Chapter 1

Elements II and Babylonian Metric Algebra

The enigmatic nature of Euclid's *Elements* II and the related propositions *El.* VI.28-29[1] (Heath, *TBE I-III* (1956); *HGM 1* (1981), 379-380; Christianidis (*ed*.), *CHGM* (2004), Part 6) has given rise to a heated debate among historians of mathematics, summarized by Artmann (*Apeiron* 24 (1991)) in the following words:

> "Traditionally VI.28 and 29 have been considered under the rubric 'geometrical algebra', a concept introduced by Zeuthen (1896), 7, following Tannery (1882). Subsequently Neugebauer (1936), van der Waerden (1954), Freudenthal (1977) and Weil (1978) adapted and extended Tannery's and Zeuthen's position. Heath followed Tannery in his comments on II.5 and 6, which he interpreted as solutions to quadratic equations. This traditional position was attacked by Szabó (1969), Unguru (1975) and Unguru and Rowe (1981), (1982). Van der Waerden (1954), 118-126 gives a clear statement of the position of the proponents of 'geometrical algebra'. His main claims are:
>
> (i) The real content of VI.28 and 29 is algebraic (as solutions of quadratic equations); geometry is only a mode of expression.
>
> (ii) Geometrical algebra originated with the Pythagoreans, who took it (somehow) from the Babylonians.
>
> (iii) The Greeks had to use a geometrical formulation of the theory of quadratic equations because they had no other way to deal with incommensurable magnitudes."

Since those words were written, one of the basic premises for the whole controversy has been shown to be invalid. Thus, it has been demonstrated by Høyrup, through a detailed analysis of the technical vocabulary in mathematical cuneiform texts, that Old Babylonian (OB) mathematicians understood quadratic equations in terms of the dimensions and areas of rectangles and other *measurable geometric magnitudes*, and not primarily in terms of anything like our school algebra. (See, for instance, Høyrup, *LWS* (2001).) Subsequently, it has been shown by Friberg (*BaM* 28 (1997),

1. A useful survey of the contents of all the thirteen books of Euclid's *Elements* is given online by D. E. Joyce, <http://aleph0.clarku.edu/~djoyce/java/elements/elements.html>.

Ch. 1) that also Late Babylonian mathematicians used a similar "metric algebra" in order to visualize and solve quadratic equations. Intriguingly, the roots of the Old and Late Babylonian metric algebra can be traced back to examples of "metric squaring" and "metric division" in Old Akkadian and Early Dynastic mathematical texts, half a millennium older than the better known Old Babylonian mathematical texts (Friberg, *CDLJ* 2005/2; *RC* (2007), Apps. 6-7)), and perhaps even to the surprising "field expansion procedure" in proto-cuneiform texts from the end of the 4th millennium BCE (Friberg, *AfO* 44/45 (1997/98); *RC*, Sec. 8.1 b).

The changed premises will make it possible to resolve the mentioned controversy by showing, in this chapter, that the alleged "geometrical algebra" in Euclid's ***Elements*** **II** is of the same nature as closely related results in Old and Late Babylonian metric algebra, and that therefore the assumption that the Greeks had to use a geometric reformulation of an originally purely *algebraic* theory of quadratic equations "because they had no other way to deal with incommensurable magnitudes" must be false.[2]

1.1. Greek Lettered Diagrams *vs.* OB Metric Algebra Diagrams

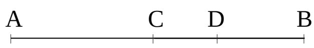
The style of Euclid's exposition in Book II of his *Elements* is shown by the following analysis of the text of one of the propositions in Book II:

El. **II.5** (Heath, *TBE 1*(1956)) begins with *a statement in general terms*:

If a straight line is cut into equal and unequal segments,
the rectangle contained by the unequal segments of the whole
together with the square on the straight line between the points of section
is equal to the square on the half.

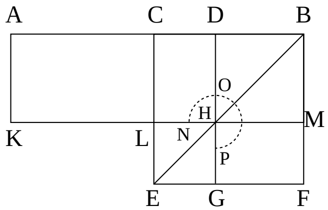
Then follows *a more comprehensible reformulation of the statement in terms of a suitable diagram*:

For let a straight line *AB* be cut into equal segments at *C*
and into unequal segments at *D*;
I say that the rectangle contained by *AD*, *DB* together with the square on *CD*
is equal to the square on *CB*.



2. The ideas discussed in this chapter were presented at the *Zeuthen-Heiberg Centenary Symposium on Current Studies in Ancient Greek Mathematics,* Copenhagen, August, 1994.

In a careful construction of the following complete diagram, step by step, this initial diagram is then extended into a combination of rectangles and squares, where lettered vertices are introduced in alphabetic order:



For let the square *CEFB* be described on *CB*, and let *BE* be joined;
through *D* let *DG* be drawn parallel to either *CE* or *BF* (cutting BE in H),
through *H* again let *KM* be drawn parallel to either *AB* or *EF*,
and again through *A* let *AK* be drawn parallel to either *CL* or *BM*.

Since the diagonal *BE* has been drawn, the proof of the statement can begin with an application of the "diagonal complements rule" in *El.* I.43:

Then, since the complement *CH* is equal to the complement *HF*,
let (the square) *DM* be added to each;
therefore the whole (rectangle) *CM* is equal to the whole (rectangle) *DF*.

Next, by a transitivity argument,

But (the rectangle) *CM* is equal to (the rectangle) *AL*,
since (the segment) *AL* is also equal to (the segment) *CB*;
therefore (the rectangle) *AL* is also equal to (the rectangle) *DF*.

Hence the following intermediate result:

Let (the rectangle) *CH* be added to each;
therefore the whole (rectangle) *AH* is equal to the gnomon *NOP*.

This intermediate result is rephrased in terms of the initial diagram:

But *AH* is (equal to) the rectangle (contained by) *AD*, *DB*, for *DH* is equal to *DB*,
therefore the gnomon *NOP* is also equal to the rectangle (contained by) *AD*, *DB*.

The last step of the procedure is the completion of the gnomon to a square:

Let (the square) *LG*, which is equal to the square on *CD*, be added to each;
therefore the gnomon *NOP* and (the square) *LG*
are equal to the rectangle contained by *AD*, *DB* and the square on *CD*.
But the gnomon *NOP* and (the square) *LG* are the whole square *CEFB*,
which is described on *CB*;
therefore the rectangle contained by *AD*, *DB* together with the square on *CD*
is equal to the square on *CB*. Therefore etc.

The consequent use of lettered vertices in all geometric diagrams is perhaps the most visually striking feature of Greek mathematics of the kind that one meets in Euclid's *Elements*. The lettered vertices are used not only in the diagrams themselves but also in the text, in all references to the diagrams. In the example above, straight lines are named after their endpoints, as in *AB*, *CD*, *etc*., rectangles or squares after their vertices, as in 'the square on *CB*', or 'the square *CEFB*', or simply '(the square) *DM*', and 'the rectangle contained by *AD*, *DB*', or simply '(the rectangle) *AL*', and so on. There are never any metrological or numerical specifications for given plane or solid figures or their parts, such as their lengths, angles, areas, or volumes. The device that is used, perhaps a bit too cleverly, in order to avoid any mention of lengths, areas, *etc*., is to say that one straight line is 'equal to' another straight line, or that one plane figure is 'equal to' another plane figure, *etc*. In the statement in the example above, for instance, a rectangle and a square are said to be equal to another square.

The situation is completely different in Babylonian mathematical cuneiform texts, where in all diagrams showing plane or solid figures, straight lines are denoted by their lengths and/or suitable names such as 'the upper length', 'the middle length', 'the lower length', 'the first length', 'the second length', *etc*., and where similarly areas or volumes are denoted by numbers and/or suitable names. (A good example is IM 55357. See Sec. 4.3 below.) The numbers or names for the lengths are normally placed alongside the figures in their proper places, while the numbers for the areas or volumes are placed inside the figures. The situation is similar in Egyptian hieratic or demotic mathematical papyri, and even in Greek-Egyptian mathematical papyri from the Ptolemaic and Roman periods. (See the many examples in Friberg, *UL* (2005).)

There is another obvious fundamental difference between the example above and a typical Babylonian mathematical text. In *El.* II.5, the object of the text is to *prove* that two geometric figures 'are equal'. The object of a Babylonian mathematical text is nearly always to *compute* something. So, how can there be any kind of relation between a Greek text like *El.* II.5 and Babylonian mathematics? To begin to see why, one has to see what becomes of the lettered diagram in *El.* II.5 *if the letters are removed and instead lengths and areas with their numerical values are explicitly* *indicated* in the Babylonian style. In Fig. 1.1.1 below, a (hypothetical) example of such a diagram in the Babylonian style is shown to the left, and a modernized version in the same style to the right.

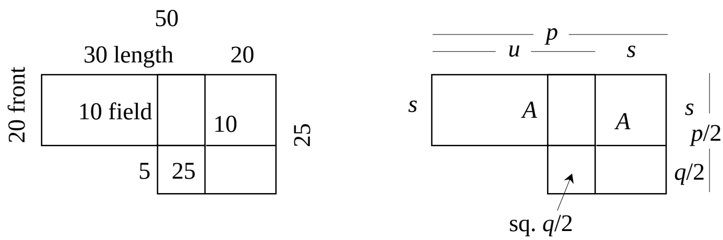


Fig. 1.1.1. A diagram in the Babylonian style (left), and a modernized version (right).

The names used for the long and short sides of a rectangle in OB mathematical texts were normally the Sumerian terms uš 'length' and sag 'front'. The most commonly chosen values for the length and the front were 30 and 20 length units (Sum. ninda = c. 6 meters, or 1/60 ninda = 1 dm). In the diagram above, to the left, the length is 30, the front 20, the sum of the length and the front 50, and half that sum 25. The area of the rectangle is 30 · 20 = 10 (· 60), the area of the small square is sq. 5 = 25, and the area of the large square is sq. 25 = 10 25 = 10 (· 60) + 25.

The numerical example shows how Babylonian mathematicians could arrive at interesting results through experimentation with numerical values for the parameters of a geometric figure. Another way in which they could find new insights was through shrewd observation. Thus, for instance, it is known that OB mathematicians were familiar with what they called a.šà *dalbani* 'the field between' two plane geometric figures.

In the example in Fig. 1.1.2, *the field between two concentric and parallel squares* is what may be called a "square band". Now, if you want to divide the square band equally into four simple pieces, you can do it in several ways. In particular, you can divide the square band into four equal rectangles, as in Fig. 1.1.2, left, or into four "square corners" (what the Greeks called "gnomons"), as in Fig. 1.1.2, right. Evidently, the area of any one of the four square corners is then equal to the area *A* of any one of the four rectangles. It is also clear from the figure that if *p* is the side of nígin *kīditum* 'the outer square' and *q* the side of nígin *qerbitum* 'the inner square', then the area of the whole square band is sq. *p* – sq. *q*, while the area of one of the square corners is sq. $p/2$ – sq. $q/2$. In Fig. 1.1.2, right, the notations p and q have been chosen for the sum $u + s$ and the difference $u - s$, respectively, where u = uš, the 'length', and s = sag, the 'front'.

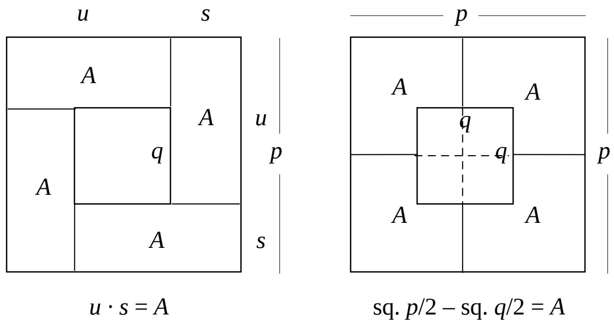


Fig. 1.1.2. Two simple ways of dividing a square band into four equal pieces.

Rectangles, squares, and square corners played a dominant role in OB metric algebra. Often, the first step in the solution of a given metric algebra problem was a transformation of the problem into one of a small number of OB "basic" metric algebra problems (Friberg, *RlA 7* (1990), Sec. 5.7 c):

Two basic *rectangular-linear systems of equations*:

B1a: $u \cdot s = A,$ $u + s = p$
B1b: $u \cdot s = A,$ $u - s = q$

Two basic *additive quadratic-linear systems of equations*:

B2a: sq. u + sq. $s = S,$ $u + s = p$
B2b: sq. u + sq. $s = S,$ $u - s = q$

Two basic *subtractive quadratic-linear systems of equations*:

B3a: sq. u – sq. $s = D,$ $u + s = p$
B3b: sq. u – sq. $s = D,$ $u - s = q$

Three basic *quadratic equations*:

B4a: sq. $s + q \cdot s = A$
B4b: sq. $u - q \cdot u = A$
B4c: $p \cdot u -$ sq. $u = A$

The important thing to remember is that all these types of rectangular-linear, quadratic-linear, or simply quadratic metric algebra problems were actually *visualized as problems for rectangles and squares*.

