Financial Markets with Asymmetric Information: Information Drift, Additional Utility and Entropy

Stefan Ankirchner and Peter Imkeller

Institut für Mathematik, Humboldt-Universität zu Berlin,
Unter den Linden 6, 10099 Berlin, Germany

We review a general mathematical link between utility and information theory appearing in a simple financial market model with two kinds of small investors: insiders, whose extra information is stored in an enlargement of the less informed agents' filtration. The insider's expected logarithmic utility increment is described in terms of the information drift, i.e. the drift one has to eliminate in order to perceive the price dynamics as a martingale from his perspective. We describe the information drift in a very general setting by natural quantities expressing the conditional laws of the better informed view of the world. This on the other hand allows to identify the additional utility by entropy related quantities known from information theory.
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1. Introduction

A simple mathematical model of two small agents on a financial market one of which is better informed than the other has attracted much attention in recent years. Their information is modelled by two different filtrations: the less informed agent has the $\sigma-$field $\mathcal{F}_t$, corresponding to the natural evolution of the market up to time t at his disposal, while the better informed insider knows the bigger $\sigma-$field $\mathcal{G}_t \supset \mathcal{F}_t$. Here is a short selection of some among many more papers dealing with this model. Investigation techniques concentrate on martingale and stochastic control theory, and methods of enlargement of filtrations (see Yor , Jeulin , Jacod in [22]), starting with the conceptual paper by Duffie, Huang [12]. The model

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is successively studied on stochastic bases with increasing complexity: e.g. Karatzas, Pikovsky [24] on Wiener space, Grorud, Pontier [15] allow Poissonian noise, Biagini and Oksendal [7] employ anticipative calculus techniques. In the same setting, Amendinger, Becherer and Schweizer [1] calculate the value of insider information from the perspective of specific utilities. Baudoin [6] introduces the concept of weak additional information, while Campi [8] considers hedging techniques for insiders in the incomplete market setting. Many of the quoted papers deal with the calculation of the better informed agent's additional utility.

In Amendinger *et al.* [2], in the setting of initial enlargements, the additional expected logarithmic utility is linked to information theoretic concepts. It is computed in terms of an energy-type integral of the *information drift* between the filtrations (see [18]), and subsequently identified with the Shannon entropy of the additional information. Also for initial enlargements, Gasbarra, Valkeila [14] extend this link to the Kullback-Leibler information of the insider's additional knowledge from the perspective of Bayesian modelling. In the environment of this utility-information paradigm the papers [16], [19], [17], [18], Corcuera *et al.* [9], and Ankirchner *et al.* [5] describe additional utility, treat arbitrage questions and their interpretation in information theoretic terms in increasingly complex models of the same base structure. Utility concepts different from the logarithmic one correspond on the information theoretic side to the generalized entropy concepts of $f-$divergences.

In this paper we review the main results about the interpretation of the better informed trader's additional utility in information theoretic terms mainly developed in [4], concentrating on the logarithmic case. This leads to very basic problems of stochastic calculus in a very general setting of enlargements of filtrations: to ensure the existence of regular conditional probabilities of σ–fields of the larger with respect to those of the smaller filtration, we only eventually assume that the base space be standard Borel. In Section 2, we calculate the logarithmic utility increment in terms of the information drift process. Section 3 is devoted to the calculation of the information drift process by the Radon-Nikodym densities of the stochastic kernel in an integral representation of the conditional probability process and the conditional probability process itself. For convenience, before proceeding to the more abstract setting of a general enlargement, the results are given in the initial enlargement framework first. In Section 4 we finally provide the identification of the utility increment in the general enlargement setting with the information difference of the two filtrations in terms of Shannon entropy concepts.

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2. Additional Logarithmic Utility and Information Drift

Let us first fix notations for our simple financial market model. First of all, to simplify the exposition, we assume that the trading horizon is given by $T = 1$. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{0\le t\le 1}$. We consider a financial market with one non-risky asset of interest rate normalized to 0, and one risky asset with price X_t at time $t \in [0,1]$. We assume that X is a continuous $(\mathcal{F}_t)-$semimartingale with values in $\mathbb{R}$ and write $\mathcal{A}$ for the set of all $X-$integrable and $(\mathcal{F}_t)-$predictable processes θ such that $\theta_0 = 0$. If $\theta \in \mathcal{A}$, then we denote by $(\theta \cdot S)$ the usual stochastic integral process. For all $x > 0$ we interpret

$$x + (\theta \cdot X)_t,\ 0 \le t \le 1,$$

as the wealth process of a trader possessing an initial wealth x and choosing the investment strategy θ on the basis of his knowledge horizon corresponding to the filtration $(\mathcal{F}_t)$.
Throughout this paper we will suppose the preferences of the agents to be described by the logarithmic utility function.

Therefore it is natural to suppose that the traders' total wealth has always to be strictly positive, i.e. for all $t \in [0,1]$

$$V_t(x) = x + (\theta \cdot X)_t > 0 \text{ a.s.} \tag{1}$$

Strategies θ satisfying Eq. (1) will be called $x-$*superadmissible*. The agents want to maximize their expected logarithmic utility from terminal wealth. So we are interested in the exact value of

$$u(x) = \sup\{E \log(V_1(x)) : \theta \in \mathcal{A},\ x - \text{superadmissible}\}.$$

Sometimes we will write $u_{\mathcal{F}}(x)$, in order to stress the underlying filtration. The expected logarithmic utility of the agent can be calculated easily, if one has a semimartingale decomposition of the form

$$X_t = M_t + \int_0^t \eta_s \, d\langle M, M\rangle_s, \tag{2}$$

where η is a predictable process. Such a decomposition has to be expected in a market in which the agent trading on the knowledge flow $(\mathcal{F}_t)$ has no arbitrage opportunities. In fact, if X satisfies the property (NFLVR), then it may be decomposed as in Eq. (2) (see [10]). It is shown in [3] that finiteness of $u(x)$ already implies the validity of such a decomposition. Hence a decomposition as in (2) may be given even in cases where arbitrage exists. We state Theorem 2.9 of [5], in which the basic relationship between optimal logarithmic utility and information related quantities becomes visible.

