Symmetry reductions for an inhomogeneus nonlinear diffusion equation

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Abstract

In this work we derive symmetry reductions and exact solutions for an inhomogeneous equation that model fast diffusion. We find the connection between classes of nonclassical symmetries of the equation and of an associated system. These symmetries allow us to increase the number of solutions. Some of theses solutions are unobtainable by classical symmetries.

Keywords: Partial differential equations; Nonclassical symmetries; Potential symmetries

1 Introduction

The diffusion processes appear in many physics processes such as plasma physics, kinetic theory of gases, solid state, metallurgy and transport in porous medium [1, 12, 15].

In this work we consider a mathematical model for diffusion processes which is the generalised inhomogeneous nonlinear diffusion equation

$$f(x)u_t = [g(x)u^n u_x]_x. \tag{1}$$

In [15] P.Rosenau presented a number of remarkable features of the fast diffusion processes: for $f(x) = 1$, $g(x) = 1$ and $-2 \leq n \leq -1$, the family of fast diffusion (1) coexists with a subclass of superfast diffusions where the whole process terminates within a finite time. The special case with $n = -1$ emerges in plasma physics and reveals a surprising richness of new physic-mathematical phenomena.

In (1) $u(x, t)$ is a function of position x and time t and may represent the temperature, $f(x)$ and $g(x)$ are arbitrary smooth functions of position and may denote the density and the density-dependent part of thermal diffusion, respectively.

There is no existing general theory for solving nonlinear partial differential equations and the methods of point transformations are a powerful tool. One of the most useful point transformations are those which form a continuous group. Lie classical symmetries admitted by nonlinear partial differential equations (PDE's) are useful for finding invariant solutions.

Motivated by the fact the symmetry reductions for many PDE's are known that are not obtained by using the classical Lie method there have been several generalizations of the classical Lie group method for symmetry reductions.

Bluman and Cole [3] developed the nonclassical method to study the symmetry reductions of the heat equation. The basic idea of the method is to require that the N order PDE

$$\Delta = \Delta\left(x, t, u, u^{(1)}(x,t), ..., u^{(N)}(x,t)\right) = 0, \tag{2}$$

where $(x,t) \in I\!R^2$ are the independent variables, $u \in I\!R$ is the dependent variable and $u^{(l)}(x,t)$ denote the set of all partial derivatives of l order of u and the invariance surface condition

$$\xi u_x + \tau u_t - \phi = 0, \tag{3}$$

which is associated with the vector field

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u, \tag{4}$$

are both invariant under the transformation with infinitesimal generator (4). Since then, a great number of papers have been devoted to the study of nonclassical symmetries of nonlinear PDE's in both one and several dimensions.

In [4, 5] Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by R$\{x,t,u\}$ in a conserved form a related system denoted by S$\{x,t,u,v\}$ as additional dependent variables is obtained. Any Lie group of point transformations admitted by S$\{x,t,u,v\}$ induces a symmetry for R$\{x,t,u\}$; when at least one of the generators of the group depends explicitly of the potential, then the corresponding symmetry is neither a point nor a Lie-Bäcklund symmetry. These symmetries of R$\{x,t,u\}$ are called *potential* symmetries.

In [14], C. Sophocleous has classified the nonlocal potential symmetries of (1). He obtained that potential symmetries exists only if the parameter n takes the values -2 or $-\frac{2}{3}$ and also certain relations must be satisfied by the functions $f(x)$ and $g(x)$.

In [10], we have derived *nonclassical potential* symmetries for the special case of (1), with $f(x) = 1$ and $g(x) = 1$

$$u_t = [u^{-1}u_x]_x. \tag{5}$$

In [13] connection between classes of nonclassical symmetries of (5), and of nonclassical symmetries of an associated system as well as some new generators have been found.

