The Schwartz Classes of complex analytic singular Varieties

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One provides a detailed construction of the Schwartz classes. They are characteristic classes associated to complex analytic singular varieties. In a first step, one gives the construction of Schwartz classes by obstruction theory. Then one relates these classes to Mather's and MacPherson's ones. The third part is devoted to the computation of examples. The last section deals with polar varieties and definitions of characteristic classes via polar varieties. These are old and new results, partly obtained jointly with M.-H. Schwartz, with G. Gonzalez-Sprinberg, with G. Barthel, K.-H. Fieseler and L. Kaup and with P. Aluffi.

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1. Introduction

The Euler-Poincaré characteristic has been the first characteristic class to be introduced. For a triangulated (possibly singular) compact variety X without boundary, it has been defined, as

$$\chi(X) = \sum(-1)^i n_i\,,$$

where n_i is the number of i-dimensional simplices of the triangulation of X. It is also equal to $\sum(-1)^i b_i$ where b_i is the i-th Betti number, rank of $H_i(X)$. The Poincaré-Hopf theorem says that, if X is a manifold and v a continuous vector field with a finite number of isolated singularities a_k with indices $I(v, a_k)$, then

$$\chi(X) = \sum I(v, a_k)\,.$$

This means that the Euler-Poincaré characteristic is a measure of the obstruction to the construction of a non-zero vector field tangent to X, i.e. of a non-zero section of the tangent bundle.

The first definition of highest characteristic classes has been given in terms of obstruction of linearly independent sections of the tangent bundle. In a parallel way, Todd defined the so-called polar varieties and shown that some linear combinations of them are invariant. In fact, it appears that they coincide with Chern classes of complex analytic manifolds (see §8).

In years 1960, Hirzebruch shown that characteristic classes can be characterized by a system of axioms. During several years, the attractiveness of the axiomatic properties of Chern classes caused the viewpoints of obstruction theory and polar varieties to be somewhat forgotten. It is interesting to see that these viewpoints came back on the scene with the question of defining characteristic classes for singular varieties (see Teissier [Te]).

There are various definitions of characteristic classes for singular varieties. In the real case, there is a combinatorial definition, which simplifies the problem. In the complex case, the situation is more complicated (and certainly more interesting !), due to the fact that there is no combinatorial definition of Chern classes. Thinking of the obstruction theory point of view, one has to find a substitute to the tangent bundle. In fact there are various candidates to substitute the tangent bundle and to each of them corresponds a different definition of Chern class for singular varieties.

If X is a singular complex analytic variety, equipped with a Whitney stratification and embedded in a smooth complex analytic manifold M, one may consider the union of tangent bundles to the strata, that is a subspace E of the tangent bundle to M. The space E is not a bundle but it generalizes the notion of tangent bundle in the following sense: A section of E over X is a section v of $TM|_X$ such that in each point $x \in X$, then $v(x)$ belongs to the tangent space of the stratum containing x. To consider E as a substitute to the tangent bundle of X and to use obstruction theory is the M.-H. Schwartz point of view (1965, [Sc1]).

Another possible substitute for the tangent bundle is to consider, in each point $x \in X$, the space of all possible limits of tangent vector spaces $T_{x_i}(X_{\text{reg}})$ where x_i is a sequence of points in the regular part X_{reg} of X converging to x. That point of view leads to the notion of Mather class, which is an ingredient in the MacPherson definition, in the case of algebraic complex varieties (1974, [MP]) . The other main ingredient for these classes is the notion of Euler local obstruction.

Finally, when there exists a normal bundle N to X in M, for example in the case of local complete intersections, one can consider the virtual bundle $TM|_X \setminus N$ as a substitute to the tangent bundle of X. That point of view is the one of Fulton (1980, [Fu]).

There are relations between the classes obtained by the previous constructions. First of all, the Schwartz and MacPherson classes coincide, via Alexander duality (1979, [BS]). The relation between Mather classes on the one side and Schwartz-MacPherson classes on the other side follows form the MacPherson's definition itself: His construction uses Mather classes, taking into account the local complexity of the singular locus along Whitney strata. This is the role of the local Euler obstruction.

A natural question arose to compare the Schwartz-MacPherson and the Fulton-Johnson classes. A result of Suwa [Su] shows that in the case of isolated singularities, the difference between these classes is given by the sum of the Milnor numbers in the singular points. It was natural to call Milnor classes the difference arising in the general case (see [Yo]). This difference has been described by several authors by different methods: P. Aluffi, J.P. Brasselet-D. Lehmann-J. Seade-T. Suwa, A. Parusiński-P. Pragacz and S. Yokura. We will not discuss of these classes in the present course. Interested reader can refer to [Br2], [Br5].

We provide in §8 explicit examples and computations of Schwartz-MacPherson classes in the case of Thom spaces associated to Segre and Veronese embeddings and iterated cones. Results of this section and other examples (for instance the case of toric varieties) have been obtained in join works with G. Barthel, K.-H. Fieseler and L. Kaup (see [BBF], [BFK]).

The definition of polar varieties is classical in the smooth case. In §10, we give the construction of Chern classes using polar varieties in the context of manifolds. The notion of polar varieties has been extended in the singular case by Lê D. T. and B. Teissier [LT] and R. Piene [Pi1,Pi2] gave formula for Mather classes in terms of polar varieties, for singular varieties. The result of P. Aluffi and J.-P. Brasselet relates other definitions of classes to polar varieties, in a particular case. A general formula has been conjectured in [Br3].

Two books related to the subject will appear soon, where the reader will find a complete view on the subject ([Br5], [BSS2])

The very nice survey on Characteristic Classes of Singular Spaces, by Jörg Schürmann and Shoji Yokura [SY], *in this volume provides, for the students* and researchers, an useful and interesting complement to this course.

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2. The substitutes of the tangent bundle

A complex analytic manifold M admits a (complex) tangent bundle TM. In the case of a complex analytic singular variety X, there is no longer tangent bundle. The different notions of Chern classes, in the singular setting, correspond to different notions of substitute to the tangent bundle. There are (at least) three ways to define such a substitute in the case of a singular variety X embedded in a manifold M:

1. let us consider the union E of tangent spaces to the strata of a stratification of X and consider the sections of TM whose images are in E. This is the method used by M.-H. Schwartz, showing that it is not possible to proceed to obstruction theory, using any section, but that one has to use vector fields and frames obtained by radial extension.

2. let us consider in each point $x \in X$, the set of all possible limits of tangent spaces $T_{x_i}(X_{\text{reg}})$ to sequences of points $x_i \in X_{\text{reg}}$ converging to x. That is the Nash transformation and the Nash bundle on it.

3. let us consider the virtual bundle. That is the method used by Fulton. If X is smooth, one has the exact sequence

$$0 \to TX \to TM|_X \to N_X M \to 0$$

where $N_X M$ is the normal bundle of X in M. In the case of a singular variety such that the normal bundle $N_X M$ exists (for instance hypersurfaces or local complete intersections), one can define the virtual bundle (in the Grothendieck group $KU(X)$) as

$$\tau_X = TM|_X - N_X M.$$

A common setting of the different notions is that they define characteristic classes in homology. One has to notice that, in the case of singular spaces, there are no longer characteristic classes in cohomology.

3. Obstruction theory - The smooth case

Let M be a complex manifold of (complex) dimension m, endowed with a hermitian metric. The tangent bundle to M, denoted by TM, is a complex vector bundle of rank m, whose fiber in a point x of M is the tangent vector space to M in x, denoted by $T_x(M)$ and is isomorphic to $\mathbb{C}^m$. The vector bundle TM is locally trivial, i.e. there is a covering of M by open subsets $\{U_i\}$ such that the restriction of TM to U_i is isomorphic to $U_i \times \mathbb{C}^m$.

The Poincaré-Hopf Theorem says that $\chi(M)$ is a measure of the obstruction for the construction of a vector field tangent to the manifold M.

In the same way, the objective of the obstruction theory is, for r fixed, $1 \leq r \leq \dim_{\mathbb{C}} M$, to define a measure of the obstruction to the construction of r linearly independent vector fields tangent to the manifold, i.e. to the construction of r sections of TM linearly independent (over $\mathbb{C}$) in each point.

Definition 3.1. An *r-field* on a subset A of M is a set $v^{(r)} = \{v_1, \dots, v_r\}$ of r continuous vector fields defined on A. A singular point of $v^{(r)}$ is a point where the vectors (v_i) fail to be linearly independent. A non-singular r-field is also called an *r-frame.*

One observes that considering r-frames which are linearly independent r-fields or orthogonal r-fields provides the same construction and result (see [St]). It will be sometimes easier to consider orthogonal r-fields.

The r-frames are sections of the fibre bundle $T^r(M)$, with basis M, associated to TM and whose fiber in the point x of M is the set of r-frames of $T_x(M)$. This bundle is no longer a vector bundle. The "typical" fiber of $T^r(M)$ is the set of all r-frames of $\mathbb{C}^m$, called the Stiefel manifold and denoted by $V_{m,r}(\mathbb{C})$. These manifolds, in particular their homotopy groups, have been studied by Stiefel and by Whitney (see Steenrod [St]).