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Proposition 2.1. *Suppose X can be decomposed into $X = M + \eta \cdot \langle M, M\rangle$. Then for any $x > 0$ the following equation holds*

$$u(x) = \log(x) + \frac{1}{2} E \int_0^1 \eta_s^2 \, d\langle M, M\rangle_s. \tag{3}$$

Let us give the core arguments proving this statement in a particular setting, and for initial wealth $x = 1$. Suppose that X is given by the linear sde

$$\frac{dX_t}{X_t} = \alpha_t dt + dW_t,$$

with a one-dimensional Wiener process W, and assume that the small trader's filtration $(\mathcal{F}_t)$ is the (augmented) natural filtration of W. Here α is a progressively measurable mean rate of return process which satisfies $\int_0^1 |\alpha_t| dt < \infty$, $P-$a.s. Let us denote investment strategies per unit by π, so that the wealth process $V(x)$ is given by the simple linear sde

$$\frac{dV_t(x)}{V_t(x)} = \pi_t \cdot \frac{dX_t}{X_t}.$$

It is obviously solved by the formula

$$V_t(x) = \exp[\int_0^t \pi_s \, dW_s - \frac{1}{2} \int_0^t \pi_s^2 \, ds + \int_0^t \pi_s \, \alpha_s \, ds].$$

Due to the local martingale property of $\int_0^t \pi_s \, dW_s, t \in [0, 1]$, the expected logarithmic utility of the regular trader is deduced from the maximization problem

$$u_{\mathcal{F}}(1) = \max_{\pi} E[\int_0^1 \pi_s \, \alpha_s \, ds - \frac{1}{2} \int_0^1 \pi_s^2 \, ds]. \tag{4}$$

The maximization of

$$\pi \mapsto \int_0^1 \pi_s \, \alpha_s \, ds - \frac{1}{2} \int_0^1 \pi_s^2 \, ds$$

for given processes α is just a more complex version of the one-dimensional maximization problem for the function

$$\pi \mapsto \pi \, \alpha - \frac{1}{2} \pi^2$$

with $\alpha \in \mathbb{R}$. Its solution is obtained by the critical value $\pi = \alpha$ and thus

(5) $$u_{\mathcal{F}}(1) = \frac{1}{2}E[\int_0^1 \alpha_s^2 ds].$$

This confirms the claim of Proposition 2.1.

This proposition motivates the following definition.

Definition 2.1. A filtration $(\mathcal{G}_t)$ is called *finite utility filtration* for X, if X is a $(\mathcal{G}_t)$−semimartingale with decomposition $dX = dM + \zeta \cdot d\langle M, M\rangle$, where ζ is $(\mathcal{G}_t)$−predictable and belongs to $L^2(M)$, i.e. $E \int_0^1 \zeta^2 \, d\langle M, M\rangle < \infty$. We write

$$\mathbb{F} = \{(\mathcal{H}_t) \supset (\mathcal{F}_t) \big| (\mathcal{H}_t) \text{ is a finite utility filtration for X}\}.$$

We now compare two traders who take their portfolio decisions not on the basis of the same filtration, but on the basis of different information flows represented by the filtrations $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ respectively. Suppose that both filtrations $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ are finite utility filtrations. We denote by

(6) $$X = M + \zeta \cdot \langle M, M\rangle$$

the semimartingale decomposition with respect to $(\mathcal{G}_t)$ and by

(7) $$X = N + \beta \cdot \langle N, N\rangle$$

the decomposition with respect to $(\mathcal{H}_t)$. Obviously,

$$\langle M, M\rangle = \langle X, X\rangle = \langle N, N\rangle$$

and therefore the utility difference is equal to

$$u_{\mathcal{H}}(x) - u_{\mathcal{G}}(x) = \frac{1}{2}E \int_0^1 (\beta^2 - \zeta^2) \, d\langle M, M\rangle.$$

Furthermore, Eqs. (6) and (7) imply

(8) $$M = N - (\zeta - \beta) \cdot \langle M, M\rangle \quad a.s.$$

If $\mathcal{G}_t \subset \mathcal{H}_t$ for all $t \geq 0$, Eq. (8) can be interpreted as the semimartingale decomposition of M with respect to $(\mathcal{H}_t)$. In this case one can show that the utility difference depends only on the process $\mu = \zeta - \beta$. In fact,

$$\begin{aligned} u_{\mathcal{H}}(x) - u_{\mathcal{G}}(x) &= \frac{1}{2}E \int_0^1 (\beta^2 - \zeta^2) \, d\langle M, M\rangle \\ &= \frac{1}{2}E(\int_0^1 \mu^2 \, d\langle M, M\rangle) - E(\int_0^1 \mu \, \zeta \, d\langle M, M\rangle) \\ &= \frac{1}{2}E(\int_0^1 \mu^2 \, d\langle M, M\rangle). \end{aligned}$$

The last equation is due to the fact that $N - M = \int \mu \, d\langle M, M\rangle$ is a martingale with respect to $(\mathcal{H}_t)$, and ζ is adapted to this filtration. It is therefore natural to relate μ to a transfer of information.

Definition 2.2. Let $(\mathcal{G}_t)$ be a finite utility filtration and $X = M + \zeta \cdot \langle M, M\rangle$ the Doob-Meyer decomposition of X with respect to $(\mathcal{G}_t)$. Suppose that $(\mathcal{H}_t)$ is a filtration such that $\mathcal{G}_t \subset \mathcal{H}_t$ for all $t \in [0, 1]$. The $(\mathcal{H}_t)$–predictable process μ satisfying

$$M - \int_0^{\cdot} \mu_t \, d\langle M, M\rangle_t \quad \text{is a} \quad (\mathcal{H}_t) - \text{local martingale}$$

is called *information drift* (see [18]) of $(\mathcal{H}_t)$ with respect to $(\mathcal{G}_t)$.

The following proposition summarizes the findings just explained, and relates the information drift to the expected logarithmic utility increment.

Proposition 2.2. *Let $(\mathcal{G}_t)$ and $(\mathcal{H}_t)$ be two finite utility filtrations such that $\mathcal{G}_t \subset \mathcal{H}_t$ for all $t \in [0, 1]$. If μ is the information drift of $(\mathcal{H}_t)$ w.r.t. $(\mathcal{G}_t)$, then we have*

$$u_{\mathcal{H}}(x) - u_{\mathcal{G}}(x) = \frac{1}{2} E \int_0^1 \mu^2 \, d\langle M, M\rangle.$$

3. The Information Drift and the Law of Additional Information

In this section we aim at giving a description of the information drift between two filtrations in terms of the laws of the information increment between two filtrations. This is done in two steps. First, we shall consider the simplest possible enlargement of filtrations, the well known *initial enlargement*. In a second step, we shall generalize the results available in the initial enlargement framework. In fact, we consider general pairs of filtrations, and only require the state space to be standard Borel in order to have conditional probabilities available.