The aim of this paper is to obtain nonclassical symmetries for (1) and for the associated system given by

$$\begin{aligned} v_x &= f(x)u, \\ v_t &= g(x)u^{-1}u_x, \end{aligned} \tag{6}$$

as well as the connection between these symmetries. These symmetries lead to new solutions, some of these solution exhibit an interesting behaviour.

2 Nonclassical symmetries

2.1 Nonclassical symmetries of the PDE (1)

To obtain nonclassical symmetries of (1), with $n = -1$, we apply the algorithm described in [6, 7] for calculating the determining equations. We can distinguish two different cases:
In the case $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,u) = 1$. The generators that we obtain can be obtained by Lie classical method consequently the nonclassical method, with $\tau \neq 0$ applied to (1) gives only rise to the classical symmetries.

In the case $\tau = 0$, without loss of generality, we may set $\xi = 1$ and we get that the determining equation for the infinitesimal ϕ is

$$\begin{aligned} & u^{(n+2)}\left(fg\phi_{xx} + 2fg'\phi_x - f'g\phi_x + fg\phi^2\phi_{uu} + 2fg\phi\phi_{ux} + fg'\phi\phi_u - f'g\phi_u\right) + fg''\phi - f'g'\phi) \\ & + n\phi u^{(n+1)}\left(3fg\phi_x + fg\phi\phi_u + 2fg'\phi - f'g\phi\right) + fg(n-1)n\phi^3u^n - f^2\phi_t u^2 = 0. \end{aligned} \tag{7}$$

The complexity of this equation is the reason why we cannot solve (7) in general. Thus we proceed, by making an ansatz on the form of $\phi(x,t,u)$, to solve (7) for $n = -1$. In this way we found, choosing $\phi = \alpha(x,t)u^2 + \beta(x,t)u$, after substituting into the determining equation and splitting with respect to u we obtain an overdetermined system for the functions α and β. So, for equation (1) with $n = -1$, we obtain the infinitesimal generator

$$\mathbf{v} = \partial_x + (\alpha(x,t)u^2 + \beta(x,t)u)\partial_u,$$

where f, g, α and β satisfy the system

$$\begin{aligned} & fg'\alpha^2 - fg\alpha\alpha_x + f^2\alpha_t = 0, \\ & f'g'\alpha - fg''\alpha^2 + (fg' + f'g)\alpha\beta + (f'g - 2fg')\alpha_x + fg(\alpha\beta_x - \alpha_x\beta - \alpha_{xx}) + f^2\beta_t = 0, \\ & f\beta\beta_x - fg\beta_{xx} + f'g\beta_x = 0, \\ & \beta(f'g' - fg'') + \beta_x(f'g - 2fg') + \beta^2 fg' + fg(\beta\beta_x - \beta_{xx}) = 0. \end{aligned} \tag{8}$$

2.2 Nonclassical symmetries of the system (6)

We now consider the associated auxiliary system (6) augmented with the invariance surface condition

$$\xi v_x + \tau v_t - \psi = 0, \tag{9}$$

which is associated with the vector field

$$\mathbf{w} = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \phi(x,t,u,v)\partial_u + \psi(x,t,u,v)\partial_v. \tag{10}$$

By requiring both (6) and (9) to be invariant under the transformation with infinitesimal generator (10) one obtains an over determined, nonlinear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\phi(x,t,u,v)$, $\psi(x,t,u,v)$. When at least one of the generators of the group depend explicitly of the potential, that is if

$$\xi_v^2 + \tau_v^2 + \phi_v^2 \neq 0 \tag{11}$$

then (10) yields a nonlocal symmetry of (1).

A nonclassical potential symmetry of (1) is a nonclassical symmetry of the associated potential system (6) that satisfies (11).

We are considering $\tau \neq 0$, and without loss of generality, we set $\tau = 1$. The nonclassical method, with $\tau \neq 0$, applied to (6), give rise to nonlinear determining equations for the infinitesimals.