Let us consider the following situation: (D) is a cell decomposition of M sufficiently small so that every cell d lies in an open subset U over which $T^r(M)$ is trivial. We notice that trivialization open sets for $T^r(M)$ are the same that those of $T(M)$.

Let us consider the following question:

Let us suppose that one has a section $v^{(r)}$ of $T^r(M)$ on the boundary ∂d of the k-dimensional cell d. Is it possible to extend this section in the interior of d ? If the answer is no, what is the obstruction for such an extension ?

The section $v^{(r)}$, defined on the boundary of d, provides a map

$$\partial d \xrightarrow{v^{(r)}} T^r(M)|_U \cong U \times V_{m,r}(\mathbb{C}) \xrightarrow{pr_2} V_{m,r}(\mathbb{C}) \tag{1}$$

where pr_2 is the second projection. One obtains a map

$$\mathbb{S}^{k-1} \cong \partial d \xrightarrow{pr_2 \circ v^{(r)}} V_{m,r}(\mathbb{C})$$

hence an element of $\pi_{k-1}(V_{m,r}(\mathbb{C}))$ denoted by $[\gamma(v^{(r)}, d)]$.

Let us suppose that $[\gamma(v^{(r)}, d)] = 0$, then, by classical homotopy theory, the map $\mathbb{S}^{k-1} \to V_{m,r}(\mathbb{C})$ defined on the boundary $\mathbb{S}^{k-1}$ of the ball $\mathbb{B}^k$ can be extended inside the ball.

$$\begin{array}{ccc} \partial d \cong \mathbb{S}^{k-1} & \longrightarrow & V_{m,r}(\mathbb{C}) \\ \cap & ? \nearrow & \\ d \cong \mathbb{B}^k & & \end{array}$$

In another words, if $[\gamma(v^{(r)}, d)] = 0$, then the map $\partial d \to V_{m,r}(\mathbb{C})$ can be extended inside d. This means that there is no obstruction to the extension of the section $v^{(r)}$ inside d. This happens, in particular, in the case $\pi_{k-1}(V_{m,r}(\mathbb{C})) = 0$.

In order to answer to the previous question, we need to know the homotopy groups of $V_{m,r}(\mathbb{C})$. They are equal to (see [St]):

$$\pi_{k-1}(V_{m,r}(\mathbb{C})) = \begin{cases} 0 & \text{for } k < 2(m-r+1) \\ \mathbb{Z} & \text{for } k = 2(m-r+1) \end{cases} \tag{2}$$

Let us denote $2p = 2(m - r + 1)$. A generator of the first non-zero homotopy group $\pi_{2p-1}(V_{m,r}(\mathbb{C}))$ can be described in the following way. Let us fix a $(r-1)$-frame in $\mathbb{C}^m$. It defines a $(r-1)$-subspace of $\mathbb{C}^m$ whose complementary is a complex space $\mathbb{C}^p$. The unit sphere in $\mathbb{C}^p$, denoted by $\mathbb{S}^{2p-1}$, is oriented with orientation induced by the natural one of $\mathbb{C}^p$. Let us consider, for every point w of the sphere, the r-frame consisting of the fixed $(r-1)$-frame and the vector w, one obtains an element of $V_{m,r}(\mathbb{C})$. The induced map from the oriented sphere $\mathbb{S}^{2p-1}$ to $V_{m,r}(\mathbb{C})$ defines a generator of $\pi_{2p-1}(V_{m,r}(\mathbb{C}))$.

One obtains:

Proposition 3.1. *Let $v^{(r)}$ be a r-frame defined on the boundary ∂d of the k-cell d.*

(i) If $k < 2(m-r+1)$, one has $[\gamma(v^{(r)}, d)] = 0$, then one can extend the r-frame, already defined on ∂d, inside d without singularity.

(ii) If $k = 2(m-r+1)$, the r-frame defines an integer $[\gamma(v^{(r)}, d)]$ that we denote by $I(v^{(r)}, \hat{d})$. That index which measures the obstruction to the extension of $v^{(r)}$ inside d.

The dimension $2p = 2(m-r+1)$ is called the obstruction dimension for the construction of an r-frame tangent to M.

If $v^{(r)}$ is a r-frame defined on the boundary ∂d of the $2p$-cell d, there are many ways to extend $v^{(r)}$ inside d with an isolated singularity. We will proceed in the following way: let us consider the $(r-1)$-frame $v^{(r-1)} = (v_1, \dots, v_{r-1})$ corresponding to the first $(r-1)$ vectors of $v^{(r)}$. It defines a section of $T^{r-1}(M)$ over ∂d. The obstruction dimension for the extension of

$v^{(r-1)}$ is $2(m-(r-1)+1) = 2p+2$. That means that one can extend $v^{(r-1)}$ inside d without singularity. The extension defines a $(r-1)$-sub-bundle of $TM|_d$, whose complementary is a sub-bundle Q of (complex) rank p. The last vector v_r of $v^{(r)}$ defines a section of Q over ∂d. The obstruction dimension for the extension of v_r as a section of Q over d is $2(p-1+1) = 2p$, that is the dimension of d. That means that one can extend v_r inside d as a section of Q with an isolated singularity at the barycenter $\hat{d}$. The index of the r-frame $v^{(r)} = (v^{(r-1)}, v_r)$ at the singular point $\hat{d}$ is defined as $[\gamma(v^{(r)}, d)]$, we denote it by $I(v^{(r)}, \hat{d})$. One observes that this corresponds to the classical definitions of the index of a r-frame at a singular point.

The Chern classes can be defined now in the following way: One choose arbitrary r-frames on the 0-cells and one extends them without singularity, i.e. as a section of $T^r M$, on the 1-cells. By (i) of Proposition 3.1 one can extend that section by induction process on higher dimensional cells till we reach the obstruction dimension $2p$. For each $2p$-cell d, the section $v^{(r)}$ being defined on the boundary, provides an index $I(v^{(r)}, \hat{d})$. The generators of $\pi_{2p-1}(V_{m,r}(\mathbb{C}))$ being consistent (see [St]), one define a cochain

$$\gamma \in C^{2p}(M; \pi_{2p-1}(V_{m,r}(\mathbb{C}))), \qquad \text{such that } \gamma(d) = I(v^{(r)}, \hat{d}),$$

for each $2p$-cell d, and then extend by linearity. This cochain is actually a cocycle, called the obstruction cocycle.

Proposition 3.2. *The cohomology class of the obstruction cocycle γ does not depend on the various choices involved in its definition.*

Definition 3.2. The p-th (cohomology) *Chern class* of M,

$$c^p(M) \in H^{2p}(M; \mathbb{Z})$$

is the class of the obstruction cocycle.

By Poincaré duality isomorphism, cap-product by the fundamental class $[M]$ of M

$$H^{2p}(M; \mathbb{Z}) \longrightarrow H_{2(r-1)}(M; \mathbb{Z})$$

the image of $c^p(M)$ in $H_{2(r-1)}(M)$ is the $(r-1)$-st homology Chern class of M, denoted by $c_{r-1}(M)$. It is represented by the cycle

$$\sum_{\dim s = 2(r-1)} I(v^{(r)}, \hat{d})\ s. \qquad (3)$$

where s is the simplex in the simplicial triangulation (K) of which d is dual. The barycenter $\hat{d}$ is the intersection point of s and d, that is the barycenter of s, as well.

In particular, the evaluation of $c^m(M)$ on the fundamental class $[M]$ of M yields the Euler-Poincaré characteristic.

4. The Schwartz classes

The first definition of Chern class for singular varieties has been given in 1965 by M.-H. Schwartz in two "Notes aux CRAS" [Sc1].

In order to define characteristic classes of singular varieties, it is necessary to know the local structure of the singular variety. That is given by the structure of stratified space and by suitable definition of triangulation on the variety.

4.1. *Stratifications, triangulations and cell decompositions*

In the following, M will be a complex analytic manifold equipped with a semi-analytic stratification $\{V_\alpha\}$, i.e. a partition into analytic manifolds V_α, called the strata such that, for each stratum V_α, the closure $\overline{V}_\alpha$ and the boundary $\dot{V}_\alpha = \overline{V}_\alpha \setminus V_\alpha$ are semi-analytic sets, union of strata. We denote by $X \subset M$ a complex analytic compact subset stratified by $\{V_\alpha\}$.

As we know, on a singular variety, there is no more tangent space in the singular points. One way to find a substitute for the tangent bundle is to stratify the singular variety into submanifolds. One can proceed to the following construction: If X is a singular complex analytic variety, equipped with a stratification and embedded in a smooth complex analytic manifold M one can consider the union of tangent bundles to the strata. That is a subspace E of the tangent bundle to M. The space E is not a bundle but it generalizes the notion of tangent bundle in the following sense: A section of E over X is a section v of $TM|_X$ such that in each point $x \in X$, then $v(x)$ belongs to the tangent space of the stratum containing x. Such a section is called a *stratified vector field* over X:

Definition 4.1. A *stratified vector field* v on a part A of X is a (continuous) section of the tangent bundle TM defined on A and such that, for every $x \in A$, one has $v(x) \in T(V_{\alpha(x)})$ where $V_{\alpha(x)}$ is the stratum containing x.