Below, *the thirteen propositions El.* II.2-14 *will be compared with this list of nine OB basic metric algebra problems.*

1.2. *El.* II.2-3 and the Three Basic Quadratic Equations

The proposition ***El.* II.1** states that if two straight lines are given, and if one of them is divided into a number of segments, then the rectangle contained by the given lines is 'equal to' the (sum of) the rectangles contained by the second line and the segments of the first. The purpose of this proposition is not at all clear, although it is likely that the proposition is meant as a reminder of *the additivity of area*s. In this sense, it paves the way for the following two propositions, *El.* II.2 and *El.* II.3.[3]

El. II.2

If a straight line is cut at random,
the rectangles[4] contained by the whole and both of the segments
are equal to the square on the whole.

El. II.3

If a straight line is cut at random,
the rectangle contained by the whole and one of the segments
is equal to the rectangle contained by the segments,
and the square on the mentioned segment.

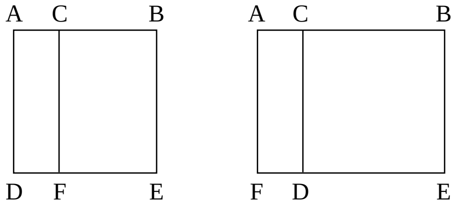


Fig. 1.2.1. Diagrams in *El.* II.2 (left), and *El.* II.3 (right).

The diagram in ***El.* II.2** (Fig. 1.2.1, left) is replaced in Fig. 1.2.2 below by a diagram in the (modernized) Babylonian style, which shows that for any triple of straight lines (of length) u, s, and q, with $u - s = q$, the statement in *El.* II.2 saying, essentially, that, by the additivity of areas,

$u \cdot s + u \cdot (u - s) = \text{sq. } u$

can be reformulated[5] as *a quadratic equation of type B4b*:

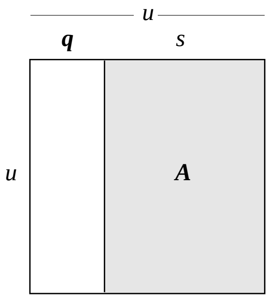
$\text{sq. } u - q \cdot u = A$, where $A = u \cdot s.$

3. All translations of propositions in the *Elements* are borrowed from Heath, *TBE* (1956).
4. Note the plural. *Cf.* the remark in Vitrac, *EA* (1990), I: 328, fn. 3.
5. Contrary to Euclid who avoids talking about one plane figure *subtracted* from another.

Alternatively, the same statement can be reformulated as *a rectangular-linear system of equations of type B1b*:

$u \cdot s = A, \quad u - s = q.$

See again the diagram in Fig. 1.2.2.



***El.* II.2 :** $u \cdot s + u \cdot (u - s) = \text{sq. } u$
B4b : sq. $u - \boldsymbol{q} \cdot u = \boldsymbol{A}$
B1b : $u \cdot s = \boldsymbol{A}, \quad u - s = \boldsymbol{q}$
(Here *u, s, q* are *straight lines*, sq. *u* a *square* with the side *u*, and $u \cdot s$ a *rectangle* with the sides *u*, *s*. Simultaneously, *u, s, q* denote the *lengths* of the straight lines with these names, while sq. *u* and $u \cdot s$ denote the *areas* of the square and the rectangle with these names.)

Fig. 1.2.2. The diagram in *El.* II.2 replaced by a diagram in the Babylonian style.

Therefore, the purpose of *El.* II.2 may have been, essentially, to demonstrate that any *quadratic equation of type B4b*:[6]

sq. $u - q \cdot u = A$

is equivalent to a *rectangular-linear system of equations of type B1b*:

$u \cdot s = A, \quad u - s = q.$[7]

Fig. 1.2.3 below shows that there are *two* ways of similarly replacing the diagram in ***El.* II.3** with a diagram in the Babylonian style. According to the interpretation in Fig. 1.2.3, left, the statement in *El.* II.3, saying, essentially, that

$u \cdot s = (u - s) \cdot s + \text{sq. } s$

can be reformulated as a *quadratic equation of type B4a*:

sq. $s + q \cdot s = A$, where $A = u \cdot s.$

Alternatively, the same statement can be reformulated as *a rectangular-linear system of equations of type B1b*:

$u \cdot s = A, \quad u - s = q.$

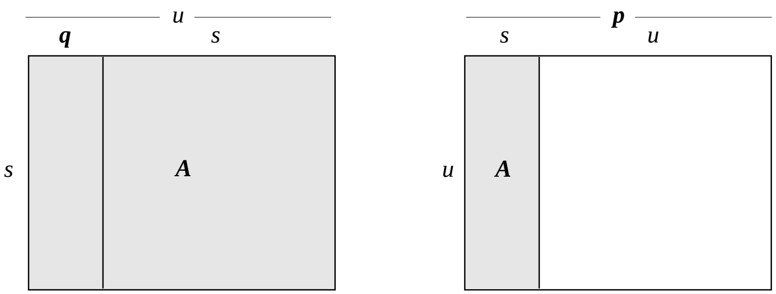
6. Necessarily with *q* and *A* positive, if *u* and *q* are interpreted as lengths and *A* as an area.
7. Necessarily with *s* positive and *s* less than *u*.

Therefore, one purpose of *El.* II.3 may have been to demonstrate that any *quadratic equation of type B4a*:[8]

$\text{sq. } s + q \cdot s = A$

is equivalent to a *rectangular-linear system of equations of type B1b*:[9]

$u \cdot s = A, \quad u - s = q.$



***El.* II.3 :** $u \cdot s = (u - s) \cdot s + \text{sq. } s$

B4a : $\text{sq. } s + \boldsymbol{q} \cdot s = \boldsymbol{A}$

B1b : $u \cdot s = \boldsymbol{A}, \quad u - s = \boldsymbol{q}$

***El.* II.3 :** $(u + s) \cdot u = u \cdot s + \text{sq. } u$

B4c : $\boldsymbol{p} \cdot u - \text{sq. } u = \boldsymbol{A}$

B1a : $u \cdot s = \boldsymbol{A}, \quad u + s = \boldsymbol{p}$

Fig. 1.2.3. Two possible interpretations of the diagram in *El.* II.3.

According to the interpretation in Fig. 1.2.3, right, the statement in *El.* II.3 can be reformulated as a *quadratic equation of type B4c*:

$p \cdot u - \text{sq. } u = A, \quad \text{where} \quad A = u \cdot s.$

Alternatively, the same statement can be reformulated as *a rectangular-linear system of equations of type B1a*:

$u \cdot s = A, \quad u + s = p.$

Therefore, another purpose of *El.* II.3 may have been to demonstrate that any *quadratic equation of type B4c*:

$p \cdot u - \text{sq. } u = A$

is equivalent to a *rectangular-linear system of equations of type B1a*:

$u \cdot s = A, \quad u + s = p.$

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8. With some obvious restrictions because of the geometric interpretation.
9. With some obvious restrictions because of the geometric interpretation.

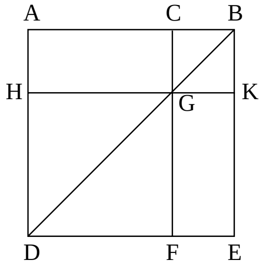
1.3. *El.* II.4, II.7 and the Two Basic Additive Quadratic-Linear Systems of Equations

El. II.4

If a straight line is cut at random,
the square on the whole is equal to the squares on the segments,
and twice the rectangle contained by the segments.

El. II.7

If a straight line is cut at random,
the square on the whole and that on one of the segments, both together,
are equal to twice the rectangle contained by the whole and the said segment,
and the square on the remaining segment.



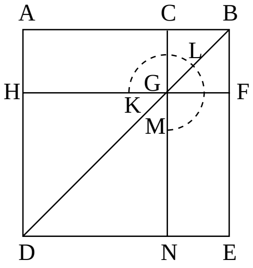


Fig. 1.3.1. Diagrams in *El.* II.4 (left), and *El.* II.7 (right).

In Fig. 1.3.2, left, below, the line AB is called p, its segments u and s. The statement in ***El.* II.4** can then be interpreted as saying that

sq. $(u + s)$ = sq. u + sq. $s + 2\ u \cdot s$.

This equation, in its turn, can be reformulated in the following way:

sq. $p = S + 2\ A$ where $p = u + s$, S = sq. u + sq. s, and $A = u \cdot s$.

Therefore, the purpose of *El.* II.4 may have been, essentially, to demonstrate that any *quadratic-linear system of equations of type B2a*:

sq. u + sq. $s = S$, $u + s = p$

is equivalent to a *rectangular-linear system of equations of type B1a*:

$u \cdot s = A$, $u + s = p$ where A = (sq. $p - S$)/2.

The interpretation of ***El.* II.7** in Fig. 1.3.2, right, is not quite as straightforward, since in order to get an interpretation where *El.* II.4 and *El.* II.7 are closely related, one has to assume that the diagram in *El.* II.7 is only the upper right corner of a larger diagram, based on two concentric and parallel squares. If this assumption is allowed, the given straight line AB in *El.* II.7 can be called *u*, and its arbitrary segments *s* and *q*, where *q* is the side of the inner square. The statement in *El.* II.7 can then be interpreted as saying that

$$\text{sq. } u + \text{sq. } s = 2\, u \cdot s + \text{sq. } (u - s).$$

This equation, in its turn, can be reformulated in the following way:

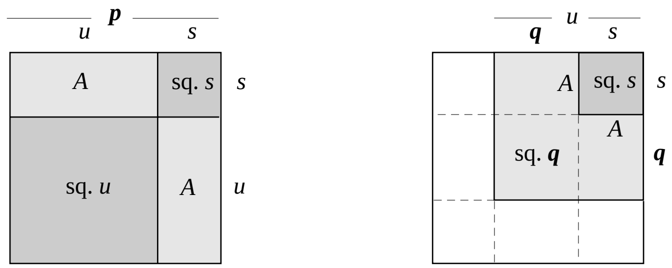
$$S = 2\,A + \text{sq. } q \quad \text{where} \quad q = u - s, \quad S = \text{sq. } u + \text{sq. } s, \quad \text{and} \quad A = u \cdot s.$$

Therefore, the purpose of *El.* II.7 may have been, essentially, to demonstrate that any *quadratic-linear system of equations of type B2b*:

$$\text{sq. } u + \text{sq. } s = S, \quad u - s = p$$

is equivalent to a *rectangular-linear system of equations of type B1b*:

$$u \cdot s = A, \quad u - s = q \quad \text{where} \quad A = (S - \text{sq. } q)/2.$$



El. II.4 :

$\text{sq. } (u + s) = \text{sq. } u + \text{sq. } s + 2\, u \cdot s$

B2a :

$\text{sq. } u + \text{sq. } s = \boldsymbol{S}, \quad u + s = \boldsymbol{p}$

$\cong \quad u \cdot s = A = (\text{sq. } \boldsymbol{p} - \boldsymbol{S})/2$

El. II.7 :

$\text{sq. } u + \text{sq. } s = 2\, u \cdot s + \text{sq. } (u - s)$

B2b :

$\text{sq. } u + \text{sq. } s = \boldsymbol{S}, \quad u - s = \boldsymbol{q}$

$\cong \quad u \cdot s = A = (\boldsymbol{S} - \text{sq. } \boldsymbol{q})/2$

Fig. 1.3.2. Interpretations of the diagrams in *El.* II. 4 and *El.* II.7.

In Sec. 1.4 below it will be shown how systems of equations of type B2a (or B2b) can be solved by use of *El.* II.4 in combination with *El.* II. 5 (or by use of *El.* II.7 in combination with *El.* II.6).

Similarly, it will be shown how quadratic equations of type B4a (or B4b or B4c) can be solved by use of *El.* II. 3 in combination with *El.* II.6 (or *El.* II.2 in combination with *El.* II.6, or *El.* II.3 in combination with *El.* II.5). See Figs. 1.2.2 and 1.2.3 above.

1.4. *El.* II.5-6 and the Two Basic Rectangular-Linear Systems of Equations

El. II. 5

If a straight line is cut into equal and unequal segments,
the rectangle contained by the unequal segments of the whole,
together with the square on the straight line between the points of section,
is equal to the square on the half.

El. II. 6

If a straight line is bisected and a straight line is added to it in a straight line,
the rectangle contained by the whole with the added straight line, and the added straight line, together with the square on the half, is equal to
the square on the straight line made up of the half and the added straight line.

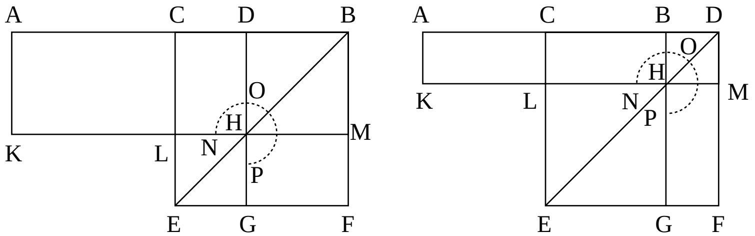


Fig. 1.4.1. The diagrams in *El.* II.5 (left), and *El.* II.6 (right).

The proofs of *El.* II. 5 and *El.* II. 6, respectively, both start by assuming that the straight line AB in the associated diagram is the given line.

In Fig. 1.4.2, left, the given straight line AB is called p, and so on, as above. Then, the statement in ***El.* II.5** can be interpreted as saying that

$(p - s) \cdot s + \text{sq.}\ (p/2 - s) = \text{sq.}\ p/2.$

This equation, in its turn, can be reformulated in the following way:

$A + \text{sq.}\ q/2 = \text{sq.}\ p/2$ where $A = u \cdot s$, $p = u + s$, and $q = u - s$.