If we require that $\xi_u = \psi_u = 0$, we obtain that

$$\phi = -f\xi_v u^2 + \left(\psi_v - \frac{f_x\xi}{f} - \xi_x\right)u + \frac{\psi_x}{f} \tag{12}$$

and $f(x)$, $g(x)$, $\xi(x,t,v)$ and $\psi(x,t,v)$, must satisfy the following equations:

$$g\xi_{vv} - \xi\xi_v = 0, \tag{13}$$

$$-fg\xi\xi_x + 2fg^2\xi_{vx} + fg\psi\xi_v + 2f'g^2\xi_v - fg\xi_t + fg'\xi^2 - fg\psi_v\xi - fg^2\psi_{vv} = 0, \tag{14}$$

$$\begin{aligned} f^2 g^2\xi_{xx} + f^2g\psi\xi_x + ff'g^2\xi_x - f^2g\psi_x\xi - f^2g'\psi\xi + ff''g^2\xi - f'^2g^2\xi - 2f^2g^2\psi_{vx} \\ +f^2g\psi\psi_v + f^2g\psi_t = 0, \end{aligned} \tag{15}$$

$$-fg\psi_{xx} + f\psi\psi_x + f'g\psi_x = 0. \tag{16}$$

We can distinguish the following cases:
If $\xi_v \neq 0$ by solving (13) and substituting into (14), (15), (16) leads to generators for which (11) is satisfied, consequently they are nonclassical potential generators and have been considered in [11].
If ξ does not depend on v, by substituting $\xi = \xi(x,t)$ in (14) and (15) we obtain that

$$\psi = v\left(-\xi_x - \frac{\xi_t}{\xi} + \frac{g'\xi}{g}\right) + g\left(\frac{\xi_x}{\xi} + \frac{\xi_t}{\xi^2}\right) - g' + \theta(x,t), \tag{17}$$

By substituting into (12) we get that $\phi_v = 0$.
We observe that in this case condition (11) is not satisfied, consequently

$$\mathbf{v} = \xi(x,t)\partial_x + \partial_t + \left((\delta(t) - \frac{f_x\xi}{f} - \xi_x)u + \frac{\psi_x}{f}\right)\partial_u + \psi\partial_v \tag{18}$$

is not a nonclassical potential generator.

2.3 Connection between symmetries of the PDE (1) and of the system (6)

If we assume that ξ and ψ do not depend on v, the system (13-16) becomes

$$\begin{aligned} &-g\xi\xi_x - g\xi_t + g'\xi^2 = 0, \\ &f^2 g^2\xi_{xx} + f^2g\psi\xi_x + ff'g^2\xi_x - f^2g\psi_x\xi - f^2g'\psi\xi + ff''g^2\xi - f'^2g^2\xi + f^2g\psi_t = 0, \\ &-fg\psi_{xx} + f\psi\psi_x + f'g\psi_x = 0. \end{aligned} \tag{19}$$

It is easy to check that denoting $\alpha = -\frac{f}{g}\xi$, $\beta = \frac{\psi}{g}$ systems (8) and (19) coincides. Consequently we can state:

$$\mathbf{w} = \xi(x,t)\partial_x + \partial_t + \psi(x,t)\partial_v - \left(\frac{f'}{f}\xi u + \frac{\psi_x}{f}\right)\partial_u$$

is a generator for system (6) *if and only if*

$$\mathbf{v} = \partial_x + \left(-\frac{f}{g}\xi u^2 + \frac{\psi}{g}u\right)\partial_u$$

is a generator for equation (1).

3 Some exact solutions

In this section we derive some exact solutions by using some generators:

1- From generator

$$\xi = k_4\sqrt{x}, \qquad \psi = 0, \tag{20}$$

for $f = \frac{k_3k_4}{2\sqrt{x}}e^{\frac{k_1k_4\sqrt{x}}{k_2}}$ and $g = \frac{2k_2\sqrt{x}}{k_4}$, we obtain the similarity solution and the ODE that gives rise to the exact solution

$$u = \frac{2\,k_1}{k_3\,k_4\,\left(k_4\,e^{\frac{k_1\,k_4\,\sqrt{x}}{k_2}} - e^{\frac{k_1\,k_4{}^2\,t}{2\,k_2}+\frac{k_1\,k_4\,k_5}{2\,k_2}}\right)}. \tag{21}$$

We observe that the solution (21), for $x = \frac{(k_4 t + k_5)^2}{4}$, blows up at a parabola.