To consider E as the substitute to the tangent bundle of X and to use obstruction theory is the M.-H. Schwartz point of view (1965, [Sc1]), in the case of analytic varieties.

When one considers stratifications of singular varieties, it is natural to ask for conditions with which the strata glue together. The so-called Whitney conditions [Wh] are the one which allow to proceed to the construction of radial extension vector fields. According to a result of Whitney, every analytic complex variety can be equipped with a Whitney stratification.

Definition 4.2. One says that the Whitney conditions are satisfied for the stratification $\{V_\alpha\}$ of X if, for any pair of strata (V_α, V_β) such that V_α is in the closure of V_β, one has:
a) if (x_n) is a sequence of points in V_β with limit $y \in V_\alpha$ and if the sequence of tangent spaces $T_{x_n}(V_\beta)$ admits a limit T (in the suitable Grassmanian space) when n goes to $+\infty$, then T contains $T_y(V_\alpha)$.
b) if (x_n) is a sequence of points in V_β with limit $y \in V_\alpha$ and if (y_n) is a sequence of points in V_α with limit y, such that the sequence of tangent spaces $T_{x_n}(V_\beta)$ admits a limit T for n going to $+\infty$ and such that the sequence of directions $\overline{x_n\, y_n}$ admits a limit λ when n goes to $+\infty$, then λ lies in T.

Let $X \subset M$ be a singular n-dimensional complex analytic variety embedded in a complex m-dimensional manifold. Let us consider a Whitney stratification $\{V_\alpha\}$ of M such that X is a union of strata and let us denote by (K) a triangulation of M compatible with the stratification, i.e. each open simplex is contained in a stratum.

The first nice observation of M.-H. Schwartz concerns the triangulations:

We denote by (K') a barycentric subdivision of (K) and by (D) the associated dual cell decomposition. Each cell in (D) is transverse to the strata. This implies that if d is a cell of (real) dimension k and V_α is a stratum of (complex) dimension n_α, then $d \cap V_\alpha$ is a cell whose (real) dimension is

$$\dim(d \cap V_\alpha) = k - 2(m - n_\alpha).$$

Consequence: This means that if d is a cell whose dimension is the dimension of obstruction to the construction of an r-frame tangent to M, i.e. $2p = 2(m - r + 1)$, then $d \cap V_\alpha$ is a cell whose dimension is exactly the dimension of obstruction to the construction of an r-frame tangent to the stratum V_α, i.e. $2(n_\alpha - r + 1)$.

We will use two important properties of the dual cells:

The (D)-cells which meet X are duals of (K)-simplices lying in X. Union of such cells is a neighbourhood $N(X)$ around X. That is not a fibre bundle on X but one has the following construction:

Dual cells are union of simplices of the barycentric subdivision (K'). For each (closed) simplex τ in (K') such that $\tau \cap X \neq \emptyset$, then one calls $\tau_X = \{a_0, a_1, \ldots, a_i\}$ the set of vertices in τ which are in X and $\tau'_X =$

$\{a_{i+1}, \dots, a_k\}$ the set of vertices in τ which are not in X. Let us call $N_\varepsilon(\tau)$ the set of points in τ such that $\sum_{j=0}^{i} \lambda_j \leq \varepsilon$ and we use the following notation:

$$N_\varepsilon(X) = \bigcup_{\tau \subset N(X)} N_\varepsilon(\tau). \tag{4}$$

That is a "tube" around X and there is a retraction of $N_\varepsilon(X)$ on X along "rays": two points x and x' in τ belong to the same ray if their barycentric coordinates corresponding to vertices in τ_X are proportional on the one hand and their barycentric coordinates corresponding to vertices in τ'_X are proportional on the other hand (see Figure 1, Page 30 in [Sc3]).

The second nice construction of M.-H. Schwartz is the construction of radial extension of vector fields that we explicit below.

4.2. *Radial extension process - the local case*

One gives a description of the local radial extension process. This will be used for the global process in the next section.

Let us consider $X, M, \{V_\alpha\}, (K), (D)$ as before. Let $V_\alpha \subset X$ be a stratum, with complex dimension n_α, let a be the barycenter of a $2(r-1)$-simplex s of the triangulation (K), lying in V_α. One denotes by $d = d(s)$ the dual cell. Then $d_\alpha = d \cap V_\alpha$ is a $2(n_\alpha - r + 1)$-cell in V_α. Note that in general, $d \cap X$ is not a cell.

Let $v^{(r)}$ a r-frame defined in d_α with an isolated singularity at a. One will construct inside d the parallel extension of $v^{(r)}$ and a particular vector field, the transversal vector field:

4.2.1. *Parallel Extension*

Provided the simplices of (K) are sufficiently small, the cell d can be identified with $d_\alpha \times \mathbb{D}^{k_\alpha}$ where $\mathbb{D}^{k_\alpha}$ is a disk which is transverse to V_α and whose dimension is $k_\alpha = 2(m - n_\alpha)$. For a precise identification, one works by induction on the dimension of cells in $d_\alpha = d \cap V_\alpha$: the 0-cells in the boundary of d_α are barycenters of $2n$-simplices s_i^{2n} containing s in their boundary, the dual cells $d(s_i^{2n})$ are homeomorphic to $\mathbb{D}^{k_\alpha}$. By induction one extends the identification on cells of the boundary, then on d_α itself.

Let us consider the parallel extension $\hat{v}^{(r)}$ of $v^{(r)}$ in d along the fibers $\mathbb{D}^{k_\alpha}$. Let V_β be a stratum such that $a \in \overline{V_\beta}$. At a point $x \in d \cap V_\beta$ the parallel extension $\hat{v}^{(r)}(x)$ is not necessarily tangent to V_β. However, the Whitney condition (a) guarantees that if d is sufficiently small, then the angle between $T_a(V_\alpha)$ and $T_x(V_\beta)$ is small. That implies that the orthogonal projection of $\hat{v}^{(r)}(x)$ on $T_x(V_\beta)$ does not vanish. Of course, considering for each stratum the projection of the parallel extension on the tangent space to the stratum at the given point does not provide a continuous frame. In order to obtain a continuous frame, one has to consider a slight modification of the construction, in the neighbourhood of the strata, which is easy to understand, but complicated to describe into details. The good extension will be $\hat{v}^{(r)}(x)$ away from V_β and continuously going to the projection of $\hat{v}^{(r)}(x)$ on $T_x(V_\beta)$ when approaching V_β, using a suitable partition of unity. That construction is correctly and entirely described in M.-H. Schwartz book [Sc2]. In fact, one has to work simultaneously for all strata V_β such that $a \in \overline{V_\beta}$, that complicates a detailed construction.

In conclusion, the Whitney (a) condition implies that one can proceed to the construction of a stratified r-frame, denoted by $\hat{v}^{(r)}(x) = \{\hat{v}_1, \dots, \hat{v}_r\}$ which is a "parallel extension" of the given frame on V_α, in the cell d, identified to a tube around d_α.

One observes that the singular locus of $\hat{v}^{(r)}$ corresponds to a k_α-dimensional disk which is transversal to d_α at the point $\hat{d} = a$.

4.2.2. *Transversal vector field*

Let us consider the transversal vector field $g(x)$, which is the gradient of the square of the function distance to V_α, for an appropriate Riemannian metric. The vector field $g(x)$ is not necessarily tangent to the strata V_β such that $a \in \overline{V_\beta}$. However, the Whitney condition (b) guarantees that in d, which is identified to a sufficiently small "tube" around d_α and for $x \in d \cap V_\beta$, the angle between $g(x)$ and $T_x(V_\beta)$ is small. That means that the orthogonal projection of $g(x)$ on $T_x(V_\beta)$ does not vanish. In the same way than for the parallel extension, considering for each stratum the projection of $g(x)$ on the tangent space to the stratum at the given point does not provide a continuous vector field. In order to obtain a continuous vector field, one has to consider a similar modification of the construction. The good vector field will be $g(x)$ away from V_β and continuously going to the projection of $g(x)$ on $T_x(V_\beta)$ when approaching V_β. That construction is also completely described in M.-H. Schwartz book [Sc2], and one has to work simultaneously for all strata V_β such that $a \in \overline{V_\beta}$.

Let us call horizontal part of the boundary of the tube $d \cong d_\alpha \times \mathbb{D}^{k_\alpha}$, the part of the boundary corresponding to $d_\alpha \times \partial\mathbb{D}^{k_\alpha} = d_\alpha \times \mathbb{S}^{k_\alpha - 1}$ by the previous identification. The vector field g is pointing outward d along the horizontal part of the boundary.

In conclusion, one obtains a stratified "transversal" vector field still denoted by g which vanishes along V_α, which is growing with the distance to V_α and which is pointing outward d along the horizontal part of the boundary of the "tube" d provided that the tube is sufficiently small.