Therefore, the purpose of *El.* II.5 may have been, essentially, to demonstrate that any *rectangular-linear system of equations of type B1a*:

$u \cdot s = A, \quad u + s = p$

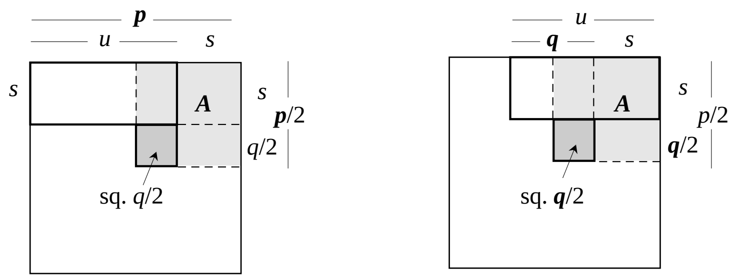
can be solved as follows (with sqs. meaning "the square-side of"):[10]

$(u - s)/2 = q/2 = \text{sqs.}\ (\text{sq.}\ p/2 - A),$

10. With some obvious restrictions because of the geometric interpretation.

$u = p/2 + q/2 = p/2 + \text{sqs. (sq. } p/2 - A),$
$s = p/2 - q/2 = p/2 - \text{sqs. (sq. } p/2 - A).$

Here sqs. (short for "square side") stands for the side of a given square. Note that when both *p* and *q* are known, *u* and *s* can be found as the "half-sum" and "half-difference", respectively, of *p* and *q*.



***El..* II.5 :**

$(p - s) \cdot s + \text{sq. } (p/2 - s) = \text{sq. } p/2$

B1a : $u \cdot s = \boldsymbol{A}, \quad u + s = \boldsymbol{p}$

$\cong \quad \text{sq. } q/2 = \text{sq. } \boldsymbol{p}/2 - \boldsymbol{A},$ *etc.*

***El.* II.6 :**

$(q + s) \cdot s + \text{sq. } q/2 = \text{sq. } (q/2 + s)$

B1b : $u \cdot s = \boldsymbol{A}, \quad u - s = \boldsymbol{q}$

$\cong \quad \text{sq. } p/2 = \boldsymbol{A} + \text{sq. } \boldsymbol{q}/2,$ *etc.*

Fig. 1.4.2. Interpretations of the diagrams in *El.* II. 5 and *El.* II.6.

In Fig. 1.4.2, right, the given straight line AB is called *q*, and so on. Then, the statement in ***El.* II.6** can be interpreted as saying that

$(q + s) \cdot s + \text{sq. } q/2 = \text{sq. } (q/2 + s).$

This equation, too, can be reformulated in the following way:

$\text{sq. } p/2 = A + \text{sq. } q/2$ where $A = u \cdot s$, $p = u + s$, and $q = u - s$.

Therefore, the purpose of *El.* II.6 may have been, essentially, to demonstrate that any *rectangular-linear system of equations of type B1b*:

$u \cdot s = A, \quad u - s = q$

can be solved as follows:

$(u + s)/2 = p/2 = \text{sqs. } (A + \text{sq. } q/2),$
$u = p/2 + q/2 = \text{sqs. } (A + \text{sq. } q/2) + q/2,$
$s = p/2 - q/2 = \text{sqs. } (A + \text{sq. } q/2) - q/2.$

As mentioned above, the solution to *a quadratic-linear system of equations of type B2a* can be obtained by use of *El.* II.4 in combination with *El.* II.5. Indeed, suppose that

$\text{sq. } u + \text{sq. } s = S, \quad u + s = p.$

Then *El.* II.4 can be used to show that

$u \cdot s = A, \quad u + s = p \quad$ where $\quad A = (\text{sq. } p - S)/2.$

In combination with *El.* II.5, this shows that

$(u - s)/2 = q/2 = \text{sqs. } (\text{sq. } p/2 - A) = \text{sqs. } (S/2 - \text{sq. } p/2).$

Consequently,

$u = p/2 + q/2 = p/2 + \text{sqs. } (S/2 - \text{sq. } p/2),$
$s = p/2 - q/2 = p/2 - \text{sqs. } (S/2 - \text{sq. } p/2).$

Similarly, of course, in the case of a system of equations of type B2b.

In the same way, *a quadratic equation of, for instance, type B4a* can be solved by use of *El.* II. 3 in combination with *El.* II.6. Indeed, if

$\text{sq. } s + q \cdot s = A,$

then it can be shown by use of *El.* II.3 that if $u = s + q$, then

$u \cdot s = A, \quad u - s = q.$

Therefore, in view of *El.* II. 6,

$s = \text{sqs. } (A + \text{sq. } q/2) - q/2.$

1.5. *El.* II.8 and the Two Basic Subtractive Quadratic-Linear Systems of Equations

El. II.8

If a straight line is cut at random,
four times the rectangle contained by the whole and one of the segments,
together with the square on the remaining segment, is equal to
the square described on the whole and the mentioned segment as on one straight line.

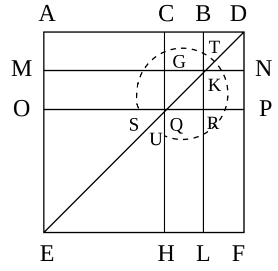


Fig. 1.5.1. The diagram in *El.* II.8.

In Fig. 1.5.2 below, the given straight line AB is called *u* and the two segments into which it is cut are called *s* and *q*. If also, as usual, *u* + *s* is called *p*, then the statement in ***El.* II.8** can be interpreted as saying that

$4\,u \cdot s + \text{sq.}\,(u - s) = \text{sq.}\,(u + s)$ (*cf.* Fig. 1.1.2!)

This equation, in its turn, can be reformulated in the following way:

$4\,A = D$ where $A = u \cdot s,\;\; p = u + s,\;\; q = u - s,$ and $D = \text{sq.}\,p - \text{sq.}\,q.$

In other words, if

$\text{sq.}\,p - \text{sq.}\,q = D,$

then

$D = 4\,A = 4\,u \cdot s = 2\,u \cdot 2\,s.$

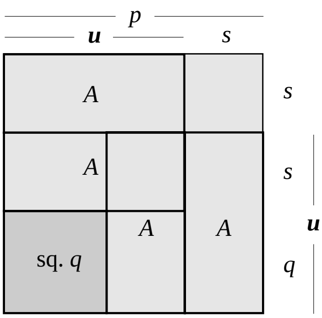
(Note that 2 *u* and 2 *s* can be interpreted as *the length of the mid-line* and *the width*, respectively, of the square corner formed by removing a square of side *p* from a square of side *p*, as in Fig. 1.5.2.)

Therefore, *any rectangular-linear system of equations of type B3a*:

$\text{sq.}\,p - \text{sq.}\,q = D,\;\; p + q = 2\,u$

can be solved by use of *El.* II.8 as follows:

$p - q = 2\,s = D/2u,$
$p = u + s = u + D/(4\,u),$
$q = u - s = u - D/(4\,u).$



***El.* II.8 :**

$4\,u \cdot s + \text{sq.}\,(u - s) = \text{sq.}\,(u + s)$

B3a-b :

$\text{sq.}\,p - \text{sq.}\,q = \boldsymbol{D}$

$p + q = \mathbf{2}\,\boldsymbol{u}$ (or $p - q = \mathbf{2}\,\boldsymbol{s}$)

$\cong\;\; 4\,A = 2\,u \cdot 2\,s = \boldsymbol{D},$

$p - q = 2\,s = \boldsymbol{D}/\mathbf{2}\,\boldsymbol{u}$

(or $p + q = 2\,u = \boldsymbol{D}/\mathbf{2}\,\boldsymbol{s}$), *etc.*

Fig. 1.5.2. Interpretation of the diagram in *El.* II. 8.

One would now expect a further proposition related to the case of a rectangular-linear system of equations of *type B3b*. However, this additional proposition was omitted by the author of *El.* II, obviously because this case, too, can be taken care of by use of *El.* II.8.

1.6. *El.* II.9-10, Constructive Counterparts to *El.* II.4 and II.7

It Secs. 1.2-1.5 above, it was demonstrated that *the first half of Elements II, comprising the seven propositions El. II.2-8, can be interpreted as a catalog of various steps in the geometric solution procedures for the nine basic problems of OB metric algebra, six kinds of quadratic-linear or rectangular-linear systems of equations, and three kinds of quadratic equations*. In this first half of *El*. II, all the proofs are based on *manipulations with squares and rectangles*.

It will be shown below that *the second half of Elements II, comprising the six propositions El. II.9-14, can be interpreted as a parallel catalog of various steps in geometric solution procedures for six of the nine basic problems of OB metric algebra, namely the six kinds of quadratic-linear or rectangular-linear systems of equations.* In this second half of *El*. II, all the proofs are based on *manipulations with right triangles and circles.*

El. II.9

If a straight line is cut into equal and unequal segments,
the squares on the unequal segments of the whole
are double of the square on the half and of the square on the straight line
between the points of section.

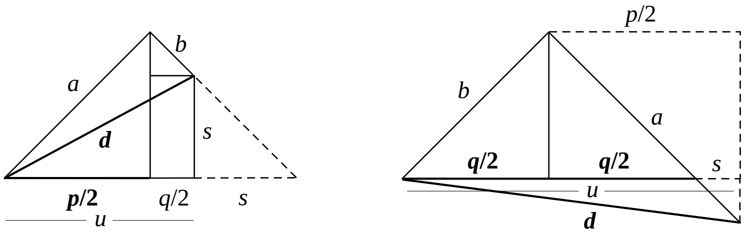
El. II.10

If a straight line is bisected, and a straight line is added to it in a straight line,
the square on the whole with the added straight line
and the square on the added straight line, both together,
are double of the square on the half,
and of the square described on the straight line made up of the half
and the added straight line as on one straight line.

In Fig. 1.6.1 left, below, the given straight line in *El*. II.9 is called *p*, and the unequal parts of *p* are called *u* and *s*, just as in the interpretation of the diagram in *El*. II.4, in Fig. 1.3.2 above.

A considerable part of the proof of *El*. II.9 is devoted to a careful construction of the various parts of the plane figure shown in the diagram. The most essential part of that plane figure consists of two right triangles with the sides *u*, *s* and *a*, *b*, respectively, joined along a common diagonal of length *d*. A plane figure of this kind can be called a “birectangle”, because it has two right angles.

The most essential part of the plane figure appearing in the diagram for *El.* II.10 consists of two *partly overlapping* right triangles with the sides *u*, *s* and *a*, *b*, respectively, joined along a common diagonal of length *d*. A plane figure of this kind can be called an "overlapping birectangle".



***El.* II.9 :**

sq. *u* + sq. *s* = 2 (sq. *p*/2 + sq. *q*/2)

B2a :

sq. *u* + sq. *s* = ***S*** = **sq. *d*,** *u* + *s* = ***p***

≅ *q*/2 = sqs. (***S***/2 – sq. ***p***/2), *etc.*

***El.* II.10 :**

sq. *u* + sq. *s* = 2 (sq. *p*/2 + sq. *q*/2)

B2b :

sq. *u* + sq. *s* = ***S*** = **sq. *d*,** *u* – *s* = ***q***

≅ *p*/2 = sqs. (***S***/2 – sq. ***q***/2), *etc.*

Fig. 1.6.1. Interpretations of the diagrams in *El.* II.9, and *El.* II.10.

The simple proof of the proposition in *El.* II.9 is based on repeated applications of the "diagonal rule" in *El.* I.47. On one hand,

sq. *d* = sq. *a* + sq. *b* = 2 sq. *p*/2 + 2 sq. *q*/2,

since *a* and *b* are the diagonals of two *half-squares* with the sides *p*/2 and *q*/2. On the other hand,

sq. *d* = sq. *u* + sq. *s*.

Therefore,

sq. *u* + sq. *s* = 2 (sq. *p*/2 + sq. *q*/2).

The proof of the similar proposition in *El.* II.10 is similar.

The purpose of ***El.* II.9** may have been to show that any *quadratic-linear system of equations of type B2a*:

sq. *u* + sq. *s* = *S*, *u* + *s* = *p*, with *S* and *p* given,

can be solved as follows: The diagram in Fig. 1.6.1, left, is constructed, with *d* = sqs. *S*. Then it can be shown, as in the proof of *El.* II. 9, that

S = sq. *u* + sq. *s* = 2 (sq. *p*/2 + sq. *q*/2).

Consequently, *u* and *s* can be computed in the following way:

$(u - s)/2 = q/2 =$ sqs. $(S/2 -$ sq. $p/2)$,
$u = p/2 + q/2 = p/2 +$ sqs. $(S/2 -$ sq. $p/2)$,
$s = p/2 - q/2 = p/2 -$ sqs. $(S/2 -$ sq. $p/2)$.

Similarly, of course, ***El.* II.10** can be interpreted as a geometric solution procedure for *a quadratic-linear system of equations of type B2b*:

sq. u + sq. $s = S$, $u - s = q$, with S and q given.

The reason why *El.* II. 9 and 10 can be understood as "constructive counterparts" to *El.* II.4 and 7 will be disclosed below, in Sec. 1.9.

1.7. *El.* II.11* and II.14*, Constructive Counterparts to *El.* II.5-6

El. II.11

To cut a given straight line so that
the rectangle contained by the whole and one of the segments
is equal to the square on the remaining segment.