2- From generator

$$\xi = k, \quad \psi = \frac{-2}{x + kt},$$

for $g = 1$, $f = exp(x)$ and the surface condition we obtain the similarity solution and the ODE that gives rise to the exact solution

$$u = e^{-x}\left(\frac{1}{x - t + k_1} - \frac{1}{x + t}\right). \tag{22}$$

We observe that solution (22) blows up in two straight lines $x - t + k_1 = 0$ and $x + t = 0$.

3- From generator

$$\xi = \tfrac{1}{f}, \quad \psi = -2k_1 \tanh[k_1(t + \textstyle\int f(x)dx)],$$

for $g = \frac{1}{f}$,we obtain the similarity solution and the ODE that gives rise to the exact bounded solution

$$u = -k_1 \tanh(k_1\, t + \int f(x)dx) - k_1 \tanh(k_1\, t - \int f(x)dx). \tag{23}$$

In figure 1 we plot (23) which represents two front waves that evolves changing their shape with opposite velocities with $f(x) = x^2$, $f(x) = sec(x)^2$ respectevely and $k_1 = 1$.

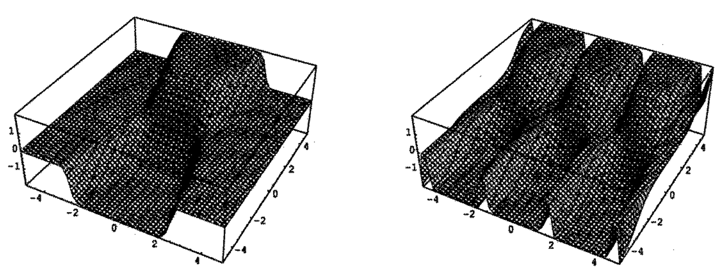

Figure 1: Solution (23) for $g = \frac{1}{f}$, $f(x) = x^2$, $f(x) = sec^2(x)$.

It was pointed out in [14] that the transformation

$$x' = \textstyle\int \frac{dx}{g(x)} = G(x), \quad t' = t, \quad u' = u \tag{24}$$

connects equation (1), and the PDE

$$g(G^{-1}(x'))f(G^{-1}(x'))u'_{t'} = [u'^n u'_{x'}]_{x'}, \tag{25}$$

where G^{-1} is the inverse function of G.

It is clear that we can equivalently use an equation of the form

$$f(x')u'_{t'} = [u'^n u'_{x'}]_{x'}, \tag{26}$$

and then transform the results for equation (1) by using the corresponding point transformation.

We also observe that the transformation

$$x' = x, \quad t' = t, \quad u' = \frac{u}{f} \tag{27}$$

connects equation (26) with $f = k_2 e^{k_1 x}$ and $n = -1$, and the PDE (5).

Consequently, it is clear that we can equivalently use for the subsequent analysis an equation of the form (5), which appear in [10] and then transform the results for equation (1) by using the transformation (24) and (27).

Concluding remarks

In this paper, for $n = -1$ we have derived *nonclassical* symmetries for (1) and for (6). We have proved that the nonclassical method with $\tau \neq 0$ applied to (1) gives only rise to classical symmetries. For $\tau = 0$ we have applied an extension [6] of the Bila and Nielsen procedure. We found the connection between classes of nonclassical symmetries of (1) and classes of nonclassical symmetries of (6). We have proved that (1) when the parameter n takes the value -1, that is when the equation model fast diffusion, admits nonclassical symmetries that yield new solutions. Some of theses solutions are unobtainable by classical symmetries and exhibit and interesting behaviour.

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