4.2.3. *Local radial extension*

Definition 4.3. [Sc3] [BS] Let s be a simplex in V_α and let d be the dual cell of s. Let $v^{(r)} = \{v_1, \ldots, v_r\}$ be an r-frame defined in $d_\alpha = d \cap V_\alpha$, possibly with an isolated singularity at the barycenter of d_α, the (local) radial extension of $v^{(r)}$ is the r-frame $\widetilde{v}^{(r)}$ defined in the cell d as the parallel extension $\hat{v}^{(r)}$ of $v^{(r)}$ to which one adds the transversal vector field on the last coordinate, i.e.

$$\widetilde{v}^{(r)} = (\widetilde{v}^{(r-1)}, \widetilde{v}_r) = \{\hat{v}_1, \ldots, \hat{v}_{r-1}, \hat{v}_r + g(x)\}.$$

Proposition 4.1. *[Sc3] [BS] Let $v^{(r)} = \{v_1, \ldots, v_r\}$ be an r-frame defined in $d_\alpha = d \cap V_\alpha$, with an isolated singularity at the barycenter a of d_α, then the (local) radial extension of $v^{(r)}$ is defined inside the cell d and it has an isolated singularity at a. The index of $\widetilde{v}^{(r)}$ at a, computed in the cell d as a section of $T^r M$, is the same than the index of $v^{(r)}$ at a, computed in the cell $d_\alpha = d \cap V_\alpha$ as a section of $T^r V_\alpha$. We write*

$$I(\widetilde{v}^{(r)}, a; d) = I(v^{(r)}, a; d \cap V_\alpha).$$

That property is the main property of the radial extension, that is precisely the property which allows to construct the obstruction classes for singular varieties.

4.3. *Chern classes for singular varieties*

In that section, one proceeds to the construction of a "global" radial extension of an r-frame and one shows the following Theorem (see also [BS]):

Theorem 4.1. *[Sc1], [Sc3] One can construct, on the cells d of the $2p$-skeleton $(D)^{2p}$ which intersect X, a stratified r-frame $v^{(r)} = (v^{(r-1)}, v_r)$ called radial extension frame, whose singularities satisfy the following properties:*

(i) $v^{(r)}$ has only isolated singular points, which are zeroes of the last vector v_r. On $(D)^{2p-1}$, the r-frame $v^{(r)}$ has no singular point. On $(D)^{2p}$ the $(r-1)$-frame $v^{(r-1)}$ has no singular point.

(ii) Let $a \in V_\alpha \cap (D)^{2p}$ be a singular point of $v^{(r)}$ in the n_α-dimensional stratum V_α. If $n_\alpha > r-1$, the index of $v^{(r)}$ at a, denoted by $I(v^{(r)}, a)$, is the same as the index at a of the restriction of $v^{(r)}$ to $V_\alpha \cap (D)^{2p}$ considered as an r-frame tangent to V_α. If $n_\alpha = r-1$, then $I(v^{(r)}, a) = +1$.

(iii) Inside a 2p-cell d which meets several strata, the only singularities of $v^{(r)}$ lie in the lowest dimensional one (in fact located at the barycenter of d).

(iv) The last vector v_r of the r-frame is pointing outward (particular) regular neighbourhoods U of X in M. The r-frame $v^{(r)}$ has no singularity on the boundary ∂U.

Proof. The "global" construction of the radial extension frame is as follows:

One consider on M a Whitney stratification compatible with X.

The proof will go by induction on the dimension of the strata. We will show that, for each stratum V_α, the theorem is true for $X = \overline{V}_\alpha$.

a) Let us denote by n_0 the lowest dimension of strata in X such that $n_0 \geq r-1$. The strata whose dimension is less than n_0 do not contribute to the corresponding Chern class. The reason is that such a stratum does not meet the $2p$-skeleton $(D)^{2p}$.

a) Let us prove the theorem for the n_0-dimensional stratum, denoted by V_{α_0}. We distinguish the cases $n_0 = r-1$ and $n_0 > r-1$:

a1) If $n_0 = r-1$, let d be a $2p$-cell which intersects V_{α_0}. The intersection is a point: the barycenter $\hat{s}$ of the $2(r-1)$-simplex s of which d is dual. At each such point $\hat{s}$, let us fix a $(r-1)$- frame $v^{(r-1)}(\hat{s})$ tangent to V_{α_0}. One can extend $v^{(r-1)}(\hat{s})$ on d by the local parallel extension process (4.2.1) as a stratified $(r-1)$-frame $\widehat{v}^{(r-1)}$ on such a cell d. Using the transversal vector field g constructed in 4.2.2, one obtains a stratified r-frame $\widetilde{v}^{(r)}(x) = (\widehat{v}^{(r-1)}(x), g(x))$ on each d. One has

$$I(\widetilde{v}^{(r)}, \hat{s}; d) = +1$$

a2) If $n_0 > r-1$, on the one hand, one observes that if a $2p$-cell d intersects V_{α_0}, then the intersection is a $2(n_0 - (r-1))$-cell which is dual, in V_{α_0} of a $2(r-1)$-simplex s. On the other hand, the (D)-cells with dimension less than $2p$ do not meet the strata which lie in $\overline{V}_{\alpha_0} \setminus V_{\alpha_0}$.

Let us call (D_{α_0}) the set of cells in V_{α_0}, intersections $d_{\alpha_0} = d \cap V_{\alpha_0}$ for all (D)-cells d whose dimension k satisfies $2(m-n_0) \leq k \leq 2p = 2(m-(r-1))$. If $\dim d = k$, then $\dim d_{\alpha_0} = k - 2(m-n_0)$. So, if $d_{\alpha_0} \in (D_{\alpha_0})$, one has $0 \leq \dim d_{\alpha_0} \leq 2(n_0 - (r-1))$. One observes that the obstruction dimension for the construction of an r-frame tangent to V_{α_0} is $2(n_0 - (r-1))$.

One chooses an r-frame $v^{(r)}$ in each 0-dimensional cell d_{α_0}, one can extend these frames as a section of $T^r V_{\alpha_0}$ on higher dimensional cells d_{α_0} by classical obstruction theory, by induction on dimensions of cells till we reach the obstruction dimension $2(n_0 - (r-1))$. One obtain an r-frame $v^{(r)}$ with isolated singularities at the barycenters $\hat{d}$ of the $2(n_0 - (r-1))$-dimensional cells d_{α_0}, with index $I(v^{(r)}, \hat{d}; d_{\alpha_0})$.

The r-frame can be extended by the local extension process 4.2.3 as an r-frame $\widetilde{v}^{(r)}$ on the $2p$-cells d such that $d \cap V_{\alpha_0} = d_{\alpha_0}$, with an isolated singularity at $\hat{d}$ in each cell d and such that

$$I(v^{(r)}, \hat{d}; d_{\alpha_0}) = I(\widetilde{v}^{(r)}; \hat{d}, d).$$

That proves the theorem for the lowest dimensional strata V_{α_0}, of dimension bigger than $2(r-1)$. In both cases a1) and a2), the neighbourhood U is the set of cells in (D) which intersect V_{α_0}.

b) Let us now consider a stratum V_γ with (complex) dimension n_γ and let us suppose that the theorem has been proved for all strata of dimension lower than (and equal to) n_γ, i.e. that the theorem is true for $X = \overline{V}_\gamma$. One denotes by $N_\varepsilon(\overline{V}_\gamma)$ the neighbourhood defined in (4). Let us denote by $v^{(r)}$ an r-frame satisfying conditions of the Theorem for $X = \overline{V}_\gamma$ and for $U = N_\varepsilon(\overline{V}_\gamma)$.

Let us call V_δ the next stratum, i.e. the one whose dimension n_δ is strictly bigger than n_γ and such that there is no other strata whose dimension is between n_δ and n_γ. One has to show that the theorem is true for $X = \overline{V}_\delta$.

The r-frame $v^{(r)}$ is defined on the $2p$-skeleton of $N_\varepsilon(\overline{V}_\gamma)$ with singularities situated in V_γ. That means that $v^{(r)}$ is defined on the skeleton of dimension $2(n_\delta - r + 1)$ of $U_\delta = V_\delta \cap N_\varepsilon(\overline{V}_\gamma)$. Moreover, the last vector of $v^{(r)}$ is pointing inward V_δ along $\partial U_\delta = \overline{U}_\delta \setminus U_\delta$. By classical obstruction theory, one can extend $v^{(r)}$ inside V_δ on the $2p_\delta = 2(n_\delta - r + 1)$-skeleton and such that:

- $v^{(r)}$ has only isolated singularities, which are zeroes of the last vector v_r,
- on $(D)^{2p-1}$, the r-frame $v^{(r)}$ has no singular point,
- on $(D)^{2p}$ the $(r-1)$-frame $v^{(r-1)}$ has no singular point.

One can extend, by the local extension process 4.2.3, the obtained r-frame $v_\delta^{(r)}$ on the $2(n_\delta - r + 1)$-cells of (D_δ) as a stratified r-frame $\widetilde{v}_\delta^{(r)}$ on the $2p$-cells in $(D)^{2p}$ which meet $V_\delta \setminus \Omega_\delta$. One has:

$$I(\widetilde{v}_\delta^{(r)}; \hat{d}, d) = I(v_\delta^{(r)}, \hat{d}; d \cap V_\delta)$$

for each $2p$-cell d which meet $V_\delta \setminus \Omega_\delta$.