3.1 Initial enlargement, Jacod's condition

In this setting, the additional information in the larger filtrations is at all times during the trading interval given by the knowledge of a random variable which, from the perspective of the smaller filtration, is known only at the end of the trading interval. To establish the concepts in fair simplicity, we again assume that the smaller underlying filtration $(\mathcal{F}_t)$ is the augmented filtration of a one-dimensional Wiener process W. Let G be an $\mathcal{F}_1$–measurable random variable, and let

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G), \quad t \in [0, 1].$$

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Suppose that $(\mathcal{G}_t)$ is small enough so that W is still a semimartingale with respect to this filtration. More precisely, suppose that there is an information drift μ^G such that

$$\int_0^1 |\mu_s^G|\, ds < \infty \quad \text{P-a.s.},$$

and such that

$$W = \tilde{W} + \int_0^{\cdot} \mu_s^G\, ds \tag{9}$$

with a $(\mathcal{G}_t)-$ Brownian motion $\tilde{W}$. To clarify the relationship between the additional information G and the information drift μ^G, we shall work under a condition concerning the laws of the additional information G which has been used as a standing assumption in many papers dealing with *grossissement de filtrations.* See Yor [27], [26], [28], Jeulin [21]. The condition was essentially used in the seminal paper by Jacod [20], and in several equivalent forms in Föllmer and Imkeller [13]. To state and exploit it, let us first mention that all stochastic quantities appearing in the sequel, often depending on several parameters, can always be shown to possess measurable versions in all variables, and progressively measurable versions in the time parameter (see Jacod [20]).

Denote by P^G the law of G, and for $t \in [0,1], \omega \in \Omega$, by $P_t^G(\omega, dl)$ the regular conditional law of G given $\mathcal{F}_t$ at $\omega \in \Omega$. Then the condition, which we will call *Jacod's condition*, states that

(10) $P_t^G(\omega, dg)$ is absolutely continuous with respect to $P^G(dg)$ for $P-$ a.e. $\omega \in \Omega$.

Also its reinforcement

$$P_t^G(\omega, dg) \quad \text{is equivalent to} \quad P^G(dg) \quad \text{for } P-\text{ a.e.}\, \omega \in \Omega, \tag{11}$$

will be of relevance. Denote the Radon-Nikodym density process of the conditional laws with respect to the law by

$$p_t(\omega, g) = \frac{dP_t^G(\omega, \cdot)}{dP^G}(g), \quad g \in \mathbb{R}, \omega \in \Omega.$$

By the very definition, $t \mapsto P_t(\cdot, dg)$ is a local martingale with values in the space of probability measures on the Borel sets of $\mathbb{R}$. This is inherited to $t \mapsto p_t(\cdot, g)$ for (almost) all $g \in \mathbb{R}$. Let the representations of these martingales with respect to the $(\mathcal{F}_t)-$Wiener process W be given by

$$p_t(\cdot, g) = p_0(\cdot, g) + \int_0^t k_u^g\, dW_u, \quad t \in [0,1]$$

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with measurable kernels k. To calculate the information drift in terms of these kernels, take $s, t \in [0,1], s \leq t$, and let $A \in \mathcal{F}_s$ and a Borel set B on the real line determine the typical set $A \cap G^{-1}[B]$ in a generator of $\mathcal{G}_s$. Then we may write

$$\begin{aligned}
E([W_t - W_s]\, 1_A\, 1_B(G)) &= E(\int_B 1_A\, [W_t - W_s]\, P_t^G(\cdot, dg)) \\
&= \int_B E(1_A\, [W_t - W_s]\, [p_t - p_s](\cdot, g))\, P^G(dg) \\
&= \int_B E(1_A \int_s^t k_u^g\, du)\, P^G(dg) \\
&= \int_B E(1_A \int_s^t \frac{k_u^g}{p_u(\cdot, g)}\, p_u(\cdot, g)\, du)\, P^G(dg) \\
&= \int_B E(1_A \int_s^t \frac{k_u^g}{p_u(\cdot, g)}\, du\, p_t(\cdot, g))\, P^G(dg) \\
&= E(\int_B 1_A \frac{k_u^g}{p_u(\cdot, g)}\, P_t^G(\cdot, dg)) \\
&= E(1_A\, 1_B(G) \int_s^t \frac{k_u^g}{p_u(\cdot, g)}|_{g=G}\, du).
\end{aligned}$$

The bottom line of this chain of arguments shows that

$$\tilde{W} = W - \int_0^{\cdot} \frac{k_u^l}{p_u(\cdot, g)}|_{g=G}\, du$$

is a $(\mathcal{G}_t)$−martingale, hence a $(\mathcal{G}_t)$−Brownian motion provided that $\int_0^1 |\frac{k_u^g}{p_u(\cdot,g)}|_{g=G}|\, du < \infty$ P−a.s.. This completes the deduction of an explicit formula for the information drift of G in terms of quantities related to the law of G in which we use the common oblique bracket notation to denote the covariation of two martingales (for more details see Jacod [20]).

Theorem 3.1. *Suppose that Jacod's condition (10) is satisfied, and furthermore that*

$$\mu_t^G = \frac{k_t^g}{p_t(\cdot, g)}|_{g=G} = \frac{\frac{d}{dt}\langle p(\cdot, g), W\rangle_t}{p_t(\cdot, g)}|_{g=G}, \quad t \in [0,1], \tag{12}$$

satisfies

$$\int_0^1 |\mu_u^G|\, du < \infty \quad P\text{−}a.s.. \tag{13}$$

Then

$$W = \tilde{W} + \int_0^{\cdot} \mu_s^G \, ds$$

is a $\mathbb{G}-$*semimartingale with a* $\mathbb{G}-$*Brownian motion* $\tilde{W}$*.*

To see how restrictive condition (10) may be, let us illustrate it by looking at two possible additional information variables G.

Example 1:
Let $\epsilon > 0$ and suppose that the stock price process is a regular diffusion given by a stochastic differential equation with bounded volatility σ and drift α, $\sigma_t = \sigma(X_t), t \in [0,1]$, where σ is a smooth function without zeroes. Let $G = X_{1+\epsilon}$. Then in particular X is a time homogeneous Markov process with transition probabilities $P_t(x, dy), x \in \mathbb{R}_+, t \in [0,1]$, which are equivalent with Lebesgue measure on $\mathbb{R}_+$. For $t \in [0,1]$, the regular conditional law of G given $\mathcal{F}_t$ is then given by $P_{1+\epsilon-t}(X_t, dy)$, which is equivalent with the law of G. Hence in this case, even the strong version of Jacod's hypothesis (11) is verified.