El. II.14

To construct a square equal to a given rectilineal figure.

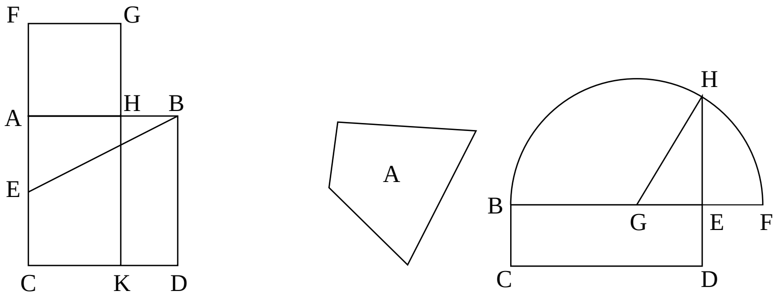


Fig. 1.7.1. The diagrams in *El.* II.11 and *El.* II.14.

In these two propositions, the author of *Elements* II has chosen to consider *two particularly important constructions* related to two propositions that would have been the "constructive counterparts" to *El.* II. 5, 6.

In the diagram for ***El.* II.11** (Fig. 1.7.1, left), AB is the given straight line. The first step of the solution to the stated construction problem is to construct the square ABDC with sides of length *h*. AC is bisected at E, and the diagonal BE is drawn. A point F on the extension of AC is found such that EF = BE. (This can be done most easily by finding the intersection of an extension of AC with a circle through B with center E.) The square AFGH is drawn on AF, and the side GH is extended to K. With this, the construction is completed, and it remains to prove that the given line AB is cut by H in the desired way.

The construction in *El.* II.11 can be explained as follows: In Fig. 1.7.1, left, let h be the length of AB = AC, let s be the length of AF, let $u = s + h$ be the length of CF, and let $p/2 = s + h/2$ be the length of EF = BE. Then,

$s + h/2 = \text{EF}$ and $h/2 = \text{EA}$.

Now, according to *El.* II.6,

CF · AF + sq. EA = sq. EF = sq. EB.

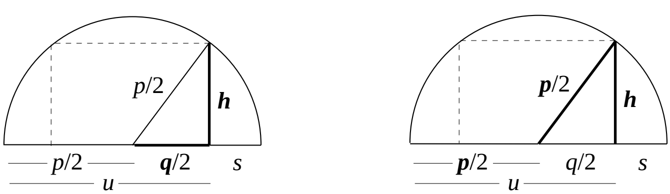
In other words,

$u \cdot s$ + sq. $h/2$ = sq. $p/2$.

An application of the diagonal rule in *El.* I.47 then shows that

$u \cdot s$ = sq. h where h is the given length of AB.

This means that *the rectangle FGKC is equal to the square ABDC*. Hence, if the rectangle AHKC is subtracted from both, it follows that also the rectangle HBDK is equal to the square FGHA. Therefore the point H divides AB in the desired way.



***El.* II.11* :** sq. h + sq. $q/2$ = sq. $p/2$

B1b : $u \cdot s = \boldsymbol{A}$ = **sq. *h*,** $u - s = \boldsymbol{q}$

≅ $p/2$ = sqs. ($\boldsymbol{A}$ + sq. $\boldsymbol{q}/2$), *etc.*

***El.* II.14*:** sq. h + sq. $q/2$ = sq. $p/2$

B1a: $u \cdot s = \boldsymbol{A}$ = **sq. *h*,** $u + s = \boldsymbol{p}$

≅ $q/2$ = sqs. (sq. $\boldsymbol{p}/2 - \boldsymbol{A}$), *etc.*

Fig. 1.7.2. The general ideas behind *El.* II.11 and *El.* II.14.

What is going on here is revealed in ***El.* VI.30**, where it is shown that if a straight line AB is divided in the way described in *El.* II.11, then it is "divided in extreme and mean ratio". Note that the diagram in *El.* II.11 is nearly identical with the diagram in *El.* VI.30.

Now, consider the construction of the diagram in Fig. 1.7.2, left. Begin by assuming that q is a given length and A a given area, and construct a right triangle with the sides $q/2$ and h = sqs. A. Then, according to the diagonal rule in *El.* I.47, the diagonal of the right triangle is also known. If it is called $p/2$, then

A + sq. $q/2$ = sq. h + sq. $q/2$ = sq. $p/2$.

Next, construct a semicircle with the radius $p/2$ and with its center at the lower left vertex of the right triangle. The result is the diagram shown in Fig. 1.7.2, left. Let

$u = p/2 + q/2$ = sqs. (A + sq. $q/2$) + $q/2$,
$s = p/2 - q/2$ = sqs. (A + sq. $q/2$) – $q/2$.

Then

$u + s = p, \quad u - s = q,$

and it can be shown geometrically, as in Fig.1.1.2 above, that

$u \cdot s$ = sq. $p/2$ – sq. $q/2$ so that $u \cdot s$ = sq. $h = A$.

Therefore, the lengths u and s constructed in this way with departure from the given quantities q and A are solutions to the following *rectangular-linear system of equations of type B1b*:

$u \cdot s = A$ = sq. h, $u - s = q$.

It is important to realize that *proposition El. II.11 in the form that Euclid gave to it is, essentially, the special case when h = q of the more general proposition El. II.11*, illustrated by the diagram in Fig. 1.7.2, left.*

Now, consider instead the diagram in Fig. 1.7.2, right, related to the diagram in Fig. 1.7.2, left. Begin by assuming that p is a given length and A a given area, and construct a right triangle with the diagonal $p/2$ and the upright h = sqs. A. Then, according to the diagonal rule in *El.* I.47, the length of the base of the right triangle is also known. Call it $q/2$. Then

sq. $p/2 - A$ = sq. $p/2$ – sq. h = sq. $q/2$.

Next, construct a semicircle with the radius $p/2$ and with its center at the lower left vertex of the right triangle. The result is the diagram shown in Fig. 1.7.2, right. Let

$u = p/2 + q/2 = p/2$ + sqs. (sq. $p/2 - A$),
$s = p/2 - q/2 = p/2$ – sqs. (sq. $p/2 - A$).

Then

$u + s = p, \quad u - s = q,$

and it can be proved as above that the lengths *u* and *s* constructed in this way with departure from the given quantities *p* and *A* are solutions to *a rectangular-linear system of equations of type B1a*:

$u \cdot s = A = \text{sq. } h, \quad u + s = p.$

What does this result have to do with ***El.* II.14**, where Euclid shows how to “construct a square equal to a given rectilineal figure”? The proposition is illustrated by the diagram in Fig. 1.7.1, right. Euclid begins by constructing a rectangle equal to the given figure (which is a paraphrase for *a rectangle of given area A*) by use of *El.* I.45. How he then continues can be explained as follows: He lets *u* (BE) and *s* (ED) be the sides of the rectangle with the given area *A*, and constructs a semicircle with the diameter $p = u + s$ (BF). Next, he constructs a perpendicular, whose length may be called *h*, in the semicircle from the point (E) where the diameter of the semicircle is divided into two segments of lengths *u* and *s*, and draws a right triangle with the given upright side *h* (EH), the given diagonal $p/2$ (HG), and the base $p/2 - s = q/2$ (GE). This is, essentially, the same construction as in Fig. 1.7.2, right. Then he notes that, according to *El.* II.5,

$u \cdot s + \text{sq. } q/2 = \text{sq. } p/2.$

In view of the diagonal rule in *El.* I.47, this means that

$\text{sq. } h = \text{sq. } p/2 - \text{sq. } q/2 = u \cdot s = A,$

where *h* is the length of the upright side of the right triangle, and where *A* is the given area. Therefore, *h* is the side of a square with the given area.

Essentially, what Euclid does in his construction in *El.* II.14 is that he starts with *any* rectangle with the given area *A*, say one with the sides $u, s = A, 1$. He then constructs the diagram in Fig. 1.7.2, right, in the case when $p = u + s$. In this way, he manages to construct the side *h* of a square with the given area *A*, as the upright side of a right triangle. Therefore, proposition *El.* II.14 in the *inverted* form that Euclid chose to give to it (with *u* and *s*, hence also *p* and *q*, given from the beginning rather than *A* and *p*) may very well have replaced an original proposition *El.* II.14* in some earlier, now lost, version of the *Element*s, one which showed how to construct a solution *u*, *s* to *a rectangular-linear system of equations of type B1a*.

1.8. *El.* II.12-13, Constructive Counterparts to *El.* II.8

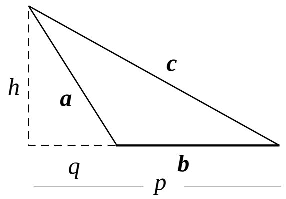
El. II.12

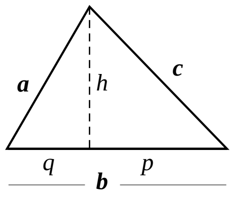
In obtuse-angled triangles the square on the side subtending the obtuse angle
is greater than the squares on the sides containing the obtuse angle
by twice the rectangle contained by one of the sides about the obtuse angle,
namely that on which the perpendicular falls, and the straight line cut off outside
by the perpendicular towards the obtuse angle.

El. II.13

In acute-angled triangles the square on the side subtending the acute angle
is less than the squares on the sides containing the acute angle
by twice the rectangle contained by one of the sides about the acute angle,
namely that on which the perpendicular falls, and the straight line cut off within
by the perpendicular towards the acute angle.

Just as the pair of propositions *El.* II. 9-10 were shown above to be concerned with pairs of right triangles *joined in two different ways along a common diagonal,* so the pair of propositions *El.* II.12, 13 are concerned with pairs of right triangles *joined in two different ways along a common upright side (perpendicular)*. Thus, in Fig. 1.8.1, right (below), two right triangles are *added* to each other, joined along a common upright side, while in Fig. 1.8.1, left, one right triangle is *subtracted* from another right triangle, to which it is joined along a common upright side.





El. II.12 :

sq. c = sq. a + sq. b + 2 $b \cdot q$

B3b :

sq. p – sq. q = $\boldsymbol{D}$, $p - q = \boldsymbol{b}$,

with $\boldsymbol{D}$ = **sq.** $\boldsymbol{c}$ **– sq.** $\boldsymbol{a}$

≅ 2 $\boldsymbol{b} \cdot q = \boldsymbol{D}$ – sq. $\boldsymbol{b}$, *etc.*

El. II.13 :

sq. c = sq. a + sq. b – 2 $b \cdot q$

B3a :

sq. p – sq. q = $\boldsymbol{D}$, $p + q = \boldsymbol{b}$,

with $\boldsymbol{D}$ = **sq.** $\boldsymbol{c}$ **– sq.** $\boldsymbol{a}$

≅ 2 $\boldsymbol{b} \cdot q$ = sq. $\boldsymbol{b} - \boldsymbol{D}$, *etc.*

Fig. 1.8.1. Interpretations of the diagrams in *El.* II.12 and *El.* II.13.

With the notations introduced in Fig. 1.8.1, left, the proof of *El.* II.12 proceeds as follows:

sq. p = sq. b + sq. q + 2 $b \cdot q$ *El.* II.4
sq. p + sq. h = sq. b + sq. q + sq. h + 2 $b \cdot q$
sq. c = sq. b + sq. a + 2 $b \cdot q$ *El.* I.47

Similarly in the case of *El.* II.13, with the notations in Fig. 1.8.1, right:

sq. b + sq. q = 2 $b \cdot q$ + sq. p *El.* II.7
sq. b + sq. q + sq. h = 2 $b \cdot q$ + sq. p + sq. h
sq. b + sq. a = 2 $b \cdot q$ + sq. c *El.* I.47
sq. c = sq. b + sq. a – 2 $b \cdot q$

The purpose of ***El.* II.12** may have been to demonstrate that any *subtractive quadratic-linear system of equations of type B3b*:

sq. p – sq. $p = D$, $p - q = b$, with D and b given,

can be solved as follows: Express D as a square-difference, for instance as

$D = D \cdot 1$ = sq. c – sq. a with $c = (D + 1)/2$, $a = (D - 1)/2$.

(*Cf.* Fig. 1.1.2.) Then it follows from the result in *El.* II. 12 that

2 $b \cdot q = D$ – sq. b.

Therefore,

$q = (D$ – sq. $b)/(2\ b)$, $p = (p - q) + q = b + q = (D$ + sq. $b)/(2\ b)$.

In a similar way, the purpose of ***El.* II.13** may have been to demonstrate that any *subtractive quadratic-linear system of equations of type B3a*:

sq. p – sq. $p = D$, $p + q = b$, with D and b given,

can be solved as follows: Express D as a square-difference,

D = sq. c – sq. a.

Then it follows from the result in *El.* II. 13 that

2 $b \cdot q$ = sq. $b - D$,

so that

q = (sq. $b - D)/(2\ b)$, $p = (p + q) - q = b - q$ = (sq. $b + D)/(2\ b)$.

It may seem a bit strange that in *El.* II.12-13 the case of the obtuse-angled triangle precedes the case of the acute-angled triangle. The reason can be that, as pointed out above, the proof of *El.* II.12 makes use of *El.* II.4, while the proof of *El.* II.13 makes use of the *later* proposition *El.* II.7.