One obtains an r-frame $v^{(r)}$ defined in $(D)^{2p} \cap U(V_\gamma)$ and an r-frame $\widetilde{v}_\delta^{(r)}$ defined in $(D)^{2p} \cap (V_\delta \setminus \Omega_\delta)$. The problem is that, while these two frames agree on V_δ, i.e. on $(D)^{2p} \cap \partial U(V_\gamma) \cap V_\delta$, they do not agree a priori on the common part $(D)^{2p} \cap \partial U(V_\gamma) \cap U'(V_\delta)$ where $U'(V_\delta)$ is the union of the (D)-cells which meet $V_\delta \setminus \Omega_\delta$.

The solution is rather technical. One has to work with two systems of neighbourhood on the following way:

The previous construction can be performed for all $0 < \varepsilon \leq 1$. Let us suppose that the construction has been performed for say, $N_\varepsilon(\overline{V}_\gamma)$ and $N_1(\overline{V}_\gamma)$, the r-frame on the first neighbourhood being restriction of the r-frame on the second. Then one obtains :

- an r-frame inside $N_\varepsilon(\overline{V}_\gamma)$, we will not modify it,
- two r-frames inside $\big(N_1(\overline{V}_\gamma) \setminus N_\varepsilon(\overline{V}_\gamma)\big) \cap U(V_\gamma)$, the first one being $v^{(r)}$ defined in $N_1(\overline{V}_\gamma)$, the second one is $\widetilde{v}_\delta^{(r)}$. In fact, they coincide on V_γ.

Let us call λ the radius of $N_1(\overline{V}_\gamma)$ and

$$\lambda' = \frac{1}{1-\varepsilon}\lambda - \frac{\varepsilon}{1-\varepsilon}$$

In $\big(N_1(\overline{V}_\gamma) \setminus N_\varepsilon(\overline{V}_\gamma)\big) \cap U(V_\gamma)$, one considers the r-frame

$$\lambda' \widetilde{v}_\delta^{(r)} + (1-\lambda') v^{(r)}$$

and one obtains a continuous stratified r-frame in

$$N_\varepsilon(\overline{V}_\delta) = N_\varepsilon(\overline{V}_\gamma) \cup U'_\varepsilon(V_\delta)$$

which solves the problem. As we said, to verify all conditions is rather tedious. That is completely performed in M.-H. Schwartz's work. □

4.4. *Obstruction cocycles and classes*

Let us denote by $N = N(X)$ the tubular neighborhood of X in M consisting of the (D)-cells which intersect X. The dual cell of a (K)-simplex s in X is denoted by $d = d(s)$ and their common barycenter by $\hat{s} = d(s) \cap s$. Let us denote by d^* the elementary (D)-cochain whose value is 1 at d and 0 at all other cells. We can define a $2p$-dimensional (D)-cochain in $C^{2p}(N, \partial N)$ by:

$$\sum_{\substack{d(s) \in N \\ \dim d(s) = 2p}} I(v^{(r)}, \hat{s})\, d^*.$$

This cochain actually is a cocycle whose class $c^p(X)$ lies in

$$H^{2p}(N, \partial N) \cong H^{2p}(N, N \setminus X) \cong H^{2p}(M, M \setminus X),$$

where the first isomorphism is given by retraction along the rays of N and the second by excision (by $M \setminus N$).

Definition 4.4. [Sc1], [Sc3] The p-th Schwartz class of X is the class

$$c^p(X) \in H^{2p}(M, M \setminus X).$$

The Schwartz class is independent of the choices : stratification, triangulation, dual cell decomposition. The direct proof is rather tedious (see [Sc2]). The fact that the Schwartz class is dual of the MacPherson one helps a lot for such a proof.

5. Euler local obstruction

5.1. *Nash transformation*

Let M be an analytic manifold, of complex dimension m. Let X be an subanalytic complex variety in M of complex dimension n, equipped with a Whitney stratification. Let us denote by $\Sigma = X_{\text{sing}}$ the singular part of X and by $X_{\text{reg}} = X \setminus \Sigma$ the regular part.

The Grassmanian manifold of complex n-planes in $\mathbb{C}^m$ is denoted by $G(n,m)$. Let us consider the Grassmann bundle of n (complex) planes in TM, denoted by G. The fibre G_x over $x \in M$ is the set of n-planes in $T_x(M)$, it is isomorphic to $G(n,m)$. An element of G is denoted by (x, P) where $x \in M$ and $P \in G_x$.

On the regular part of X, one can define the Gauss map $\sigma : X_{\text{reg}} \longrightarrow G$ by

$$\sigma(x) = (x, T_x(X_{\text{reg}})).$$

Definition 5.1. The Nash transformation $\widetilde{X}$ is defined as the closure of the image of σ in G. It is equipped with a natural analytic projection $\nu : \widetilde{X} \longrightarrow X$.

$$\begin{array}{ccc} & & G \\ & \nearrow \sigma & \downarrow \\ X_{\text{reg}} & \hookrightarrow & M \end{array} \qquad \begin{array}{ccc} \widetilde{X} = \overline{\text{Im}\sigma} & \hookrightarrow & G \\ \nu \downarrow & & \downarrow \\ X & \hookrightarrow & M \end{array} \qquad (5)$$

In general, $\widetilde{X}$ is not smooth, nevertheless, it is an analytic variety and the restriction $\nu : \widetilde{X} \to X$ of the bundle projection $G \to M$ is analytic.

Let us denote by Θ the tautological bundle over G. The fiber Θ_P of the tautological bundle Θ over G, in a point $(x,P) \in G$, is the set of the vectors v of the n-plane P.

$$\Theta_P = \{v(x) \in T_xM \,:\, v(x) \in P, \quad x = \nu(P)\}$$

Let us define $\widetilde{\Theta} = \Theta|_{\widetilde{X}}$, then $\widetilde{\Theta}|_{\widetilde{X}_{\rm reg}} = T(X_{\rm reg})$ where $\widetilde{X}_{\rm reg} = \nu^{-1}(X_{\rm reg}) \cong X_{\rm reg}$ and

$$\widetilde{\Theta} = \Theta \times_G \widetilde{X} = \{(v(x), \tilde{x}) \in \Theta \times \widetilde{X} \,:\, v(x) \in \tilde{x}\}$$

$\tilde{x} \in \widetilde{X}$ is an n-complex plane in $T_x(M)$ and $x = \nu(\tilde{x})$.

One has a diagram:

$$\begin{array}{ccc} \widetilde{\Theta} & \hookrightarrow & \Theta \\ \downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & G \\ \nu\downarrow & & \downarrow \\ X & \hookrightarrow & M \end{array} \tag{6}$$

We will denote by $\widetilde{\Theta}^r$ the bundle of "r-frames" associated to $\widetilde{\Theta}$, i.e. the bundle whose fiber in a point (x,P) of $\widetilde{X}$ is the set of r linearly independent vectors in P.

The following lemma is fundamental for the understanding of the geometrical definition of the local Euler obstruction.

Lemma 5.1. *([BS], Proposition 9.1) A stratified vector field v on a subset $A \subset X$ admits a canonical lifting $\tilde{v}$ on $\nu^{-1}(A)$ as a section of $\widetilde{\Theta}$.*

One defines the map $\nu_* : \widetilde{\Theta} \to TM|_X$ by $\nu_*(v(x), \tilde{x}) = v(\nu(\tilde{x})) = v(x)$. One has a commutative diagram:

$$\begin{array}{ccc} \widetilde{\Theta} & \xrightarrow{\nu_*} & TM|_X \\ \tilde{v}\uparrow\downarrow & & v\uparrow\downarrow \\ \widetilde{X} & \xrightarrow{\nu} & X \end{array}$$

Let us recall that a *radial* vector field v in a neighbourhood of the point $a \in X$ is a stratified vector field so that there exists $\varepsilon_0 > 0$ such that for all ε, $0 < \varepsilon < \varepsilon_0$, the vector $v(x)$ is pointing outward the ball $B_\varepsilon = B_\varepsilon(a)$ over the boundary $S_\varepsilon = S_\varepsilon(a) = \partial B_\varepsilon$. By the Bertini-Sard theorem, S_ε is transverse to the strata V_α if ε is small enough, so the definition takes sense.

Theorem 5.1. *Theorem-Definition [BS] Let v be a radial vector field over $X \cap S_\varepsilon$ and $\tilde{v}$ the lifting of v over $\nu^{-1}(X \cap S_\varepsilon)$. The local Euler obstruction*

$\mathrm{Eu}_a(X)$ *is the obstruction to extend* $\tilde{v}$ *as a nowhere zero section of* $\tilde{\Theta}$ *over* $\nu^{-1}(X \cap B_\varepsilon)$, *evaluated on the orientation class* $O_{\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon)}$:

$$Eu_a(X) = Obs(\tilde{v}, \widetilde{\Theta}, \nu^{-1}(X \cap B_\varepsilon)).$$

Theorem 5.2. *([BS], Théorème 11.1) (Proportionality Theorem). Let* $v^{(r)}$ *be a stratified* r*-frame on the* $2p$*-cell* $d = d(s)$ *with an isolated singularity with index* $I(v^{(r)}, \hat{s})$ *at the barycenter* $\{\hat{s}\} = d \cap s$. *Let us denote by* $\tilde{v}^{(r)}$ *the lifting of* $v^{(r)}$ *on* $\nu^{-1}(\partial d \cap X)$. *The obstruction to the extension of* $\tilde{v}^{(r)}$ *as a section of* $\widetilde{\Theta}^r$ *on* $\nu^{-1}(d \cap X)$ *is equal to:*

$$Obs(\tilde{v}^{(r)}, \widetilde{\Theta}^r, \nu^{-1}(d \cap X)) = Eu_{\hat{s}}(X) \cdot I(v^{(r)}, \hat{s}).$$

6. MacPherson and Mather classes

Let us recall firstly some basic definitions. In this section, one considers the category of complex algebraic varieties.