Example 2:
Let

$$G = \sup_{t \in [0,1]} W_t.$$

To abbreviate, denote for $t \in [0,1]$

$$G_t = \sup_{0 \le s \le t} W_s, \quad \tilde{G}_{1-t} = \sup_{t \le s \le 1} (W_s - W_t).$$

Finally, let p_{1-t} denote the density function of $\tilde{G}_{1-t}$. Then we may write for every $t \in [0,1]$

$$G = G_t \vee [W_t + \tilde{G}_{1-t}]. \tag{14}$$

Now G_t is $\mathcal{F}_t-$measurable, independent of $\tilde{G}_{1-t}$, and therefore for Borel sets A on the real line we have

$$P_t^G(\cdot, A) = \int_{-\infty}^{G_t - W_t} p_{1-t}(y) dy \cdot \delta_{G_t}(A) + \int_{A \cap [G_t - W_t, \infty[} p_{1-t}(y) dy. \tag{15}$$

Note now that the family of Dirac measures in the first term of (15) is supported on the random points G_t, and that the law of G_t is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$. Hence there cannot be any common reference measure equivalent with δ_{G_t} $P-$a.s. Therefore in this example Jacod's condition is violated.

It can be seen that there is an extension of Jacod's framework into which example 2 still fits. This is explained in [18], [19], and resides on a version of Malliavin's calculus for measure valued random elements. It yields a description of the information drift in terms of traces of logarithmic Malliavin gradients of conditional laws of G. We shall not give details here, since we will go a considerable step ahead of this setting. In fact, in the following subsection we shall further generalize the framework beyond the Wiener space setting.

3.2 General enlargement

Assume again that the price process X is a semimartingale of the form

$$X = M + \eta \cdot \langle M, M \rangle$$

with respect to a finite utility filtration $(\mathcal{F}_t)$. Moreover, let $(\mathcal{G}_t)$ be a filtration such that $\mathcal{F}_t \subset \mathcal{G}_t$, and let α be the information drift of $(\mathcal{G}_t)$ relative to $(\mathcal{F}_t)$. We shall explain how the description of α by basic quantities related to the conditional probabilities of the larger $\sigma-$algebras $\mathcal{G}_t$ with respect to the smaller ones $\mathcal{F}_t, t \geq 0$ generalizes from the setting of the previous subsection. Roughly, the relationship is as follows. Suppose for all $t \geq 0$ there is a regular conditional probability $P_t(\cdot, \cdot)$ of $\mathcal{F}$ given $\mathcal{F}_t$, which can be decomposed into a martingale component orthogonal to M, plus a component possessing a stochastic integral representation with respect to M with a kernel function $k_t(\cdot, \cdot)$. Then, provided α is square integrable with respect to $d\langle M, M \rangle \otimes P$, the kernel function at t will be a signed measure in its set variable. This measure is absolutely continuous with respect to the conditional probability itself, if restricted to $\mathcal{G}_t$, and α coincides with their Radon-Nikodym density.

As a remarkable fact, this relationship also makes sense in the reverse direction. Roughly, if absolute continuity of the stochastic integral kernel with respect to the conditional probabilities holds, and the Radon-Nikodym density is square integrable, the latter turns out to provide an information drift α in a Doob-Meyer decomposition of X in the larger filtration.

To provide some details of this fundamental relationship, we need to work with conditional probabilities. We therefore assume that $(\Omega, \mathcal{F}, P)$ is standard Borel (see [23]). Unfortunately, since we have to apply standard techniques of stochastic analysis, the underlying filtrations have to be assumed completed as a rule. On the other hand, for handling conditional probabilities it is important to have countably generated conditioning $\sigma-$fields. For this reason we shall use small versions $(\mathcal{F}_t^0)$, $(\mathcal{G}_t^0)$ which are countably generated, and big versions $(\mathcal{F}_t)$, $(\mathcal{G}_t)$ that are obtained as the smallest right-continuous and completed filtrations containing the small ones, and thus satisfy the usual conditions of stochastic calculus. We further suppose that $\mathcal{F}_0$ is trivial and that every $(\mathcal{F}_t)-$local martingale has a continuous modification, and of course $\mathcal{F}_t^0 \subset \mathcal{G}_t^0$ for all $t \geq 0$. We assume that M a $(\mathcal{F}_t^0)-$local martingale. The regular conditional probabilities relative to the $\sigma-$algebras $\mathcal{F}_t^0$ are denoted by P_t. For any set $A \in \mathcal{F}$ the process

$$(t, \omega) \mapsto P_t(\omega, A)$$

is an $(\mathcal{F}_t^0)-$martingale with a continuous modification adapted to $(\mathcal{F}_t)$ (see e.g. Theorem 4, Chapter VI in [11]). We may assume that the processes $P_t(\cdot, A)$ are modified in such a way that $P_t(\omega, \cdot)$ is a measure on $\mathcal{F}$ for P_M-almost all (ω, t), where P_M is given on $\Omega \times [0, 1]$ defined by $P_M(\Gamma) = E \int_0^\infty 1_\Gamma(\omega, t) d\langle M, M\rangle_t, \Gamma \in \mathcal{F} \otimes \mathcal{B}_+$. It is known that each of these martingales may be described in the unique representation (see e.g. [25], Chapter V)

$$P_t(\cdot, A) = P(A) + \int_0^t k_s(\cdot, A) dM_s + L_t^A, \tag{16}$$

where $k(\cdot, A)$ is $(\mathcal{F}_t)-$predictable and L^A satisfies $\langle L^A, M\rangle = 0$.

Note that trivially each $\sigma-$field in the left-continuous filtration $(\mathcal{G}_{t-}^0)$ is also generated by a countable number of sets.

We claim that the existence of an information drift of $(\mathcal{G}_t)$ relative to $(\mathcal{F}_t)$ for the process M depends on the validity of the following condition, which is the generalization of Jacod's condition (10) to arbitrary stochastic bases on standard Borel spaces.

Condition 3.1. $k_t(\omega, \cdot)\Big|_{\mathcal{G}_{t-}^0}$ is a signed measure and satisfies

$$k_t(\omega, \cdot)\Big|_{\mathcal{G}_{t-}^0} \ll P_t(\omega, \cdot)\Big|_{\mathcal{G}_{t-}^0}$$

for P_M-a.a (ω, t).

