3, 4, 5, 6, 7$, find a cycle of length k in G.
(c) Find a circuit of length 6 in G that is not a cycle.
(d) Find a circuit of length 8 in G that does not contain t.
(e) Find a circuit of length 9 in G that contains t and v.

(3) Is the following graph H disconnected? If it is so, find its number of components.



(4) Let G be a graph with $V(G) = \{1, 2, \cdots, n\}$, where $n \geq 5$, such that two numbers i and j in $V(G)$ are adjacent if and only if $|i - j| = 5$. How many components does G have?

(5) (+) Show that any $u - v$ walk in a graph contains a $u - v$ path.

(6) (+) Show that any circuit in a graph contains a cycle.

(7) (+) Show that any graph G with $\delta(G) \geq k$ contains a path of length k.

(8) (+) Let G be a graph of order $n \geq 2$ such that $\delta(G) \geq \frac{1}{2}(n-1)$. Show that $d(u, v) \leq 2$ for any two vertices u, v in G.

(9) (+) Let G be a graph of order n and size m such that $m > \binom{n-1}{2}$. Show that G is connected.

(10) For $n \geq 2$, construct a disconnected graph of order n and size $\binom{n-1}{2}$.

(11) Let G be a disconnected graph of order 5. What is the largest possible value for $e(G)$? If G is a disconnected graph of order $n \geq 2$, what is the largest possible value for $e(G)$? Construct one such extremal graph of order n.

(12) (+) Let G be a graph of order $n \geq 2$ and u, v be two non-adjacent vertices in G such that $d(u) + d(v) \geq n + r - 2$. Show that u and v have at least r common neighbours.

(13) (+) Let G be a connected graph that is not complete. Show that there exist three vertices x, y, z in G such that x and y, y and z are adjacent,

but x and z are not adjacent in G.

(14) (+) Let G be a graph of order n and size m such that $\Delta(G) = n - 2$ and $d(u, v) \leq 2$ for any two vertices u, v in G. Show that $m \geq 2n - 4$.

(15) Let G be a graph such that $N(x) \cup N(y) = V(G)$ for every pair of vertices x, y in G. What can be said of G?

(16) (+) Let H be a graph of order $n \geq 2$. Suppose that H contains two distinct vertices u, v such that (i) $N(u) \cup N(v) = V(H)$ and (ii) $N(u) \cap N(v)$ is non-empty.
What is the least possible value of $e(H)$?

(17) Suppose G is a disconnected graph which contains exactly two odd vertices u and v. Must u and v be in the same component of G? Why?

(18) ($*$) Show that any two longest paths in a connected graph have a vertex in common.

(19) (+) Show that a graph G is connected if and only if for any partition of $V(G)$ into two non-empty sets A and B, there is an edge in G joining a vertex in A and a vertex in B.

(20) ($*$) Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \cdots, k$ in such a way that at each vertex which belongs to two or more edges (i.e. which is of degree at least two), the greatest common divisor of the integers labeling those edges is 1 (32nd IMO, 1991/4).