A *constructible set* in a variety X is a subset obtained by finitely many unions, intersections and complements of subvarieties. A *constructible function* $\alpha : X \to \mathbb{Z}$ is a function such that $\alpha^{-1}(n)$ is a constructible set for all n. The constructible functions on X form a group denoted by $\mathbb{F}(X)$. If $A \subset X$ is a subvariety, we denote by $\mathbf{1}_A$ the characteristic function whose value is 1 over A and 0 elsewhere.

If X is triangulable, α is a constructible function if and only if there is a triangulation (K) of X such that α is constant on the interior of each simplex of (K). Such a triangulation of X is called α-adapted.

The correspondence $\mathbb{F} : X \to \mathbb{F}(X)$ defines a contravariant functor when considering the usual pull-back $f^* : \mathbb{F}(Y) \to \mathbb{F}(X)$ for a morphism $f : X \to Y$. One interesting fact is that it can be made a covariant functor when considering the pushforward f_* defined on characteristic functions by:

$$f_*(\mathbf{1}_A)(y) = \chi(f^{-1}(y) \cap A), \qquad y \in Y$$

for a morphism $f : X \to Y$, and linearly extended to elements of $\mathbb{F}(X)$. The following result was conjectured by Deligne and Grothendieck in 1969:

Theorem 6.1. *[MP] Let* $\mathbb{F}$ *be the covariant functor of constructible functions and let* $H_*(\ ;\mathbb{Z})$ *be the usual covariant* $\mathbb{Z}$*-homology functor. Then there exists a unique natural transformation*

$$c_* : \mathbb{F} \to H_*(\ ;\mathbb{Z})$$

satisfying $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ *if* X *is a manifold.*

The theorem means that for every algebraic complex variety, one has a functor $c_* : \mathbb{F}(X) \to H_*(X;\mathbb{Z})$ satisfying the following properties:

(1) $c_*(\alpha+\beta) = c_*(\alpha) + c_*(\beta)$ for α and β in $\mathbb{F}(X)$,
(2) $c_*(f_*\alpha) = f_*(c_*(\alpha))$ for $f : X \to Y$ morphism of algebraic varieties and $\alpha \in \mathbb{F}(Y)$,
(3) $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ if X is a manifold.

6.1. *Mather classes*

The first approach to the proof of the Deligne-Grothendieck's conjecture is given by the construction of Mather classes. Let $X \subset M$ a possibly singular algebraic complex variety embedded in a smooth one. Let us define the Nash transformation $\widetilde{X}$ of X, as in section 5.1 and the Nash bundle $\widetilde{\Theta}$ on $\widetilde{X}$.

Definition 6.1. The Mather class of X is defined by:

$$c^M(X) = \nu_*(c^*(\widetilde{\Theta}) \cap [\widetilde{X}])$$

where $c^*(\widetilde{\Theta})$ denotes the usual (total) Chern class of the bundle $\widetilde{\Theta}$ in $H^*(\widetilde{X})$ and the cap-product with $[\widetilde{X}]$ is the Poincaré duality homomorphism (in general not an isomorphism, see [Br1]).

The Mather classes do not satisfy the Deligne-Grothendieck's conjecture. One has to take into account the complexity of the singular variety along the strata. That is the role of the local Euler obstruction, used in MacPherson's construction.

6.2. *MacPherson classes*

The MacPherson's construction uses both the constructions of Mather classes and local Euler obstruction.

For a Whitney stratification $\{V_\alpha\}$ of X, we have the following lemma:

Lemma 6.1. *[MP] There are integers n_α such that, for every point $x \in X$, one has:*

$$\sum_\alpha n_\alpha \mathrm{Eu}_x(\overline{V_\alpha}) = 1.$$

Definition 6.2. [MP] The MacPherson class of X is defined by

$$c_*(X) = c_*(\mathbf{1}_X) = \sum_\alpha n_\alpha \; i_* c_*^M(\overline{V_\alpha})$$

where i denotes the inclusion $\overline{V_\alpha} \hookrightarrow X$.

Theorem 6.2. *([BS], see also [AB2]) The MacPherson class is image of the Schwartz class by the Alexander duality isomorphism [Br1]*

$$H^{2(m-r+1)}(M, M \setminus X) \xrightarrow{\cong} H_{2(r-1)}(X).$$

One calls Schwartz-MacPherson class the class $c_*(X)$ in $H_*(X)$.

Corollary 6.1. *The Schwartz-MacPherson class $c_{r-1}(X)$ is represented by the cycle:*

$$\sum_{\substack{s \subset X \\ \dim s = 2(r-1)}} I(v^{(r)}, \hat{s}) \; s$$

Corollary 6.2. *[BS] The Chern-Mather class $c^M_{r-1}(X)$ is represented by the cycle:*

$$\sum_{\substack{s \subset X \\ \dim s = 2(r-1)}} Eu_{\hat{s}}(X) I(v^{(r)}, \hat{s}) \; s$$

7. Schwartz-MacPherson classes of Thom spaces associated to embeddings

In this section and as a matter of example, we compute the Schwartz-MacPherson classes of the Thom spaces associated to Segre and Veronese embeddings. Results of this section have been obtained with Gerard Gonzalez-Sprinberg following ideas of Jean-Louis Verdier and Mark Goresky and with Gottfried Barthel, Karl-Heinz Fieseler and Ludger Kaup.

7.1. *The projective cone*

Let us consider an n-dimensional projective variety Y in $\mathbb{P}^m = \mathbb{P}\mathbb{C}^m$ and let us denote by H_Y the restriction of the hyperplane bundle of $\mathbb{P}^m$ to Y. We denote by Q the completed projective space of the total space of H_Y, i.e. $Q = \mathbb{P}(H_Y \oplus 1_Y)$ where 1_Y is the trivial bundle of complex rank 1 on Y. The canonical projection $p \colon Q \to Y$ admits two sections, zero and infinite, with images $Y_{(0)}$ and $Y_{(\infty)}$. The Thom space, i.e. the projective cone $X = KY$ is obtained as a quotient of Q by contraction of $Y_{(\infty)}$ in a point $\{s\}$. It is the Thom space associated to the bundle H_Y, with basis Y.

Let us consider $p \colon Q \to Y$ as a sphere bundle with fiber $\mathbb{S}^2$, subbundle of a bundle $\bar{p} \colon \bar{Q} \to Y$ with fiber the ball $\mathbb{B}^3$. We denote by $\theta_{\bar{Q}} \in H^3(\bar{Q}, Q)$ the associated Thom class; one has a Gysin exact sequence

$$\ldots \to H_{j+1}(Y) \to H_{j-2}(Y) \xrightarrow{\gamma} H_j(Q) \xrightarrow{p_j} H_j(Y) \to \ldots;$$

in which the Gysin map γ is the composition

$$H_{j-2}(Y) \xrightarrow{(\bar{p}_{j-2})^{-1}} H_{j-2}(\bar{Q}) \xrightarrow{(\cap\theta_{\bar{Q}})^{-1}} H_{j+1}(\bar{Q}, Q) \xrightarrow{\partial} H_j(Q)$$

and can be explicited in the following way: If ζ is a cycle in the class $[\zeta] \in H_{j-2}(Y)$, then $\gamma([\zeta])$ is the class of the cycle $p^{-1}(\zeta)$ in $H_j(Q)$.

Let us denote by π the canonical projection $\pi \colon Q \to KY$.

Proposition 7.1. *The Chern classes of Q and Y are related by the formula*

$$c_*(Q) = (1 + \eta_0 + \eta_\infty) \cap \gamma(c_*(Y)), \tag{7}$$

where $\eta_j := c^1(\mathcal{O}(Y_{(j)})) \in H^2(Q)$ for $j = 0,\ \infty$, and $\cap$ denotes the usual cap-product.