1.9. Summary. The Three Parts of *Elements* II

The discussion above aimed to demonstrate that *Elements* II can be divided into three distinct parts with obvious relations to the nine basic equations or systems of equations in OB metric algebra:

A. *El.* II.(1), 2, 3: related to the basic quadratic equations
B. *El.* II.4-8: related to the basic quadratic- or rectangular-linear systems of equations
C. *El.* II.9-14: related to the same quadratic- or rectangular-linear systems of equations

The question then naturally arises why work that was already done in part B of *Elements* II is repeated in a different way in part C. The answer to this question may be as follows:

It is possible that a lost Greek forerunner to *Elements* II, call it *Elements* II*, was written in imitation of a Babylonian theme text with the same subject. (See below, Sec. 1.12, for examples of OB theme texts.) Presumably, *Elements* II* contained only parts A and B, possibly with Babylonian style metric algebra diagrams rather than the lettered diagrams preferred by Euclid, and with solutions to concrete metric algebra problems instead of abstract geometric propositions. Then, somebody may have reacted to the circumstance that the solutions to the metric algebra problems in part B of *Elements* II* were *analytic and non-constructive*, in the sense that the diagrams associated with the forerunners to *El.* II.4-8 cannot be drawn accurately until *after* the solutions to the stated metric algebra problems have been found. Therefore, the non-constructive solutions in part B were complemented with alternative *synthetic and constructive* solutions in part C, consisting of forerunners to *El.* II.9-14.

Take, for instance, a renewed look at the pair ***El.* II.9-10**. Suppose that p is a *given length* and that S = sq. d is the area of a square with *sides of given length* d. Then a solution to the metric algebra problem

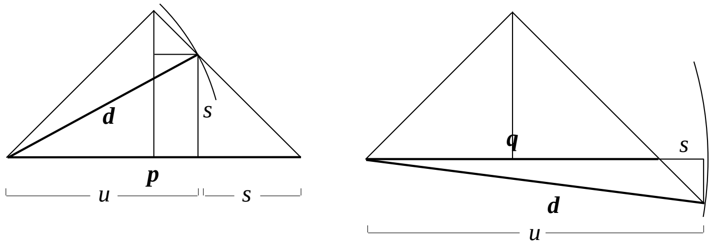
B2a: sq. u + sq. s = S = sq. d, $u + s = p$

can be constructed in the following way:

Draw a straight line of length p, as in Fig. 1.9.1, left. Bisect the straight line, and erect a perpendicular of length $p/2$ at its midpoint. Complete a half-square with the straight line of length p as its diagonal and base. Then draw a circle of radius d with its center at one of the endpoints of the given straight line. Draw a perpendicular to the given straight line from the point where the circle intersects that half-square. This perpendicular cuts the given straight line into two segments. Call the lengths of the segments *u* and *s*. Then *s* is also the length of the perpendicular from the point of intersection of the circle and the half-square. Therefore, it is clear that *u*, *s* is a solution to the stated metric algebra problem of type B2a.

As mentioned above (Fig. 1.6.1), this constructive *geometric* solution to the problem can be transformed into the following *metric* solution:

$u = p/2 + \text{sqs. } (S/2 – \text{sq. } p/2), \quad s = p/2 – \text{sqs. } (S/2 – \text{sq. } p/2.$



B2a: sq. *u* + sq. *s* = ***S*** = **sq. *d***
u + *s* = ***p*** (***p*** > ***d***)

B2b: sq. *u* + sq. *s* = ***S*** = **sq. *d***
u – *s* = ***q*** (***q*** < ***d***)

Fig. 1.9.1. Geometric constructions of solutions in possible forerunners to *El.* II.9, 10.

A similar constructive solution to the metric algebra problem of type B2b is illustrated in Fig. 1.9.1, right. It is a likely forerunner to *El.* II.10.

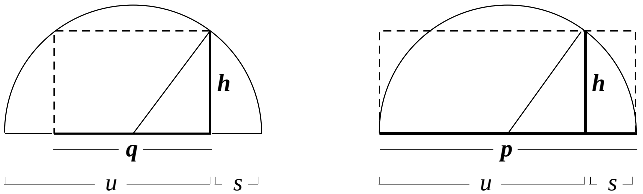
In a similar way, consider the following likely forerunners to the pair ***El.* II.11*** and ***El.* II.14*** (Fig. 1.7.2), the proposed forerunners to *El.* II.11 and II.14. First, suppose that *q* is a *given length* and that *A* = sq. *h* is the *given area* of a square. Then a solution to the metric algebra problem

B1b: $u \cdot s = A = \text{sq. } h, \quad u – s = q$

can be constructed in the following way: Draw a rectangle with sides of length *q* and *h*, as in Fig. 1.9.2, left, and draw a semicircle with its center at the midpoint of one of the sides of length *q*, and passing through the two opposite vertices of the rectangle. Then the diameter of the circle is cut into three segments of which one is the side of the rectangle of length *q*. Let *s* be the common length of the remaining two segments, let *u* = *s* + *q*, and let *p* = *u* + *s*. Then *p*/2 is the length of the radius of the semicircle. Therefore, by the diagonal rule, sq. *p*/2 – sq. *q*/2 = sq. *h*. On the other hand,

$\text{sq. } p/2 – \text{sq. } q/2 = u \cdot s.$

(See Fig. 1.1.2.) Consequently, *u* · *s* = sq. *h*, and it follows that *u*, *s* is a solution to the mentioned metric algebra problem of type B1b.



B1b: $u \cdot s = \boldsymbol{A} = \textbf{sq. } \boldsymbol{h}$, $u - s = \boldsymbol{q}$ **B1a:** $u \cdot s = \boldsymbol{A} = \textbf{sq. } \boldsymbol{h}$, $u + s = \boldsymbol{p}$

Fig. 1.9.2. Geometric constructions of solutions in possible forerunners to *El.* II.11*, 14*.

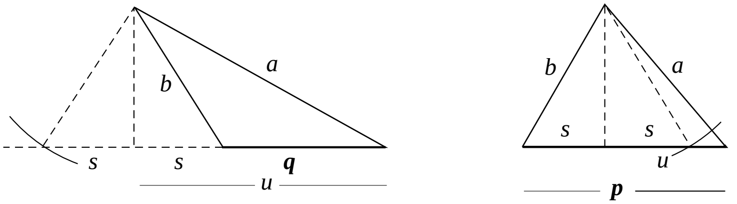
A similar constructive solution to the metric algebra problem of type B1a is illustrated in Fig. 1.9.2, right. It is a likely forerunner to *El.* II.14*.

Note: A shortcoming in the proposed constructive solutions to systems of equations of types B1a-b or B2a-b depicted in Figs. 1.9.1-2 is that they are based on the assumption that the square sides *d* and *h* of *S* and *A*, respectively, are known. Apparently, Euclid observed this shortcoming in the mentioned constructive solutions, and that is why he included a description of *the geometric construction of square sides* in his *El.* II. 14. Having inserted *El.* II.14 in *Elements* II, he did not bother to include also *El.* II.14* (Fig. 1.7.2, right), for which the diagram would be, essentially, the same.

Consider, finally, the following likely forerunners to the pair ***El.* II.12, *El.* II.13**. Suppose that *a, b, q are given lengths of the three sides of a triangle*. Then a geometric solution to the metric algebra problems

B3a-b: sq. u – sq. $s = D$ = sq. a – sq. b, $u - s = q$ (or $u + s = p$)

can be constructed as in Fig. 1.9.3. The uncomplicated details of the argument are left to the readers.



B3b: sq. u – sq. $s = \boldsymbol{D}$, $u - s = \boldsymbol{q}$
$\boldsymbol{D}$ = sq. a – sq. b

B3a: sq. u – sq. $s = \boldsymbol{D}$, $u + s = \boldsymbol{p}$
$\boldsymbol{D}$ = sq. a – sq. b

Fig. 1.9.3. Geometric constructions of solutions in possible forerunners to *El.* II.12, 13.

1.10. An Old Babylonian Catalog Text with Metric Algebra Problems

There does not exist any known OB mathematical text that is an exact parallel to *Elements* II, or to any one of the three parts of it. On the other hand, there do exist examples of *OB catalog texts or theme texts with metric algebra problems*, which therefore in a restricted sense can be called forerunners to *Elements* II. One such text is **BM 80209**, a small clay tablet from the OB city Sippar, now in the collections of the British Museum in London. The interpretation of that text given in Friberg, *JCS* 33 (1981) will be partly repeated here.

BM 80209 is a very special kind of theme text, namely a very brief but systematically arranged "catalog" of metric algebra problems of a number of different types, each represented by one or several numerical examples. There are no solution procedures and no answers to the stated problems.

Here is an abbreviated transliteration and translation of the text. In the transliteration, square brackets indicate destroyed parts of the text, Sumerian words are written in normal style, and Akkadian (that is, Babylonian) words in italics. (Sumerian terms were used in Babylonian mathematical texts in much the same way as words of Greek or Latin origin are used in modern mathematical texts.) Sexagesimal numbers are written as they are in the original text, without zeros and without any indication of where the fractional part of a number begins.

BM 80209

1. [···:ta *im*]*-ta-ḫar* a.šà *mi-nu-um*
2. [*šum-ma*] 20.ta *im-ta-ḫar* dal *mi-nu-um*
3. [*šum-m*]*a* 10.ta *im-ta-ḫar di-ik-šum mi-nu-um*
4. *šum-ma A* a.šà gúr *mi-nu-um*
5. *a-na* a.šà gúr *c* uš daḫ *P*
i-na a.šà gúr *c* uš ba.zi *Q*
6. a.šà 2 gúr ul.gar *A* gúr ugu gúr 10 diri
7. a.šà gúr dal gúr *ù sí-ḫi-ir-ti* gúr ul.gar*-ma A*

1. ··· *each it is* equalsided. The field is what?
2. *If* 20 each it is equalsided, the transversal is what?
3. *If* 10 each it is equalsided, the expansion is what?

4. If *A* is the field, the arc is what?
5. To the field of the arc *c* (times) the length I added, *P*.
   From the field of the arc *c* (times) the length I tore off, *Q*.
6. The fields of 2 arcs I joined, *S*. Arc over arc 10 beyond.
7. The field of the arc, the transversal of the arc, the go-around of the arc I joined, *S*.

(In the translation, destroyed parts of the text are written in italics.)

For various reasons, it is advisable to use *literal translations* of Babylonian mathematical texts. In the translation above, the literal translations 'equalsided', 'field', and 'arc' correspond to the modern terms 'square', 'area', and 'circle'. The 'transversal' of a square is its diagonal, while the 'transversal' of a circle is its diameter. The circumference of a circle is called its 'arc', its 'length', or its 'go-around'. 'Tear off' means 'subtract', and 'arc over arc is 10 beyond' means that the circumference of one circle is 10 (length units) longer than the circumference of another circle.

The very convenient approximation Θ = appr. 3 is used in all Babylonian mathematical texts. More precisely, the area *A* and the diameter *d* of a circle are expressed as follows in terms of the arc (circumference) *a*:

$A = 5 \cdot a$, where '5' means ;05 = 5/60 = 1/12 = appr. $1/4\Theta$,
$d = 20 \cdot a$, where '20' means ;20 = 20/60 = 1/3 = appr. $1/\Theta$.

In addition to sexagesimal fractions, such as the circle constants '5' and '20', there are also two other kinds of fractions of numbers that appear in Babylonian mathematical texts. One kind is the "basic fractions"

3' (= 1/3)
2' (= 1/2)
3" (= 2/3)
6" (= 5/6)

for which there existed special signs in the cuneiform script. Another kind are the "reciprocals"

1/*n*, where *n* = 4, 5, 6, …, often written in Sumerian in the form igi.*n*.gál.

In §§ 4-7 of the catalog text BM 80209, the coefficients *A*, *P*, *Q*, *S*, and *c* are allowed to take various values, so that there are several examples of each type of problem. In quasi-modern notations, the contents of BM 80209 can be described as follows. (The answers, which are not given in the text, are listed in the last column. A minor numerical error in the text is corrected here.)