If (3.1) is satisfied, one can show (see [4]) that there exists an $(\mathcal{F}_t \otimes \mathcal{G}_t)-$predictable process γ such that for P_M-a.a. (ω, t)

$$\gamma_t(\omega, \omega') = \frac{dk_t(\omega, \cdot)}{dP_t(\omega, \cdot)}\Big|_{\mathcal{G}_{t-}^0}(\omega'). \tag{17}$$

It is also immediate from the definition that

$$\gamma_t(\omega, \omega')\, P_t(\omega, d\omega')\, d\langle M, M\rangle_t = \gamma_t(\omega, \omega)\, d\langle M, M\rangle_t. \tag{18}$$

On the basis of these simple facts it is possible to identify the information drift, provided (3.1) is guaranteed.

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http://www.worldscibooks.com/economics/6330.html

Theorem 3.1. *Suppose Condition 3.1 is satisfied and γ is as in (17). Then*

$$\alpha_t(\omega) = \gamma_t(\omega, \omega)$$

is the information drift of $(\mathcal{G}_t)$ relative to $(\mathcal{F}_t)$.

Proof. We give the arguments in case M is a martingale. For $0 \leq s < t$ and $A \in \mathcal{G}_s^0$ we have to show

$$E\left[1_A(M_t - M_s)\right] = E\left[1_A \int_s^t \gamma_u(\omega, \omega)\, d\langle M, M\rangle_u\right].$$

Observe

$$\begin{aligned} E\left[1_A(M_t - M_s)\right] &= E\left[P_t(\cdot, A)(M_t - M_s)\right] \\ &= E\left[(M_t - M_s) \int_0^t k_u(\cdot, A)\, dM_u\right] + E[(M_t - M_s)L_t^A] \\ &= E\left[\int_s^t k_u(\cdot, A)\, d\langle M, M\rangle_u\right] \\ &= E\left[\int_s^t \int_A \gamma_u(\omega, \omega')\, dP_u(\omega, d\omega')\, d\langle M, M\rangle_u\right] \\ &= E\left[1_A(\omega) \int_s^t \gamma_u(\omega, \omega)\, d\langle M, M\rangle_u\right], \end{aligned}$$

where we used (18) in the last equation. □

We now look at the problem from the reverse direction. As an immediate consequence of (18) and Proposition 2.2 note that $(\mathcal{G}_t)$ is a finite utility filtration if and only if

$$\int \int \int \gamma_t^2(\omega, \omega')\, P_t(\omega, d\omega')\, d\langle M, M\rangle_t\, dP(\omega) < \infty.$$

Starting with the assumption that $(\mathcal{G}_t)$ is a finite utility filtration, which thus amounts to $E \int_0^1 \alpha^2\, d\langle M, M\rangle < \infty$, we derive the validity of Condition 3.1.

In the sequel, $(\mathcal{G}_t)$ denotes a finite utility filtration and α its predictable information drift, i.e.

(19) $$\tilde{M} = M - \int_0^\cdot \alpha_t\, d\langle M, M\rangle_t$$

is a $(\mathcal{G}_t)-$local martingale. To prove absolute continuity, we first define approximate Radon-Nikodym densities. This will be done along a sequence of partitions of the state space which generate the respective σ–fields of the bigger filtration. So let $t_i^n = \frac{i}{2^n}$ for all $n \geq 0$ and $0 \leq i \leq 2^n$. We denote by $\mathbb{T}$ the set of all t_i^n. It is possible to choose a family of finite partitions $(\mathcal{P}^{i,n})$ such that

- for all $t \in \mathbb{T}$ we have $\mathcal{G}_{t-}^0 = \sigma(\mathcal{P}^{i,n} : i, n \geq 0 \text{ s.t. } t_i^n = t)$,
- $\mathcal{P}^{i,n} \subset \mathcal{P}^{i+1,n}$,
- if $i < j$, $n < m$ and $i\,2^{-n} = j\,2^{-m}$, then $\mathcal{P}^{i,n} \subset \mathcal{P}^{j,m}$.

We define for all $n \geq 0$ the following approximate Radon-Nikodym densities

$$\gamma_t^n(\omega, \omega') = \sum_{i=0}^{2^n-1} \sum_{A \in \mathcal{P}^{i,n}} 1_{]t_i^n, t_{i+1}^n]}(t) 1_A(\omega') \frac{k_t(\omega, A)}{P_t(\omega, A)}.$$

Note that $\frac{k_t(\omega,A)}{P_t(\omega,A)}$ is $(\mathcal{F}_t)-$predictable and $1_{]t_i^n, t_{i+1}^n]}(t) 1_A(\omega')$ is $(\mathcal{G}_t)-$predictable. Hence the product of both functions, defined as a function on $\Omega^2 \times [0,1]$, is predictable with respect to $(\mathcal{F}_t \otimes \mathcal{G}_t)$. By the very definition, for P_M-almost all $(\omega, t) \in \Omega \times [0,1]$ the discrete process $(\gamma_t^m(\omega, \cdot))_{m \geq 1}$ is a martingale. To have a chance to see this martingale converge as $m \to \infty$, we will prove uniform integrability which will follow from the boundedness of the sequence in $L^2(P_t(\omega, \cdot))$. This again is a consequence of the following key inequality (for more details see [4]).

Lemma 3.1. *Let $0 \leq s < t \leq 1$ and $\mathcal{P} = \{A_1, \ldots, A_n\}$ be a finite partition of Ω into $\mathcal{G}_s^0-$measurable sets. Then*

$$E \int_s^t \sum_{k=1}^n \left(\frac{k_u}{P_u} \right)^2 (\cdot, A_k)\, 1_{A_k}\, d\langle M, M \rangle_u \leq 4E \left(\int_s^t \alpha_u^2\, d\langle M, M \rangle_u \right) < \infty.$$