Proof. The vertical tangent bundle T_v of $p \colon Q \to Y$ is defined by the exact sequence:

$$0 \to T_v \to TQ \to p^*TY \to 0.$$

One has, in $H^*(Q)$

$$c^*(Q) = c^*(T_v) \cup c^*(p^*(TY)). \tag{8}$$

The sheaf of sections of the bundle T_v is the sheaf canonically associated to the divisor $Y_{(0)} + Y_{(\infty)}$, it is denoted by $\mathcal{O}_Q(Y_{(0)} + Y_{(\infty)})$. By Poincaré isomorphism in Y, the divisor $[Y_{(j)}] \in H_{2n}(Q)$ is identified to the class $\eta_j \in H^2(Q)$. The Chern class of T_v is

$$c^*(T_v) = 1 + \eta_0 + \eta_\infty.$$

By definition of the Gysin map γ, one has a commutative diagram

$$\begin{array}{ccc} H^i(Y) & \xrightarrow{\cap[Y]} & H_{2n-i}(Y) \\ \downarrow{\scriptstyle p^i} & & \downarrow{\scriptstyle \gamma} \\ H^i(Q) & \xrightarrow{\cap[Q]} & H_{2n+2-i}(Q) \end{array} \tag{9}$$

and by Poincaré duality

$$c^*(p^*(TY)) \cap [Q] = p^*(c^*(TY)) \cap [Q] = \gamma(c^*(TY) \cap [Y]) = \gamma(c_*(Y)). \tag{10}$$

Using formulae (8) and (10), one obtains the formula (7):

$$c_*(Q) = (1 + \eta_0 + \eta_\infty) \cap \gamma(c_*(Y))$$

□

7.2. *Schwartz-MacPherson classes of the projective cone*

Definition 7.1. We call homological projective cone and we denote by K the composition $K = \pi_*\gamma : H_{j-2}(Y) \to H_j(KY)$ for $j \geq 2$. For $j = 0$, i.e. for $H_{-2}(Y) = 0$, we let $K(0) := [a] \in H_0(KY)$ where $\{a\}$ is the vertex of the projective cone KY.

Let us remark that, for $j \geq 2$, K is an homomorphism.

Theorem 7.1. *Let $Y \subset \mathbb{P}^m$ be a projective variety and $\imath \colon Y \hookrightarrow KY$ the canonical inclusion into the projective cone KY on Y with vertex $\{a\}$. Let us denote also by $K : H_*(Y) \to H_{*+2}(KY)$ the homological projective cone, one has*

$$c_j(KY) = \imath_* c_j(Y) + Kc_{j-1}(Y), \tag{11}$$

where $Kc_{-1}(Y)$ denotes the class $[a] \in H_0(KY)$

Proof. Let $\mathbf{1}_Q$ be the constructible function which is the characteristic function of Q, one has

$$\pi_*(\mathbf{1}_Q)(x) = \begin{cases} \chi(Y), & \text{if } x = a \\ 1, & \text{elsewhere,} \end{cases}$$

i.e.

$$\pi_*(\mathbf{1}_Q) = \mathbf{1}_{KY} + (\chi(Y) - 1)\mathbf{1}_{\{a\}}.$$

On the one hand, one has

$$\pi_* c_*(\mathbf{1}_Q) = c_*(\pi_*(\mathbf{1}_Q)),$$

one obtains

$$\pi_* c_*(Q) = c_*(KY) + (\chi(Y) - 1)\,[a]. \tag{12}$$

On the other hand, from the formula (7) one obtains:

$$\pi_* c_*(Q) = \pi_*\gamma(c_{*-1}(Y)) + \pi_*(\eta_0 \cap \gamma(c_*(Y))) + \pi_*(\eta_\infty \cap \gamma(c_*(Y))). \tag{13}$$

Let $\iota_0 : Y \hookrightarrow Q$ and $\iota_\infty : Y \hookrightarrow Q$ be the inclusions of Y as zero and infinite sections of Q respectively. By definition of γ, one has for every cycle ζ in Y and for $j = 0$ or ∞

$$\eta_j \cap \gamma([\zeta]) = (\iota_j)_*([\zeta])$$

then

$$\pi_*(\eta_j \cap \gamma(c_*(Y))) = \pi_* \iota_{j*} c_*(\mathbf{1}_Y) = \pi_* c_*(\mathbf{1}_{Y_{(j)}}) = c_* \pi_*(\mathbf{1}_{Y_{(j)}}).$$

Let us denote by $\iota = \pi \circ \iota_0 : Y \hookrightarrow KY$ the natural inclusion of Y in KY, one has

$$\pi_*(\mathbf{1}_{Y_{(0)}}) = \mathbf{1}_{\iota(Y)} \text{ and } \pi_*(\mathbf{1}_{Y_{(\infty)}}) = \chi(Y)\mathbf{1}_{\{a\}}.$$

One obtains

$$\pi_*(\eta_0 \cap \gamma(c_*(Y))) = c_*(\mathbf{1}_{\iota(Y)}) = \iota_* c_*(Y),$$

and

$$\pi_*(\eta_\infty \cap \gamma(c_*(Y))) = \chi(Y)c_*(\mathbf{1}_{\{a\}}) = \chi(Y)[a],$$

where $[a]$ is the class of the vertex a in $H_0(KY)$. The comparizon of the formulae (12) and (13) gives:

$$c_*(KY) = \imath_* c_*(Y) + \pi_* \gamma c_{*-1}(Y) + [a],$$

and the Theorem 7.1. □

7.3. *Case of the Segre and Veronese embeddings*

The previous construction associates canonically a Thom space $X = KY$ to the embedding of a smooth variety Y in $\mathbb{P}^m$. As examples, let us consider the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, defined in homogeneous coordinates by

$$(x_0 : x_1) \times (y_0 : y_1) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1),$$

and the image of the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ defined by

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2).$$

Chern classes and intersection homology of these exemples have been computed in [BG1] and, in a more systematic way in [BFK] . In the case of the Segre embedding, let d_1 and d_2 two fixed lines belonging each to a system of generatrices of the quadric $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Let us denote by ω the canonical generator of $H^2(\mathbb{P}^1)$, one has $c^*(\mathbb{P}^1) = 1 + 2\omega$ and

$$c_*(Y) = c_*(\mathbb{P}^1 \times \mathbb{P}^1) = ([Y] + 2[d_1]) * ([Y] + 2[d_2]) = [Y] + 2([d_1] + [d_2]) + 4[y]$$

where y is a point in Y and where $*$ denotes the intersection of cycles or homology classes. One has

$$K(c_*(Y)) = [KY] + 2([Kd_1] + [Kd_2]) + 4[Ky].$$

Let us denote by $\sim$ the homology relation of cycles. In KY, one has ([BG1], 3):

$$Y \sim Kd_1 + Kd_2, \quad d_1 \sim d_2 \sim Ky, \quad y \sim a,$$

and, by Theorem 7.1

$$c_*(KY) = \underbrace{[KY]}_{H_6(KY)} + \underbrace{3([Kd_1] + [Kd_2])}_{H_4(KY)} + \underbrace{8[Ky]}_{H_2(KY)} + \underbrace{5[a]}_{H_0(KY)} ,$$

which is the result of [BG1].

In the case of the Veronese embedding, let d be a projective line in $Y = \mathbb{P}^2$, one has: $c^*(\mathbb{P}^2) = 1 + 3\omega + 3\omega^2$ where ω is the canonical generator of $H^2(\mathbb{P}_2)$, and is dual, by Poincaré isomorphism of the class $[d] \in H_2(\mathbb{P}_2)$. One has, by Poincaré duality

$$c_*(Y) = [Y] + 3[d] + 3[y]$$

where y is a point in Y. One has

$$K(c_*(Y)) = [KY] + 3[Kd] + 3[Ky]$$

such that, in KY, ([BG1], 3.b), $Y \sim 2Kd$, $d \sim 2Ka$ and $y \sim a$. One has

$$c_*(KY) = \underbrace{[KY]}_{H_6(KY)} + \underbrace{5[Kd]}_{H_4(KY)} + \underbrace{9[Ky]}_{H_2(KY)} + \underbrace{4[a]}_{H_0(KY)}$$

8. Polar varieties

A very nice and interesting historical introduction and complete bibliography for relation of characteristic classes with polar variieties can be found in the Teissier's paper [Te]. We use (and abuse of) it.

History of polar varieties began with Todd, in 1936. The basic idea is to consider what Todd calls the "Polar Loci" of a projective variety $X \subset \mathbb{P}^n$. It turns out that certain formal linear combinations of the intersections of general polar loci of X with general linear sections (of various dimensions) of X are invariants of X, i.e. do not depend upon the projective embedding of X and the choices of polar loci and linear sections.

More precisely, given a non singular $d-1$-dimensional variety X in $\mathbb{P}^{N-1}$, for a linear subspace $L \subset \mathbb{P}^{N-1}$ of dimension $N - d + k - 2$, i.e. of codimension $d - k + 1$, let us set

$$P_k(X; L) = \{x \in X \mid \dim(T_{X,x} \cap L) \geq k - 1\}.$$

this is the polar variety of X associated to L. If L is general, it is either empty or the (pure) codimension in X is k.

Todd shows that the following formal linear combinations of varieties

$$X_k = \sum_{j=0}^{k} (-1)^j \binom{d-k+j+1}{j} P_{k-j}(X;L) \cap H_j$$

where H_j is a linear subspace of codimension j, are independent of all the choices made and of the embedding of X in a projective space, provided that the L's and the H_j's have been chosen general enough.