BM 80209, table of contents (sexagesimal numbers with floating values)

1.	sq. *s* = ?	*s* = [⋯]		
2.	sq. *s* = *A*, *d* = ?	*s* = 20		*d* = 20 · 1 24 51 10
3.	expansion of *s* = ?	*s* = 10	(meaning unknown)	
4.	5 sq. *a* = *A*, *a* = ?	*A* = 8 20		*a* = 10
		A = 2 13 20		*a* = 40
		A = 3 28 20		*a* = 50
		A = 5		*a* = 1 (· 60)
5.	5 sq. *a* + *c* · *a* = *P*, *a* = ?	*c* = 2'	*P* = 8 25	*a* = 10
	5 sq. *a* – *c* · *a* = *Q*, *a* = ?		*Q* = 8 15	*a* = 10
		c = 1	*P* = 8 30	*a* = 10
			Q = 8 10	*a* = 10
		c = 1 3'	*P* = 8 33 20	*a* = 10
			Q = 8 06 40	*a* = 10
		c = 1 2'	*P* = 8 35	*a* = 10
			Q = 8 05	*a* = 10
		c = 1 3"	*P* = 8 36 40	*a* = 10
			Q = 8 03 20	*a* = 10
		c = 1 1/4	*P* = 8 332 30	*a* = 10
			Q = 8 07 30	*a* = 10
		c = 1 1/5	*P* = 8 32	*a* = 10
			Q = 8 08	*a* = 10
6.	5 sq. *a* + 5 sq. *b* = *S*, *a* – *b* = 10, *a*, *b* = ?		*S* = 41 40	*a* = 20, *b* = 10
			S = 3 28 20	*a* = 40, *b* = 30
			S = 41 40	*a* = 50, *b* = 40
			S = 8 28 20	*a* =1 (· 60), *b* = 50
7.	5 sq. *a* + 20 *a* + *a* = *B*, *a* = ?		*B* = 8 33 20	*a* = 10
			B = 1	*a* = 20
			B = 1 55	*a* = 30
			B = 3 06 40	*a* = 40

1.11. A Large Old Babylonian Catalog Text of a Similar Kind

Another similar, but much more extensive, OB catalog text with metric algebra problems without answers is ***TMS* 5**, from the ancient city Susa (Western Iran). Here is an abbreviated transliteration and translation, with several corrected readings of crucial but misunderstood words in the original edition of the text in Bruins and Rutten, *TMS* (1961):

***TMS* 5**

1a.	*s* nígin *c* uš-*ia mi-nu*
1b.	[*c* uš-*ia b* nígin *mi-nu*]
1c.	nígin *ù c* u*š-ia* gar.gar-*ma e*
1d.	nígin ugu *c* uš *d* diri
2a.	*s* nígin *c* a.šà *mi-nu*
2b.	*c* a.šà *A* nígin *mi-nu*
3a.	*s* nígin a.šà *c* uš *mi-nu*
3b	a.šà *ù* a.šà *c* uš gar.gar-*ma S*
3c.	a.šà ugu a.šà *c* uš *D* diri
4a.	*a-na* a.šà nígin-*ia c* uš daḫ-*ma P*
4b.	*i-na* a.šà nígin-*ia c* uš zi-*ma Q*
4c.	*c* nígin ugu a.šà *D* diri
4d.	*c* nígin *ki-ma* a.šà [⋯]-*ma*
5.	nígin.ba a.šà *ab-ni mi-nu* íb.si *ù* [⋯]-*ma*
6.	*c* a.šà *it-ba-al* íb.tag$_4$ a.šà *D* nígin *mi-nu*
7a.	*p* nígin *ki-di-tum d me-šé-tum* nígin *qer-bi-tum mi-nu*
7b.	*q* nígin *qer-bi-tum d me-šé-tum* nígin *ki-di-tum mi-nu*
7c.	*p* nígin *ki-di-tum q* nígin *qer-bi-tum* ul.gar a.šà 2 nigin *mi-nu*
7d.	a.šà 2 nigin ul.gar-*ma S p* nígin *ki-di-tum qer-bi-tum mi-nu*
7e.	a.šà 2 nigin ul.gar-*ma S q* nígin *qer-bi-tum ki-di-tum mi-nu*
7f.	a.šà 2 nigin ul.gar-*ma S* uš-*ši-na* gar.gar-*ma b* nigin *mi-nu*
[⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯]⋯⋯⋯⋯⋯⋯⋯⋯	
8a.	[⋯]nígin *qer-bi-tim* [⋯]*qer-bi-tum* nigin *mi-nu*
8b.	[*D*] a.šà *dal-ba-ni d me-šé-tum* nígin *ki-di-tum ù qer-bi-tum mi-nu*
8c.	*D* a.šà *dal-ba-ni c* nígin *ki-di-tim* nígin *ki-di-tum qer-bi-tum mi-nu*
9a.	*p* nígin *ki-di-tum q múr r* nígin *qer-bi-tum* a.šà *dal-ba-an dal-ba-ni mi-nu*
9b.	a.šà *dal-ba-an dal-ba-ni E* uš-*ši-na* ul.gar-*ma b* nigin *mi-nu*
9c.	a.šà *dal-ba-an dal-ba-ni E* múr ugu nígin *d* nigin *mi-nu*
	4 22 mu.bi nigin.meš

1a. *s* is the equalside. *c* (times) my length is what?
1b. *c (times) my length is b. The equalside is what?*
1c. Equalside and *c* (times) my length I added, *e*.
1d. Equalside over *c* (times) the length is *d* beyond.

2a. *s* is the equalside. *c* (times) the field is what?
2b. *c* (times) the field is *A*. The equalside is what?

3a. *s* is the equalside. The field of *c* (times) the length is what?

3b. The field and the field of *c* (times) the length I heaped, *S*.

3c. The field over the field of *c* (times) the length is *D* beyond.

4a. To the field of my equalside *c* (times) the length I added, *P*.

4b. From the field of my equalside *c* (times) the length I tore off, *Q*.

4c. *c* (times) the equalside over the field is *D* beyond.

4d. *c* (times) the equalside is like the field [····]

5. (meaning not clear)

6. *c* (times) the field he took away. The remaining field is *D*. The equalside is what?

7a. *p* the outer equalside, *d* the distance. The inner equalside is what?

7b. *q* the inner equalside, *d* the distance. The outer equalside is what?

7c. *p* the outer equalside, *q* the inner equalside.
The join of the fields of the 2 equalsides is what?

7d. The fields of two equalsides I joined, *S*.
p is the outer equalside. The inner equalside is what?

7e. The fields of two equalsides I joined, *S*.
q is the inner equalside. The outer equalside is what?

7f. The fields of two equalsides I joined, *S*.
Their lengths I heaped, *b*. The equalsides are what?

····· (several problems missing)

8a. (badly preserved)

8b. *D* is the field between, *d* the distance.
The outer and inner equalsides are what?

8c. *D* is the field between.
c (times) the outer equalside is the inner equalside. The inner equalside is what?

9a. *p* is the outer equalside, *q* the middle, *r* the inner equalside.
The field between between is what?

9b. The field between between is *E*. Their lengths I joined, *b*.
The equalsides are what?

9c. The field between between is *E*. The middle over the <inner> equalside is *d*.
The equalsides are what?

The theme of *TMS* 5 is *problems for squares*. This is confirmed by the subscript which states that the text contains ‘4 22 (262) cases of squares’.

It is interesting to note that the cuneiform sign nígin, which in this text stands for ‘equalside’ has the form of a square. The related sign nigin, which stands for ‘equalsides’ has the form of two adjoining squares. Note also that it is difficult to establish the exact meaning of ‘equalside’. Thus, for instance, ‘the length of the equalside’ means *the side of the square*, while ‘the field of the equalside’ means *the area of the square*.

In quasi-modern notations, the problems in *TMS* 5 can be explained as follows:

***TMS* 5, table of contents**

1 a.	s	given	$c \cdot s = ?$	20 values for c
1 b	[$c \cdot s$	given	$s = ?$]	20 values for c
1 c	$s + c \cdot s$	given	$s = ?$	20 values for c
1 d	$s - c \cdot s$	given	$s = ?$	17 values for c
2 a.	s	given	$c \cdot$ sq. $s = ?$	19 values for c
2 b	$c \cdot$ sq. s	given	$s = ?$	19 values for c
3 a.	s	given	sq. $(c \cdot \text{s}) = ?$	20 values for c
3 b	sq. s + sq. $(c \cdot \text{s})$	given	$s = ?$	20 values for c
3 c	sq. s – sq. $(c \cdot \text{s})$	given	$s = ?$	20 values for c
4 a.	sq. $s + c \cdot s$	given	$s = ?$	27 values for c
4 b	sq. $s - c \cdot s$	given	$s = ?$	27 values for c
4 c	$c \cdot s$ – sq. s	given	$s = ?$	3 values for c
4 d	$c \cdot s$ = sq. s		$s = ?$	1 value for c
5	…………………	(meaning not clear)	1 problem	
6	sq. $s - c \cdot$ sq. s	given	$s = ?$	5 values for c
7 a.	p and $(p - q)/2$	given	$q = ?$	1 problem
7 b	q and $(p - q)/2$	given	$p = ?$	1 problem
7 c	p and q	given	sq. p + sq. $q = ?$	1 problem
7 d	sq. p + sq. q and p	given	$q = ?$	1 problem
7 e	sq. p + sq. q and q	given	$p = ?$	1 problem
7 f	sq. p + sq. q and $p + q$	given	$p, q = ?$	1 problem
…………………………………………				(10 problems missing?)
8 a.	p and $p - q$	given	$q = ?$	1 problem
8 b	sq. p – sq. q and $p - q$	given	$p, q = ?$	1 problem
8 c	sq. p – sq. q given, $q = c \cdot p$		$q = ?$	2 values for c
9 a.	p, m, q	given	sq. m – sq. $q = ?$	1 problem
9 b	sq. m – sq. q and $p + m + q$	given	$p, m, q = ?$	1 problem
9 c	sq. m – sq. q and $m - q$	given	$p, m, q = ?$	1 problem

Note: In 9b-c it is tacitly assumed that $p - q = q - r$.

In §§ 1-4 of *TMS* 5, the given values of the coefficient c are allowed to vary in the same way as the given values of the coefficient c in § 5 of BM 80209 (Sec. 1.10 above), but much more extensively. Here is a list of given values of c and the corresponding values of the solution s (the asked for length of the square side):

c	*s*	*c*	*s*	*c*	*s*	
1	30	1	35	1	10 05	35 = 5 · 7
2		7		11 11		
3		2 7		2 11 11		4 05 = 5 · 7 · 7
4		1 7		1	6 25	
3"		1 2 7		11 7		55 = 5 · 11
2'		7 1/7		2 11 7		
3'		7 2 1/7		1	12 50	10 05 = 5 · 11 · 11
4		1	4 05	3" 2' 3' 11 7		
3' 4		7 7		2 3" 2' 3' 11 7		6 25 = 5 · 7 · 11
1 3"		2 7 7				
1 2'		1 7 7				12 50 = 2 · 5 · 7 · 11
1 3'		1 2 7 7				
1 4		1	55			
1 3' 4		11				
2 2'		2 11				
3 3'						
4 4						

Probably in order to save space, the values given for *c* in this text make use of some otherwise undocumented notations for fractions. Take, for instance the most complicated examples, those of the values 3" 2' 3' 11 7 and 2 3" 2' 3' 11 7. They appear in **§ 1 c**, in the two lines

nígin *ù* 3" 2' 3' 11 7 u*š-ia* gar.gar-*ma* 12 51 06 40
nígin *ù* 2 3" 2' 3' 11 7 u*š-ia* gar.gar-*ma* 12 52 13 20.

This means that

s + 3" 2' 3' 11 7 · s = 12 51 06 40, and s + 2 3" 2' 3' 11 7 · s = 12 52 13 20.

This and other examples together show that what is meant here is

3" 2' 3' 11 7 · s = 2/3 · 1/2 · 1/3 · 1/11 · 1/7 · s

and

2 3" 2' 3' 11 7 · s = 2 · 2/3 · 1/2 · 1/3 · 1/11 · 1/7 s.

It is likely that the student who got these equations as problems to solve was assumed to make use of the rule of false value, a frequently used method in Babylonian mathematics. He would then start with a tentative value for *s*, such as

s^* = 7 · 11 = 1 17 (77).

Using this tentative value, and working with sexagesimal numbers in "relative place value notation" without zeros, he would then find that

2/3 · 1/2 · 1/3 · 1/11 · 1/7 · 1 17 = 2/3 · 1/2 · 1/3 · 1/11 · 11 = 2/3 · 1/2 · 1/3 · 1
= 2/3 · 1/2 · 20 = 2/3 · 10 = 6 40.

Therefore, keeping track of the relative size of the computed fraction of 1 17, he could conclude that

s^* + 3'' 2' 3' 11 7 · s^* = 1 17 + 6 40 = 1 17 06 40,

where

1 17 06 40 = 1/10 · 12 51 06 40.

This means that s = 10 · s^* = 10 · 1 17 = 12 50 is the correct solution to the first of the mentioned equations. It is left to the interested reader to show that it is also the solution to the second equation.

Note the following important connection between *TMS* 5 and the explanation of *Elements* II suggested in Secs. 1.2, 1.3, and 1.5 above: The problems in ***TMS* 5 § 4 a-c** are *basic quadratic equations* of types B4a, B4b, and B4c. Similarly, the problems in ***TMS* 5 § 7 f** (and probably the lost § 7 g) are *basic additive quadratic-linear system of equations* of types B2a (and B2b). Finally, the problem in ***TMS* 5 § 8 b** is a *subtractive quadratic-linear system of equations* of type B3b.

In ***TMS* 5 §§ 7-9** are also of interest in this connection, because they demonstrate quite clearly that OB mathematicians were familiar with the concepts of *concentric and parallel squares,* and s*quare bands*. (*Cf.* the discussion of Fig. 1.1.2 in Sec. 1.1 above.)

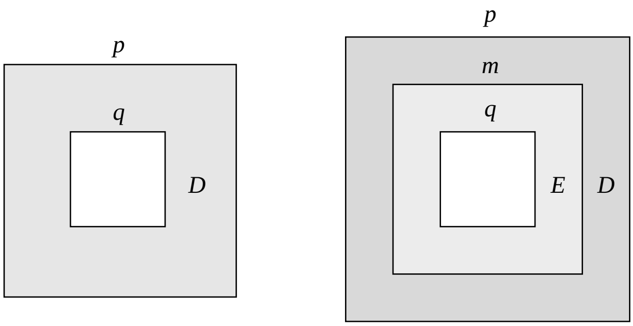


Fig. 1.11.1. The concentric squares and square bands in *TMS* 5 §§ 7, 8, and 9.