Proof. An application of Ito's formula, in conjunction with (16) and (19),

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yields

$$
\begin{aligned}
&\sum_{k=1}^{n} [1_{A_k} \log P_s(\cdot, A_k) - 1_{A_k} \log P_t(\cdot, A_k)] \\
&= \sum_{k=1}^{n} \Big[-\int_s^t \frac{1}{P_u(\cdot, A_k)} 1_{A_k}\, dP_u(\cdot, A_k) \\
&\quad + \frac{1}{2} \int_s^t \frac{1}{P_u(\cdot, A_k)^2} 1_{A_k}\, d\langle P(\cdot, A_k), P(\cdot, A_k)\rangle_u \Big] \\
&= \sum_{k=1}^{n} \Big[-\int_s^t \frac{k_u}{P_u}(\cdot, A_k)\, 1_{A_k}\, d\tilde{M}_u - \int_s^t \frac{k_u}{P_u}(\cdot, A_k)\, 1_{A_k} \alpha_u\, d\langle M, M\rangle_u \\
&\quad - \int_s^t \frac{1}{P_u(\cdot, A_k)} 1_{A_k}\, dL_u^{A_k} + \frac{1}{2} \int_s^t \left(\frac{k_u}{P_u}\right)^2 (\cdot, A_k)\, 1_{A_k}\, d\langle M, M\rangle_u \\
&\quad + \frac{1}{2} \int_s^t \frac{1}{P_u(\cdot, A_k)^2}\, 1_{A_k}\, d\langle L^{A_k}, L^{A_k}\rangle_u \Big] \qquad (20)
\end{aligned}
$$

Note that $P_t(\cdot, A_k) \log P_t(\cdot, A_k)$ is a submartingale bounded from below for all k. Hence the expectation of the left hand side in the previous equation is at most 0. One readily sees that the stochastic integral process with respect to $\tilde{M}$ in this expression is a martingale and hence has vanishing expectation, while a similar statement holds for the stochastic integral with respect to the singular parts L^{A_k}. Consequently we may deduce from Eq. (20) and the Kunita-Watanabe inequality

$$
\begin{aligned}
&E \sum_{k=1}^{n} \frac{1}{2} \int_s^t \left(\frac{k_u}{P_u}\right)^2 (\cdot, A_k)\, 1_{A_k}\, d\langle M, M\rangle_u \\
&\leq E \sum_{k=1}^{n} \left[\int_s^t \frac{k_u}{P_u}(\cdot, A_k)\, 1_{A_k} \alpha_u\, d\langle M, M\rangle_u \right] \\
&\leq E \left(\int_s^t \sum_{k=1}^{n} \left(\frac{k_u}{P_u}\right)^2 (\cdot, A_k)\, 1_{A_k}\, d\langle M, M\rangle_u \right)^{\frac{1}{2}} E \left(\int_s^t \alpha_u^2\, d\langle M, M\rangle_u \right)^{\frac{1}{2}},
\end{aligned}
$$

which implies

$$
E \int_s^t \sum_{k=1}^{n} \left(\frac{k_u}{P_u}\right)^2 (\cdot, A_k)\, 1_{A_k}\, d\langle M, M\rangle_u \leq 4E \left(\int_s^t \alpha_u^2\, d\langle M, M\rangle_u \right).
$$

This completes the proof. □

Lemma 3.1 will now allow us to obtain a Radon-Nikodym density process provided the given information drift α satisfies $E\int_0^1 \alpha^2\, d\langle M, M\rangle < \infty$. Note that our main result implicitly contains the statement that the kernel k_t is a signed measure on the σ–field $\mathcal{G}_t^0$, P_M–a.e.

Theorem 3.2. *Suppose that the information drift α satisfies $E\int_0^1 \alpha^2\, d\langle M, M\rangle < \infty$. Then the kernel k is absolutely continuous with respect to $P_t(\omega, \cdot)|_{\mathcal{G}_{t-}^0}$, for P_M–a.a. $(\omega, t) \in \Omega \times [0, 1]$. This means that Condition 3.1 is satisfied. Moreover, the density process γ provides a description of the information drift of $(\mathcal{G}_t)$ relative to $(\mathcal{F}_t)$ by the formula*

$$\alpha_t(\omega) = \gamma_t(\omega, \omega).$$

Proof. By definition and Lemma 3.1 $(\gamma_t^m(\omega, \cdot))_{m\geq 1}$ is an $L^2(P_t(\omega, \cdot))$–bounded martingale and hence, for a.a. fixed (ω, t), $(\gamma_t^m(\omega, \cdot))_{m\geq 1}$ possesses a limit γ. It can be chosen to be $(\mathcal{F}_t \otimes \mathcal{G}_t)$–predictable. Take for example

$$\gamma_t = \liminf_n (\gamma_t^n \vee 0) + \limsup_n (\gamma_t^n \wedge 0).$$

Now define a signed measure by

$$\tilde{k}_t(\omega, A) = \int 1_A(\omega') Z_t(\omega, \omega') dP_t(\omega, d\omega').$$

Observe that $\tilde{k}_t(\omega, \cdot)$ is absolutely continuous with respect to $P_t(\omega, \cdot)$ and that we have for all $A \in \mathcal{P}^{j,m}$ with $j2^{-m} \leq t$

$$\tilde{k}_t(\omega, A) = k_t(\omega, A)$$

for P_M–a.a. $(\omega, t) \in \Omega \times [0, 1]$. By integrating, we obtain the equation

$$P_t(\omega, A) = P(A) + \int_0^t \tilde{k}_s(\omega, A)\, dM_s + L_t^A(\omega) \tag{21}$$

for all $A \in \bigcup_{j2^{-m}\leq t} \mathcal{P}^{j,m}$. Since the LHS and both expressions on the RHS are measures coinciding on a system which is stable for intersections, Eq. (21) holds for all $A \in \mathcal{G}_{t-}^0$. Hence, by choosing $k_t(\cdot, A) = \tilde{k}_t(\cdot, A)$ for all $A \in \mathcal{G}_{t-}^0$, the proof is complete. □

We close this section by illustrating the method developed by means of an example.

Example 3.1. Let W be the Wiener process, P the Wiener measure, $\mathcal{F}_t^0$ the filtration generated by W, $a > 0$, $\tau(a) = 1 \wedge \inf\{t \geq 0 : W_t = a\}$, $\delta > 0$,

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$\mathcal{H}_t^0 = \sigma(\tau \wedge t + \delta)$ and $\mathcal{G}_t^0 = \mathcal{F}_t^0 \vee \mathcal{H}_t^0$. Again let $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ be the smallest respective extensions of $(\mathcal{F}_t^0)$ and $(\mathcal{G}_t^0)$ satisfying the usual conditions. An investor having access to the information represented by $(\mathcal{G}_t)$ knows at any time whether within the next δ time units the Wiener process will hit the level a, provided the level has not yet been hit. In this example, the information drift of $(\mathcal{G}_t)$ is already completely determined as the density process of $k_t(\omega, \cdot)$ relative to $P_t(\omega, \cdot)$ along the $\sigma-$algebras $\mathcal{H}_t^0$ (this follows from a slight modification of the proof of Theorem 3.1).