The linear combination is at first sight a rather awkward object to deal with. The idea is that X_k represents a generalized variety of codimension k in X, also any numerical character $e(Y)$ associated to algebraic varieties Y and which is additive in the sense that $e(Y_1 \cup Y_2) = e(Y_1) + e(Y_2)$ whenever Y_1 and Y_2 have the same dimension, can be extended by linearity to such a generalized variety. Given a partition $i_1, \ldots, i_k$ of $d-1$, the intersection numbers

$$(X_{i_1} * \ldots * X_{i_k})$$

are well defined since the intersection of the corresponding varieties is zero dimensional. Here each X_i is assumed to be a general representative obtained by taking general and independant linear spaces. The intersection numbers depend only upon the structure of X as an algebraic variety.

Todd considered an equivalence relation between varieties, called rational equivalence. One of the main results of Todd is that the numbers $X_{i_1} * \ldots * X_{i_k}$ depend only upon X, that they are independent invariants and the arithmetic genus of X is a function of them.

The topological Euler-Poincaré characteristic of X can be computed to show the equality

$$\chi(X) = \deg X_d = \sum_{j=0}^{d} (j+1)(P_{d-j}(X).H_j)$$

where $(a.b)$ denotes the intersection number. In this case since we intersect with a linear space of complementary dimension, it is the degree of the projective variety $P_{d-j}(X)$.

After Nakano, Hirzebruch, Serre, Gamkrelidze, the invariants X_k of Todd (or rather their cohomology classes) coincide with the Chern classes of the tangent bundle of X.

9. Chern classes via polar varieties (smooth case)

The construction of Chern classes using Schubert varieties was already present in Chern's original paper. This construction was emphasized by

Gamkrelidze in [Ga1] and [Ga2].

The Schubert cell decomposition of the Grassmann manifold $G = G(n, m)$ of n-planes in $\mathbb{C}^m$ has been described by Ehresmann [Eh] and has been used by Chern to give an alternative definition of his characteristic classes. Let $(\mathcal{D})$

$$\{0\} = D_m \subset D_{m-1} \subset \cdots \subset D_1 \subset D_0 = \mathbb{C}^m \tag{14}$$

be a flag in $\mathbb{C}^m$, with $\operatorname{codim}_{\mathbb{C}} D_j = j$.

For each integer k, with $0 \leq k \leq n$, the k-th Schubert variety associated to (14), defined by

$$M_k(\mathcal{D}) = \{T \in G(n, m) : \dim(T \cap D_{n-k+1}) \geq k\}$$

is an algebraic subvariety of $G(n, m)$ of pure codimension k. The inequality condition is equivalent to saying that T and D_{n-k+1} do not span $\mathbb{C}^m$.

Let θ^n be the universal bundle over $G(n, m)$. The cycle $(-1)^k M_k(\mathcal{D})$ represents the image, under the Poincaré duality isomorphism, of the Chern class $c^k(\theta^n) \in H^{2k}(G(n, m))$. If V is an n-dimensional complex analytic manifold and $f : V \to G(n, m)$ is the classifying map for TV, i.e. such that $TV \cong f^*(\theta^n)$, then the cohomological Chern classes of V are $c^k(V) = c^k(TV) = f^*(c^k(\theta^n))$ (see [MS]).

Let us now consider the projective situation. We denote by $\tilde{G}(n, m)$ the Grassmann manifold of n-dimensional linear subspaces in $\mathbb{P}^m$. We fix a flag of projective linear subspaces $(\mathcal{L})$

$$L_m \subset L_{m-1} \subset \cdots \subset L_1 \subset L_0 = \mathbb{P}^m \tag{15}$$

where $\operatorname{codim}_{\mathbb{C}} L_j = j$. The k-th Schubert variety associated to $\mathcal{L}$ is defined by

$$M_k(\mathcal{L}) = \{\tilde{T} \in \tilde{G}(n, m) : \dim(\tilde{T} \cap L_{n-k+2}) \geq k - 1\}$$

Let us remark that we always have $\dim(\tilde{T} \cap L_{n-k+2}) \geq k - 2$. The Schubert variety $M_k(\mathcal{L})$ has codimension k in $\tilde{G}(n, m)$.

Let us denote $N = nm = \dim_{\mathbb{C}} \tilde{G}(n, m)$ and fix $0 \leq \alpha \leq m$. The Schubert variety

$$\begin{aligned} M_k^{N-\alpha} &= \{(x, \tilde{T}) : x \in L_{\alpha-k},\ x \in \tilde{T},\ \dim(\tilde{T} \cap L_{n-k+2}) \geq k - 2\} \\ &= L_{\alpha-k} \cap M_k(\mathcal{L}) \end{aligned} \tag{16}$$

is the intersection of $M_k(\mathcal{L})$ with a general $(\alpha - k)$-codimensional plane and it has codimension α in $\tilde{G}(n,m)$. The (homological) Chern classes of $\tilde{G}(n,m)$ are

$$c_{N-\alpha}(\tilde{G}(n,m)) = \sum_{k=0}^{\alpha}(-1)^k \binom{n-\alpha+1}{n-k+1} M_k^{N-\alpha}. \tag{17}$$

Let us now consider the case of an n-projective manifold $V \subset \mathbb{P}^m$.

The k-th polar variety is defined by

$$P_k = \{x \in V : \dim(T_x(V) \cap L_{n-k+2}) \geq k-1\},$$

where $T_x(V)$ is the projective tangent space to V at x. For L_{n-k+2} sufficiently general, the codimension of P_k in V is equal to k. Also, the class $[P_k]$ of P_k modulo rational equivalence in the Chow group $A_{n-k}(V)$ does not depend on L_{n-k+2} for L_{n-k+2} sufficiently general. This class is called the k-th polar class of V.

Let $\gamma : V \to \tilde{G}(n,m)$ be the Gauss map, i.e. the map defined by

$$\gamma(x) = T_x(V) \subset \mathbb{P}^m.$$

Then

$$P_k = \gamma^{-1}(M^k(\mathcal{L})).$$

The relation between Chern classes and Todd invariant has been described by Nakano [Na], Gamkrelidze [Ga1], [Ga2] and indirectly by Hirzebruch and Serre.

If $\mathcal{L} = \mathcal{O}_{\mathbb{P}^m}(1)|_V$, then one obtains the Todd formula (compare with (3)):

$$c_{n-\alpha}(V) = \sum_{k=0}^{\alpha}(-1)^k \binom{n-\alpha+1}{n-k+1} c^1(\mathcal{L})^{\alpha-k} \cap [P_k] \tag{18}$$

where the cap-product with $c^1(\mathcal{L})^{\alpha-k}$ is equivalent to the intersection with a general $(\alpha-k)$-codimensional plane.

10. Mather classes via polar varieties

The Mather classes have be defined in §6.1. One can provide an alternative definition, by using polar varieties. Let us firstly consider the situation of an affine variety $X^n \subset \mathbb{C}^m$. For a general flag $\mathcal{D}$ as in (14), one define (see diagramme (9))

$$\begin{array}{ccccc} & \tilde{X} & \hookrightarrow G(n,m)\times\mathbb{C}^m & \xrightarrow{\pi_1} & G(n,m) \\ & \sigma\nearrow\downarrow\nu & & \downarrow\pi_2 & \\ X_{\text{reg}} \hookrightarrow & X \hookrightarrow & & \mathbb{C}^m & \end{array}$$

and we denote by $\tilde{\gamma} = \pi_1|_{\tilde{X}} : \tilde{X} \to G(n,m)$ the Gauss map.

Let us define the following analytic subspace of X [LT]:

$$N_k(\mathcal{D}) = \nu \circ \tilde{\gamma}^{-1}(M_k(\mathcal{D})) = \overline{\nu(\tilde{\gamma}^{-1}(M_k(\mathcal{D})) \cap \sigma(X_{\text{reg}}))}.$$

We will say that the flag $\mathcal{D}$ is good if it is sufficiently general, i.e. if $\tilde{\gamma}$ is transverse to the strata

$$M_{k,i}(\mathcal{D}) = \{W \in G(n,m) : \text{codim}(W + D_{n-k+i-1}) = k+1\}$$

of $M_k(\mathcal{D})$. In that case, the cycle $N_k(\mathcal{D})$ is well defined and independent of the choice of the good flag, it is called the polar variety (Lê - Teissier).

If the flag $\mathcal{D}$ is good, and still in the affine situation, let $\pi : X \to \mathbb{C}^{n-k+1}$ be the restriction to X of a linear projection with kernel D_{n-k+1}, then $N_k(\mathcal{D})$ is the closure (in X) of the critical locus of the restriction of π to X_{reg} [LT].

Let us consider now the projective case, the Mather class can also be defined using polar varieties [LT]. Let us denote by $X^n \subset \mathbb{P}^m$ a projective variety, we define the k-th polar variety P_k as the closure of

$$\{x \in X_{\text{reg}} : \dim(T_x(X_{\text{reg}}) \cap L_{n-k+2}) \geq k-1\}.$$

Then one has [Pi2]:

$$c^M_{n-s}(X) = \sum_{k=0}^{s} (-1)^k \binom{n-s+1}{n-k+1} c^1(\mathcal{L})^{s-k} \cap [P_k],$$

where $\mathcal{L} = \mathcal{O}_{\mathbb{P}^m}(1)|_X$.

That provides an expression of Schwartz-MacPherson classes in terms of polar varieties. In particular cases, the Fulton and Milnor classes can also be expressed in terms of polar varieties (see [AB1]).

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