In *TMS* 5 § 7, and § 8 b, two squares have the sides 30 and 20, respectively. It is silently assumed that the two squares are concentric and parallel. The distance between the squares is 5.

In § 8 c, two cases are considered. In the first case, the area of the 'space between', that is of the square band, is 20 (· 60), and the length of the inner square is 1/7 of the length of the outer square, with 1/7 written simply as '7'. The solution procedure, which is not given in the text, is simple, since the length q of the inner square can be found as the solution to the equation

sq. (7 q) – sq. q = 20 (· 60), and sq. 7 – sq. 1 = 48,
so that 48 sq. q = 20 (· 60), hence sq. q = 25, and
$q = 5$, $p = 35$.

In the second case, the area of the square band is 16 40 (· 60), and the length of the inner square is 1/7 · 1/7 of the length of the outer square, with 1/7 · 1/7 written simply as '7 7'. In this case,

sq. (49 q) – sq. q = 20 (· 60), and sq. 49 – sq. 1 = 40 01 – 1 = 40 (· 60),
so that 40 sq. q = 16 40, hence sq. q = 25, and
$q = 5$, p = 4 05.

In TMS 5 § 9, there are three concentric squares and two square bands. It is silently assumed that the middle square is halfway between the other two. The sides of the three squares are simply 30, 20, and 10.

1.12. Old Babylonian Solutions to Metric Algebra Problems

1.12 a. Old Babylonian problems for rectangles and squares

The two OB catalog texts with metric algebra problems discussed in Secs. 1.10 and 1.11 above are well organized but lack both answers and explicit solution procedures to the stated problems.

BM 13901 (Neugebauer, *MKT 3* (1937); Høyrup, *LWS* (2002), 288) is of a different type, a *theme text* with metric algebra problems. It is a large text containing 23 exercises for squares, *each with a complete solution procedure*. The table of contents below, where the exercises are listed in their order of appearance in the text, reveals that BM 13901 is a mathematical "recombination text", by which is meant *a somewhat disorganized collection of more or less closely related mathematical exercises from a number of sources.*

BM 13901, table of contents (sexagesimal numbers with floating values)

1 a.	**sq. *s* + *s* = 45**	***s* = 30**
1 b	**sq. *s* – *s* = 14 30**	***s* = 30**
1 c	sq. *s* – 3' sq. *s* + 3' *s* = 20	*s* = 30
1 d	sq. *s* – 3' sq. *s* + *s* = 4 46 40	*s* = 20
1 e	sq. *s* + *s* + 3' *s* = 55	*s* = 30
1 f	sq. *s* + 3'' *s* = 35	*s* = 30
1 g	7 s + 11 sq. *s* = 6 15	*s* = 30
2 a.	**sq. *p* + sq. *q* = 21 40, *p* + *q* = 50**	***p* = 30, *q* = 20**
2 b	**sq. *p* + sq. *q* = 21 40, *p* – *q* = 10**	***p* = 30, *q* = 20**
2 c	sq. *p* + sq. *q* = 21 15, *q* = *p* – 1/7 *p*	*p* = 3 30, *q* = 3
2 d	sq. *p* + sq. *q* = 28 15, *p* = *q* + 1/7 *q*	*p* = 4, *q* = 3 3 0
2 e	**sq. *p* + sq. *q* = 21 40, *p* · *q* = 10**	***p* = 30, *q* = 20**
2 f	sq. *p* + sq. *q* = 28 20, *q* = 1/4 *p*	*p* = 40, *q* = 10
2 g	sq. *p* + sq. *q* = 25 25, *q* = 3'' *p* + 5	*p* = 30, *q* = 25
4	sq. *p* + sq. *q* + sq. *r* + sq. *s* = 27 05, *q* = 3'' *p*, *r* = 2' *q*, *s* = 3' *r*	*p* = 30, *q* = 20, *r* = 15, *s* = 10
1 h	sq. *s* – 3' *s* = 5	*s* = 30
3 a.	sq. *p* + sq. *q* + sq. *r* = 10 12 45, *q* = 1/7 *p*, *r* =1/7 *q*	*p* = 24 30, *q* = 3 30, *r* = 30
3 b	sq. *p* + sq. *q* + sq. *r* = 23 20, *p* – *q* = *q* – *r* = 10	*p* = 30, *q* = 20, *r* = 10
2 h	sq. *p* + sq. *q* + sq. (*p* – *q*) = 23 20, *p* + *q* = 50	*p* = 30, *q* = 20
………………………………		(3 exercises lost)
1 i	4 *s* + sq. *s* = 41 40	*s* = 10
3 c	sq. *p* + sq. *q* + sq. *r* = 29 10, *q* = 3'' *p* + 5, *r* = 2' *q* + 2 30	*p* = 30, *q* = 20, *r* = 10

Four of the exercises in BM 13901 are closely associated with the theme of parts A and B of *Elements* II. (See Sec. 1.9 above.) These four exercises will be discussed separately below.

BM 13901 § 1 a, literal translation	explanation (relative values)
The field and my equalside I heaped, 45.	sq. *s* + *s* = *A* = 45
1, the going-out, you set.	Set *q* = 1
The halfpart of 1 you break.	*q*/2 = 30
30 and 30 you make eat each other.	sq. *q*/2 = sq. 30 = 15
15 to 45 you add.	*A* + sq. *q*/2 = 45 + 15 = 1
1 makes 1 equalsided.	sqs. 1 = 1
30 that you made eat itself,	*q*/2 = 30
inside 1 you tear out.	subtracted from 1 = 30
30 is the equalside	*s* = 30.

See (Høyrup, *LWS* (2002), 50) for a transliteration of this text, and for a literal translation, differing in some details from the one proposed here. It is, by the way, not easy to find adequate literal translations of the terms in a Babylonian mathematical text, since there is often *no exact correspondence between Babylonian and modern mathematical terms*. Nevertheless, it is advisable to use literal translations, for the reason that OB mathematical terminology was not standardized. The fact that crucial elements of the terminology are different in texts from different sites and different periods ought to be apparent in the translations.

Besides, the use of non-literal translations can obscure important fine points of the text, such as the fact, first pointed out by Høyrup in *AoF* 17 (1990), that OB mathematicians used different terms for several different kinds of addition, several different kinds of multiplication, *etc.* Thus, for instance, in the text above, *when two lengths are multiplied with each other*, the term used is that the numbers for the two lengths "eat each other" (and become replaced by a number for an area).

Note in the explanation the use of the abbreviations sq. for the square of a length number and sqs. for the square side of an area number. (The use of modern notations for squares and square roots would be anachronistic.)

The term 'going-out' (Akk. *wāṣītum* 'that which goes out') in this text refers to the coefficient q in the quadratic equation sq. $s + q \cdot s = A$. It has to be understood as *a length number*, which explains why it is possible to add together the area number sq. s and the product $q \cdot s$. In the present case, when q = '1', the phrase 'the field and my equalside' has to be understood as sq. $s + 1 \cdot s$, where both sq. s and $1 \cdot s$ are *area numbers*!

Note, by the way, that it is not absolutely clear what it means that in this text the going-out is equal to '1'. Høyrup is of the opinion that it means that $q = 1$ length unit, and is then forced to interpret the answer '30 is the equalside' as meaning that the computed side of the square is ;30 = 1/2 length unit. However, there is plenty of evidence that plane figures in OB mathematical texts were normally (but perhaps not always) thought of as *actual fields*, with the size of their sides in the range of *tens or sixties* of the length unit ninda (= about 6 meters). Since the situation is unclear, it may be a good idea to stay neutral on this issue and interpret '1' as *either* 1 *or* 1 00 = 1 · 60 and '30' as *either* 30 *or* ;30 = 30 · 1/60.

The quadratic equation in BM 13901 § 1a is of type **B4a**. The solution procedure can be interpreted as a combination of the ideas behind *El.* II.3 and II.6. (Fig. 1.2.3, left, and Fig. 1.4.2, right.) See Fig. 1.12.1 below:

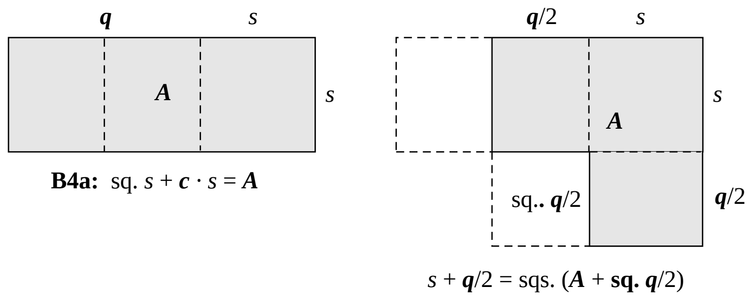


Fig. 1.12.1. A geometric explanation of the solution procedure in BM 13901 § 1 a.

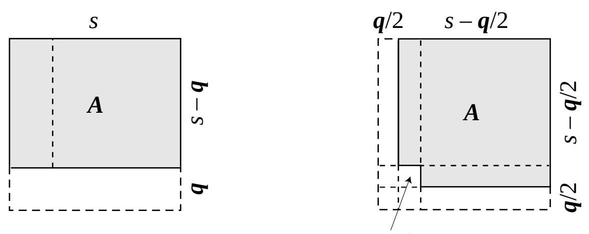
Note that if it is assumed here that s = ‘30’ = 30, then sq. s = ‘15’ = 15 00 and consequently A = ‘45’ = 45 00 and q = ‘1’ = 1 00!

Now, consider instead **BM 13901 § 1 b** (Høyrup, *LWS* (2002), 52):

BM 13901 § 1 b, literal translation	explanation (floating values)
My equalside inside the field I tore out, 14 30.	sq. $s - s = A$ = 14 30
1, the going-out, you set.	set q = 1
The halfpart of 1 you break.	q/2 = 30
30 and 30 you make eat each other.	sq. q/2 = sq. 30 = 15
15 to 14 30 you add.	sq.q/2 + A = 15 + 14 30 = 14 30 15(!)
14 30 15 makes 29 30 equalsided.	sqs. 14 30 15 = 29 30
30 that you made eat itself,	Recall that q/2 = 30
to 29 30 you add. 30 is the equalside	30 added to 29 30 = 30, s = 30

The problem in BM 13901 § 1 b can be interpreted as a quadratic equation of type **B4b**, sq. $s - q \cdot s = A$, with q = ‘1’. The most likely interpretation of the solution procedure is that it is a combination of the ideas behind *El.* II.2 and II.6 (Figs. 1.2.2 and 1.4.2, right). See Fig. 1.12.2 below:

Note that in § 1b the computed value of u is again ‘30’, but when s = 30, then in Fig. 1.12.2, left, the going-out q = ‘1’ cannot possibly have the value 1 00, which is greater than 30. Indeed, in a geometric interpretation like the one in Fig. 1.12.2, the difference $s - q$ must be a (positive) length number. For this reason, the author of BM 13901 apparently chose to interpret ‘the going-out is 1’ in § 1 b as meaning that q = 1, not q = 1 00!



B4b: sq. $s - \boldsymbol{q} \cdot s = \boldsymbol{A}$

$s - \boldsymbol{q}/2 = \text{sqs.}\ (\boldsymbol{A} + \textbf{sq.}\ \boldsymbol{q}/2)$

Fig. 1.12.2. A geometric explanation of the solution procedure in BM 13901 § 1 b.

Next, consider **BM 13901 § 2 a** (Høyrup, *op. cit*., 66):

BM 13901 § 2 a, literal translation	explanation (floating values)
The fields of my two equalsides I heaped, 21 40,	sq. p + sq. $q = S = 21\ 40$
and my equalsides I heaped, 50.	$p + q = 2\ u = 50$
The halfpart of 21 40 you break.	$S/2 = 10\ 50$
10 50 you write down.	make a note of $S/2 = 10\ 50$
The halfpart of 50 you break.	$u = (p + q)/2 = 50/2 = 25$
25 and 25 you make eat each other.	sq. u = sq. $25 = 10\ 25$
10 25 inside 10 50 you tear out.	$S/2$ – sq. $u = 10\ 50 - 10\ 25 = 25$
25 makes 5 equalsided.	sqs. ($S/2$ – sq. u) = sqs. $25 = 5 = s$
5 to the first 25 you add,	$u + s = 25 + 5 = 30$
30 is the first equalside.	$p = 30$
5 inside the second 25 you tear out,	$u - s = 25 - 5 = 20$
20 is the second equalside.	$q = 20$

The problem in BM 13901 § 2 a can be interpreted as a quadratic-linear system of equations of type B2a, sq. p + sq. $q = S$, $p + q = 2\ u$. The solution procedure is based on the identity

sq. $s = S/2$ – sq. u when sq. p + sq. $q = S$, $p = u + s$ and $q = u - s$.