Let $S_t = \sup_{0\le r\le t} W_r$, $F(a, x, u) = P(\tau(a-x) \le u)$ and recall that $F(a, x, u) = \int_0^u \frac{y}{\sqrt{2\pi y^3}} \exp(-\frac{(a-x)^2}{2y})dy$, for all $x < a$ (see Ch.III, p.107 in [25]). Note that for all $r \le u \le 1$ we have $P_r(\omega, \{\tau(a) \le u\}) = 1_{\{S_r\ge a\}} + 1_{\{S_r<a\}} F(a, W_r, u-r)$. It is straightforward to show that

$$k_r(\omega, \{\tau(a) \wedge t + \delta \le u\}) = 1_{[0,u]}(r)\Big(1_{[0,t+\delta)}(u)\frac{\partial}{\partial x}F(a, W_r, u-r) \\ +1_{[t+\delta,\infty)}(u)\left[-\frac{\partial}{\partial x}F(a, W_r, t+\delta-r)\right]\Big)$$

and consequently the density process of $k_r(\omega, \cdot)$ relative to $P_r(\omega, \cdot)$ along $\mathcal{H}_r^0$ is given by

$$\gamma_r(\omega, \omega') = 1_{[0,\tau(a)(\omega')]}(r)\Big\{1_{[0,r+\delta]}(\tau(a)(\omega'))\left(\frac{1}{a - W_r(\omega)} - \frac{a - W_r(\omega)}{\tau(a)(\omega') - r}\right) \\ +1_{[r+\delta,\infty)}(\tau(a)(\omega'))\left[\frac{-\frac{\partial}{\partial x}F(a, W_r(\omega), \delta)}{1 - F(a, W_r(\omega), \delta)}\right]\Big\}.$$

So the process $\alpha_t(\omega) = \gamma_t(\omega, \omega)$ is the information drift of $(\mathcal{G}_t)$.

4. Additional Utility and Entropy of Filtrations

As in Subsection 3.2 let $X = (X_t)_{0\le t\le 1}$ be a semimartingale, $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ two finite utility filtrations such that $\mathcal{F}_t \subset \mathcal{G}_t$, $t \in [0, 1]$, and let μ be the information drift of $(\mathcal{G}_t)$ relative to $(\mathcal{F}_t)$. As before we assume that there exist countably generated filtrations $(\mathcal{F}_t^0)$ and $(\mathcal{G}_t^0)$ such that $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ are obtained as the the smallest respective extensions satisfying the usual conditions.

Let the decomposition of X with respect to $(\mathcal{F}_t)$ be given by $X = M + \eta \cdot \langle M, M\rangle$, where $M = (M_t)_{0\le t\le 1}$ is an $(\mathcal{F}_t)-$local martingale. To simplify the analyis we will assume throughout this section that M has the predictable representation property (PRP) with respect to $(\mathcal{F}_t)$. We aim at showing that the utility difference $u_{\mathcal{G}}(x) - u_{\mathcal{F}}(x)$ can be interpreted as a conditional entropy of the enlarged filtration $(\mathcal{G}_t)$ with respect to $(\mathcal{F}_t)$. If we did not assume (PRP), then we would obtain that the additional utility is only bounded by this entropy (see [4]).

Recall that the entropy of a measure μ relative to a measure ν on a σ-algebra $\mathcal{S}$ is defined by

$$\mathcal{H}_{\mathcal{S}}(\mu\|\nu) = \begin{cases} \int \log\left(\frac{d\mu}{d\nu}\Big|_{\mathcal{S}}\right) dP, & \text{if } \mu \ll \nu \text{ on } \mathcal{S} \text{ and the integral exists,} \\ \infty, & \text{else.} \end{cases}$$

We first fix a time $s \in [0,1]$ and try to measure the entropy of the information contained in $\mathcal{G}_s^0$ relative to the filtration $(\mathcal{F}_u^0)$, conditional to the $\sigma-$algebra $\mathcal{F}_s$. To this end we introduce an auxiliary filtration obtained by enlarging $(\mathcal{F}_u)$ with $\mathcal{G}_s^0$ at time s,

$$\mathcal{K}_u = \begin{cases} \mathcal{F}_u & \text{if } 0 \le u < s \\ \bigcap_{r>u} \mathcal{F}_r \vee \mathcal{G}_s^0, & \text{if } u \in [s,1] \end{cases}$$

and we denote by μ^s the information drift of $(\mathcal{K}_u)$ relative to M. The *conditional entropy* of the $\sigma-$algebra $\mathcal{G}_s^0$ relative to the filration $(\mathcal{F}_u^0)$ on the time interval $[s,t]$, $t \in (s,1]$, will be defined by

$$\mathcal{H}(s,t) = \int \mathcal{H}_{\mathcal{G}_s^0}(P_t(\omega,\cdot)\|P_s(\omega,\cdot))dP(\omega).$$

We will now show that $2\,\mathcal{H}(s,t)$ is equal to the square-integral of μ^s on $\Omega\times[s,t]$. To this end let $(\mathcal{P}^m)_{m\ge 0}$ be an increasing sequence of finite partitions such that $\sigma(\mathcal{P}^m : m \ge 0) = \mathcal{G}_s^0$. Then

$$\begin{aligned}\mathcal{H}(s,t) &= \int \mathcal{H}_{\mathcal{G}_s^0}(P_t(\omega,\cdot)\|P_s(\omega,\cdot))dP(\omega) \\ &= E\sum_{A\in\mathcal{P}^m} [1_A \log P_s(\cdot,A) - 1_A \log P_t(\cdot,A)] \\ &= E\sum_{A\in\mathcal{P}^m} \Big[-\int_s^t \frac{k_u}{P_u}(\cdot,A)\,1_A\, d\tilde{M}_u - \int_s^t \frac{k_u}{P_u}(\cdot,A)\,1_A \mu_u^s\, d\langle M,M\rangle_u \\ &\quad + \frac{1}{2}\int_s^t \left(\frac{k_u}{P_u}\right)^2 (\cdot,A)\,1_A\, d\langle M,M\rangle_u \Big],\end{aligned}$$

where the last equation follows from (20). Since $\tilde{M}$ is a local martingale, we obtain by stopping and taking limits if necessary

$$\mathcal{H}(s,t) = E\sum_{A\in\mathcal{P}^m} \left[\int_s^t \frac{k_u}{P_u}(\cdot,A)\,1_A \mu_u\, d\langle M,M\rangle_u - \frac{1}{2}\int_s^t \left(\frac{k_u}{P_u}\right)^2 (\cdot,A)\,1_A\, d\langle M,M\rangle_u\right].$$

