Preface

Discrete Mathematics is a branch of mathematics dealing with finite or countable processes and elements. Graph theory is an area of Discrete Mathematics which studies configurations (called graphs) consisting of a set of nodes (called vertices) interconnecting by lines (called edges). From humble beginnings and almost recreational type problems, Graph Theory has found its calling in the modern world of complex systems and especially of the computer. Graph Theory and its applications can be found not only in other branches of mathematics, but also in scientific disciplines such as engineering, computer science, operational research, management sciences and the life sciences. Since computers require discrete formulation of problems, Graph Theory has become an essential and powerful tool for engineers and applied scientists, in particular, in the area of designing and analyzing algorithms for various problems which range from designing the itineraries for a shipping company to sequencing the human genome in the life sciences.

Graph Theory shows its versatility in the most surprising areas. Recently, the connectivity of the World Wide Web and the number of links needed to move from one webpage to another has been remarkably modeled with graphs, thus opening the real world internet connectivity to more rigorous studies. These studies form part of research into the phenomena of the property of a 'small world' even in huge systems such as the aforementioned internet and global human relationships (in the so-called 'Six Degrees of Separation').

This book is intended as a general introduction to Graph Theory and, in particular, as a resource book for junior college students and teachers reading and teaching the subject at H3 Level in the new Singapore Mathematics curriculum for Junior College. Graph Theory is chosen as one of four topics (the others are Differential Equations, Combinatorics and Plane Geometry) in H3 Mathematics. Together with Plane Geometry, Graph Theory is aimed at providing a vehicle for the development of mathematical reasoning and proof. These topics, by their nature, may provide the breadth and depth for more mathematically mature junior college students to develop higher skills and insight for more advanced work in mathematics. In addition, the variety of problems and applications in the book are not only useful for building up an aptitude in Graph Theory but are a rich source for honing basic skills and techniques in general problem solving and logical thinking.

Certain features of this book are worth mentioning. The book has to be written in a way that caters to pre-university students and yet is equally suitable for undergraduate use. To this end, care is specially taken so that concepts are explained clearly and developed properly; it strives to be student friendly, accessible to college students and at the same time be mathematically rigorous. At suitable junctures, questions are inserted for discussion. This is to ensure that the reader understands the preceding section fully before proceeding on to new ideas and concepts. There are many questions in the Exercise component following most sections. Some are exercises intended for reinforcing what is learnt earlier while others test the full range of understanding and problem solving in the concepts acquired. Proofs of most important theorems are given in their full mathematical rigour. Although these proofs are not stipulated in the H3 syllabus, their inclusion in the book makes for completeness and logical continuity. Each chapter concludes with applications of the concepts in real-life. Again, the applications are often optional with regard to the H3 syllabus but they are added for general interest and as substantiation of the usefulness of Graph Theory concepts. Sometimes, even concepts which are not in the syllabus such as the distance between vertices and strong digraphs are included as the authors believe that they are essential knowledge in an introductory book on Graph Theory. **Note that optional sections are indicated at the beginning with the symbol (*) and the discussion is highlighted with a border on the left**. References are cited in full at the end of the book in the References section and they are indexed with the first letter of the first author's name within square parentheses. For example, [E] is for a paper by Euler. The symbol □ is used to indicate the end of a proof or a result stated without proof. Challenging problems are indicated with the symbol (+) while problems that are not in H3 mathematics are indicated with (*).

Chapter 1 covers the fundamental concepts and basic results in Graph

Theory tracing its history from Euler's solution of the problem of the Seven Bridges of Königsberg. Fundamental concepts include those of graphs, multigraphs, vertex degrees, paths, cycles and connectedness.

When are two graphs the 'same'? Following the style of Chapter 1, Chapter 2 further exposes the student to the rigour of mathematics in constructing a theory through definitions and theorems. Since two graphs may look different and yet 'function' similarly, the empirical perspective that mathematics students are so accustomed to needs to be reconsidered. Thus, congruence is defined in terms of isomorphism rather than a vague notion of shape thus enabling a 'handle' to compare graphs. This rigour and mathematical method of definitions and theorems continues throughout the whole book.

In Chapter 3, we introduce two important families of graphs, namely trees and bipartite graphs. A tree, in some sense, forms the 'skeleton' of a connected graph and in general, a forest of trees forms the 'skeleton' of any graph. Thus, the structure and properties of trees are very important. Bipartite graphs are another family of graphs that have found applications in many real-life situations such as matching a group of job seekers with a set of potential jobs under certain conditions.

Are four colours sufficient to colour any map? This question had frustrated many great mathematicians for over a century. Chapter 4 introduces the concept of vertex colouring which rephrases the question more simply. The notion of chromatic number (minimum number of colours used) is presented and an algorithm and some techniques to estimate or enumerate it are discussed. Interesting applications of vertex-colouring to scheduling problems are given in some detail.

Chapter 5 expands on the concept of matchings in bipartite graphs introduced in Chapter 3. Here we have a beautiful classical result in Graph Theory - Hall's Theorem. The necessity of the condition is trivial but the insight leading to the condition and the proof of its sufficiency exhibits the creativity of good mathematics. Hall's Theorem is used to determine the existence of a complete matching and this is used to good effect in the Marriage Problem and to find a system of distinct representatives (SDR).

Chapter 6 returns the reader to Euler's seminal work on the Bridges of Königsberg. Euler is memorialized for his contribution by having graphs with the property that one can have a walk that traverses all edges exactly once and that returns to the starting vertex named after him - Eulerian multigraphs. This chapter gives a fuller treatment of Eulerian multigraphs. It also discusses an apparently similar concept, that of graphs with the property that one can have a walk that visits all vertices exactly once and that returns to the starting vertex. This kind of graphs is called Hamiltonian, named after another mathematical giant, William Rowan Hamilton.

The last chapter is a necessary addition in an introductory book on Graph Theory. Chapter 7 studies graphs with 'directions' indicated on the edges. Such are called directed graphs or digraphs. This addition to Graph Theory suitably models many situations where relationships between items (vertices) are directional. The chapter covers some basic concepts and provides some detail on the most basic of digraphs which are tournaments.

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For those who find this introductory book interesting and would like to know more about the subject, a recommended list of publications for further reading is provided at the end of this book.

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