Lemma 3.1 implies that $\sum_{A\in\mathcal{P}^m}\left(\frac{k_u}{P_u}\right)^2(\omega,A)\,1_A(\omega')$ is an $L^2(P_u(\omega,\cdot))$-bounded martingale for P_M−a.a. (ω,u), and therefore, by Theorem 3.1

$$\lim_m E\int_s^t \sum_{A\in\mathcal{P}^m}\left(\frac{k_u}{P_u}\right)^2(\cdot,A)\,1_A\,d\langle M,M\rangle_u = E\int_s^t (\mu_u^s)^2\,d\langle M,M\rangle_u.$$

Similarly we have

$$\lim_m E\int_s^t \sum_{A\in\mathcal{P}^m}\frac{k_u}{P_u}(\cdot,A)\,1_A\,\mu_u^s\,d\langle M,M\rangle_u = E\int_s^t (\mu_u^s)^2\,d\langle M,M\rangle_u.$$

and hence

$$\mathcal{H}(s,t) = \frac{1}{2}E\int_s^t (\mu_u^s)^2\,d\langle M,M\rangle_u. \tag{22}$$

We are now in a position to introduce a notion of conditional entropy between our filtrations $(\mathcal{G}_t^0)$ and $(\mathcal{F}_t^0)$. For any partition $\Delta : 0 = t_0 \le t_1 \le \ldots \le t_k = 1$ we will use the abbreviations $\sum_\Delta = \sum_{i=1}^k$ and $\prod_\Delta = \prod_{i=1}^k$

Definition 4.1. Let (Δ^n) be a sequence of partitions of $[0,1]$ with mesh $|\Delta^n|$ converging to 0 as $n\to\infty$. The limit of the sums $\sum_{\Delta^n}\mathcal{H}(t_{i-1},t_i)$ as $n\to\infty$ is called *conditional entropy* of $(\mathcal{G}_t^0)$ relative to $(\mathcal{F}_t^0)$ and will be denoted by $\mathcal{H}_{\mathcal{G}^0|\mathcal{F}^0}$.

Theorem 4.1. *The conditional entropy $\mathcal{H}_{\mathcal{G}^0|\mathcal{F}^0}$ is well defined and it satisfies*

$$\mathcal{H}_{\mathcal{G}^0|\mathcal{F}^0} = \frac{1}{2}E\int_0^1 \mu_u^2 d\langle M,M\rangle_u.$$

Proof. Let (Δ^n) be a sequence of partitions of $[0,1]$ with mesh $|\Delta|$ converging to 0 as $n\to\infty$. For all Δ^n we define auxiliary filtrations

$$\mathcal{D}_t^n = \bigcap_{s>t}(\mathcal{F}_s^0\vee\mathcal{G}_{t_i}^0)\quad \text{if } t\in[t_i,t_{i+1}[.$$

Since all $(\mathcal{D}_t^n)$ are subfitrations of $(\mathcal{G}_t^0)$, the respective information drifts μ^n of M exist. It follows immediately from Eq. (22) that

$$\sum_{\Delta^n}\mathcal{H}(t_{i-1},t_i) = \frac{1}{2}E\int_s^t (\mu_u^n)^2\,d\langle M,M\rangle_u.$$

As it is shown in Theorem 4.4 in [4], the information drifts μ^n converge in $L^2(M)$ to the information drift μ. Consequently, the conditional entropy of $(\mathcal{G}_t^0)$ relative to $(\mathcal{F}_t^0)$ is well defined and equals $\frac{1}{2}E\int_0^1\mu_u^2\langle M,M\rangle_u$. □

The conditional entropy $\mathcal{H}_{\mathcal{G}^0|\mathcal{F}^0}$ can be interpreted as a multiplicative integral along the filtration $(\mathcal{G}^0_t)$. More precisely, if for any $s \le t \le 1$ we define $d(s,t,\omega,\omega') = \frac{P_t(\omega,\cdot)}{P_s(\omega,\cdot)}\Big|_{\mathcal{G}^0_s}(\omega')$, and if Δ is a partition of $[0,1]$, then

$$\begin{aligned}
\sum_\Delta \mathcal{H}(t_{i-1},t_i) &= \sum_\Delta \int \left(\int \log \frac{P_{t_i}(\omega,\cdot)}{P_{t_{i-1}}(\omega,\cdot)}\Big|_{\mathcal{G}^0_{t_{i-1}}}(\omega') P_{t_i}(\omega, d\omega') \right) dP(\omega) \\
&= \sum_\Delta \int \log d(t_{i-1},t_i,\omega,\omega) dP(\omega) \\
&= \int \log \prod_\Delta d(t_{i-1},t_i,\omega,\omega) dP(\omega)
\end{aligned}$$

In the special case where $(\mathcal{G}^0_t)$ is obtained by an *initial* enlargment with a random variable G, we have $\frac{P_t(\omega,\cdot)}{P_s(\omega,\cdot)}\Big|_{\mathcal{G}^0_s} = \frac{P_t(\omega,\cdot)}{P_s(\omega,\cdot)}\Big|_{\sigma(G)}$ and hence

$$\begin{aligned}
\mathcal{H}_{\mathcal{G}^0|\mathcal{F}^0} &= \int \left(\int \log \frac{P_1(\omega, d\omega')}{P(d\omega')}\Big|_{\sigma(G)}(\omega') P_1(\omega, d\omega') \right) dP(\omega) \\
&= \mathcal{H}_{\mathcal{F}_1 \otimes \sigma(G)}(P_1(\omega, d\omega')P(d\omega) \| P \otimes P).
\end{aligned}$$

The image of the measure $P_1(\omega, d\omega')P(d\omega)$ under the mapping $(\omega,\omega') \mapsto (M(\omega), G(\omega'))$ is the joint distribution of $M = (M_t)_{0\le t\le 1}$ and G. Consequently, in the initial enlargement case, $\mathcal{H}_{\mathcal{G}^0|\mathcal{F}^0}$ is equal to the entropy of the joint distribution of M and G relative to the product of the respective distributions, which is also known as the *mutual information* between M and G. To sum up, we obtain a very simple formula for the additional logarithmic utility under initial enlargements.

Theorem 4.2. *Let G be a random variable and $\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(G)$. Then $u_\mathcal{G}(x) - u_\mathcal{F}(x)$ coincides with the* mutual information *between M and G.*

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