BM 13901 § 2 b (Høyrup, *op. cit*., 68) is similar:

BM 13901 § 2 b, literal translation	explanation (floating values)
The fields of my two equalsides I heaped, 21 40.	sq. p + sq. $q = S = 21\ 40$
Equalside over equalside is 10 beyond.	$p - q = 10$
The halfpart of 21 40 you break.	$S/2 = 10\ 50$
10 50 you write down.	make a note of $S/2 = 10\ 50$
The halfpart of 10 you break.	$(p - q)/2 = s = 5$

5 and 5 you make eat each other.	sq. *s* = sq. 5 = 25
25 inside 10 50 you tear out.	*S*/2 – sq. *s* = 10 50 – 25 = 10 25
10 25 makes 25 equalsided.	sqs. (*S*/2 – sq. *s*) = 25 = *u*
25 you write down twice.	make two notes of *u* = 25
5 that you made eat itself	Recall that *s* = 5
to the first 25 you add,	*u* + *s* = 25 + 5 = 30
30 is the equalside.	*p* = 30
5 inside the second 25 you tear out,	*u* – *s* = 25 – 5 = 20
20 is the second equalside.	*q* = 20

The problem in BM 13901 § 2 b can be interpreted as a quadratic-linear system of equations of type B2b, sq. *p* + sq. *q* = *S*, *p* – *q* = 2 *s*. The solution procedure is based on the identity

sq. *u* = *S*/2 – sq. *s* when sq. *p* + sq. *q* = *S*, *p* = *u* + *s* and *q* = *u* – *s*.

In *LWS* (2002), 67-70, Figs. 10-12, Høyrup presents three different possible configurations in terms of squares and rectangles which the OB mathematicians may have used to prove identities like the ones mentioned above. There is, however, a fourth possible, and perhaps more plausible, configuration, which Høyrup did not consider in this connection (but which he did consider elsewhere, *op. cit.*, 259, Fig. 67).

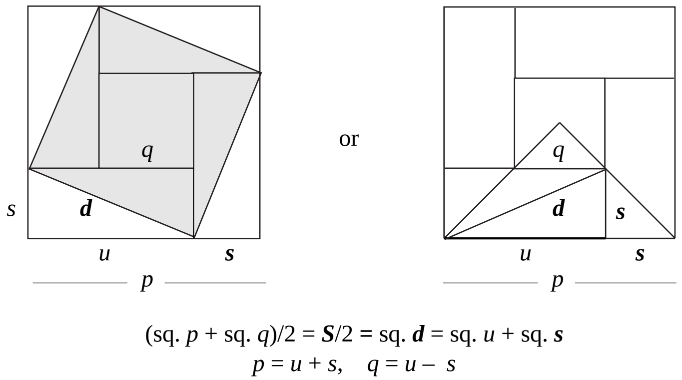


Fig. 1.12.3. Left: A geometric explanation of BM 13901 § 2 a-b. Right: *El.* II.9.

Indeed, in Fig. 1.12.3 above, left,

sq. *d* (= the area of the oblique square) = (sq. *p* + sq. *q*)/2.

This is so because sq. *d* plus the areas of four right triangles = sq. *p*, while sq. *d* minus the areas of four right triangles = sq. *q* (see Fig. 2.3.2, right).

On the other hand, in view of the diagonal rule (Sec. 2.3), it is also true that

sq. d = sq. u + sq. s, where $u = (p + q)/2$, and $s = (p - q)/2$.

Therefore,

$S/2$ = sq. u + sq. s when sq. p + sq. $q = S$, $(p + q)/2 = u$ and $(p - q)/2 = s$.

The identity that can be derived in this way by use of the configuration in Fig. 1.12.3, left, can also derived by use of a birectangle as in the proof of *El.* II.9 (Fig. 1.6.1 above, left). That this is no coincidence is shown in Fig. 1.12.3 above, right.

MS 5112 is a large fragment of a mathematical recombination text with metric algebra problems, published in Friberg, *RC* (2007), Sec. 11.2 n. The text is late OB, maybe younger. It is inscribed on the obverse with a number of metric algebra problems for *squares*, and on the reverse with metric algebra problems for *rectangles*. There are explicit solution procedures for all the problems. One of the problems on the reverse is a rectangular-linear system of equations of type B1b:

MS 5112 § 11, literal translation	explanation
Length (and) front (I) made eat each other,	$u \cdot s = A$
1 èše the field.	= 1 èše = 10 00 square ninda
The length over the front is 10 beyond.	$u - s = q$ = 10 (ninda)
The length (and) the front are what?	u, s = ?
You with your doing:	Do it like this:
1/2 of 10 that the length over the front is beyond crush,	$q/2 = 10/2 = 5$
5 steps of 5 (make) eat (each other), 25.	sq. $q/2$ = sq. 5 = 25
To 10 the field add, 10 25.	A + sq. $q/2$ = 10 00 + 25 = 10 25
What is it equalsided?	sqs. (A + sq. $q/2$) = sqs. 10 25 = ?
25 each way equalsided.	sqs. (A + sq. $q/2$) = 25 = $p/2$
Twice write it down.	Write down 25 = $p/2$ twice.
5 that was eaten to the 1st 25 add, 30.	$p/2 + q/2 = 25 + 5 = 30$
30 ninda is the length.	$u = p/2 + q/2 = 30$
From the second 25 the 5 tear off, 20	.$p/2 - q/2 = 25 - 5 = 20$
20 ninda is the front.	$s = p/2 - q/2 = 20$

The geometric model on which, apparently, both the question and the solution procedure in MS 5112 § 11 are based is obviously an OB forerunner to the construction in *El.* II.6 (see Figs. 1.4.1 and 1.4.2 above, right).

1.12 b. Old Babylonian problems for circles and chords

The examples discussed in Sec. 1.12 a above make it clear that parts A and B of *Elements* II (*El.* II.2-II.8) have many OB forerunners in the form of metric algebra problems for *squares and rectangles*. It is also easy to find examples of OB forerunners to part C of *Elements* II (*El.* II. 9-II.14), in the form of metric algebra problems for *right triangles and circles*. As suggested above, maybe the pair of exercises BM 13901 § 2 a-b is one such example. Further examples will be offered in the discussions of OB "igi-igi.bi problems" in Secs. 3.2-3 below, and in the discussion of an OB geometric algorithm in Appendix 1.

For some reason, there are few known metric algebra problems specifically for circles in the known corpus of OB mathematical texts.

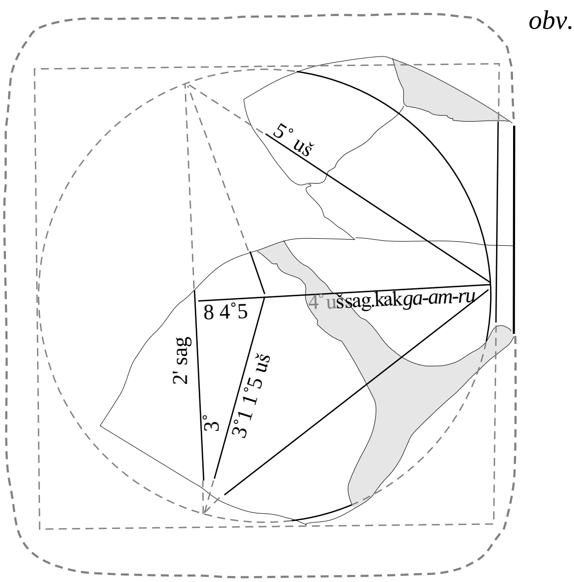


Fig. 1.12.4. *TMS* 1. A school boy's hand tablet with a diagram of a triangle in a circle.

An interesting first example is ***TMS* 1** (Bruins and Rutten, *TMS* (1961); Fig. 1.12.4 above). This is a relatively late OB "hand tablet" from the ancient city Susa, with a diagram of a "symmetric" (*isosceles*) triangle and its circumscribed circle. The triangle is constructed as two right triangles with the sides 50, 40, 30, glued together along a common side of length 40.

The object of the exercise was probably to compute the radius of the circumscribed circle. This can have been done, essentially, in the following way: Let *d*/2, *s*/2, *b*/2 be the sides of the small right triangle with *d*/2 = the radius and with *s*/2 = 1/2 the front of the symmetric triangle (in the diagram called 2' sag '1/2 of the front'). Then *d* and *b* can be found as the solutions to the following *subtractive quadratic-linear system of equations*:

sq. *d*/2 – sq. *b*/2 = sq. *s*/2 = sq. 30 = 15 00
d/2 + *b*/2 = 40 (the height of the symmetric triangle)

Apparently it was known that then

d/2 – *b*/2 = (sq. *d*/2 – sq. *b*/2)/(*d*/2 + *b*/2) = 15 00 / 40 = 15 · 1;30 = 22;30, so that
d/2 = (40 + 22;30)/2 = 31;15, *b*/2 = (40 – 22;30)/2 = 8;45.

The correctly computed values are recorded in the diagram as '31 15 the length' and '8 45', respectively. It is, by the way, easy to check that the diagram is *amazingly accurate*. The person who made the diagram must have known quite well how to work with ruler and compass.

Presumably, he started by drawing, very carefully, a triangle with the sides proportional to 1 00, 50, 50, with the front 1 00 vertical and facing to the left, in agreement with an OB convention. Next, he found the midpoint on the front. (Euclid shows how to bisect a given straight line in *El.* I.10, with reference to the constructions in *El.*I.1 and I.9.) Then he drew a line from there to the opposite vertex of the triangle, a line which necessarily turned out to be horizontal. (*Cf.* the remark in Høyrup, *LWS* (2002), 265 that "the angle between the height and the base is as right as can be controlled on the photo".) The next step of the construction must have been to find the center of the circumscribed circle. How this was done is not known, of course, but it is likely that it was done by use of the method demonstrated by Euclid in *El.* IV.5, with reference to *El.* 1.11.)

The next example is taken from **MS 3049** (Friberg, *RC* (2007), Sec. 11.1), a small fragment of an OB mathematical recombination text, where only one exercise (**§ 1 a**) is preserved on the obverse:

MS 3049 § 1 a, literal translation	explanation
An arc *I curved,*	A circle
20 the transversal,	The diameter is $d = 20$
and 2 that which I went down.[11]	A chord is $p = 2$ below the top
The upper (= left) *transversal* (is) what?	The chord $s = ?$

You:	Do it like this:
20, the transversal, break, then 10 you see,	$d/2 = 10$
10, the descent that like a string is set.	10 = the “vertical” radius
Turn back, then solve(?).	Continue like this:
20, the transversal, break, 10 you see.	$d/2 = 10$
10, a copy, lay down,	Write down $d/2 = 10$ again
let (them) eat each other, then 1 40 you see.	sq. $d/2 = 1\ 40$
2, the upper descent, from 10, the descent	$d/2 - p =$
that like a string is set tear off, then 8 you see.	$10 - 2 = 8 = b/2$
8 let eat itself, then 1 04 you see.	sq. $b/2 = 1\ 04$
1 04 from 1 40 that you sa*w*	sq. $d/2$ – sq. $b/2 = 1\ 40 - 1\ 04$
tear off, then 36 you see	= 36 = sq. $s/2$
Its likeside let come up, then 6 you see.	sqs. $36 = 6 = s/2$
To two repeat, then	$2 \cdot 6 =$
12, the upper *transversal*, you see.	s = the chord
Such is the do*ing.*	Done

The straightforward solution procedure is explained in Fig. 1.12.5, left Given are the diameter d of a circle and the distance p of a chord from the circumference of the circle along a radius orthogonal to the chord. The length of the chord is computed by use of the diagonal rule (see Ch. 2 below), applied to the triple $d/2$, $s/2$, $b/2$.

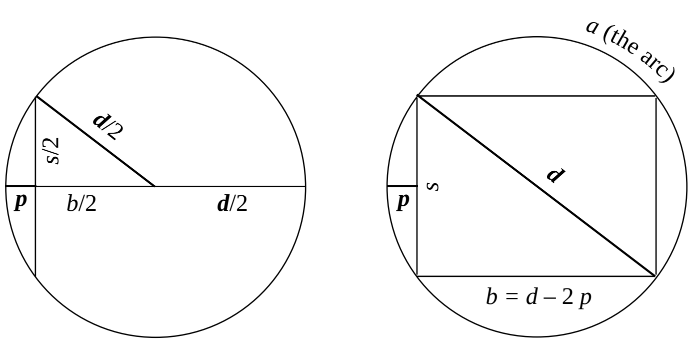


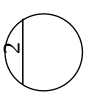
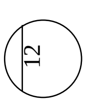
Fig. 1.12.5. Left: MS 3049 § 1 a. Right: BM 85194 # 21.

11. According to an OB convention, in cuneiform texts ‘up’ is to the left and ‘down’ to the right. The well known explanation is that the cuneiform script was originally written in vertical columns, but at some (unknown) point of time, the direction of writing seems to have changed so that texts became written in horizontal rows. After this rotation of the direction of writing, the meaning of ‘up’ and ‘down’ had changed correspondingly.

The strange way of calling half the diameter of the circle ‘that (which) like a string is set’ is not known from any other Babylonian mathematical text. It may refer to the fact that if a piece of string has one end point fixed, then upon rotation of the stretched string the other end point of the string describes a circle. Therefore a radius of a circle can be likened to a ‘string’. (There is no competing word for the radius of a circle used in any other known mathematical cuneiform text.)

On the reverse of MS 3049, a subscript says that the text (originally) contained 6 problems for circles (and also 5 problems for squares, 1 for a triangle, etc.). Although only 1 of these 6 problems has happened to be preserved, it is a reasonable conjecture that the 6 circle problems resulted from the 6 possible ways of choosing 2 of the 4 parameters d, p, $s/2$, and $b/2$ as the *given pair of parameters* in the problem. In § 1 a, the given pair of parameters is d and p. The remaining possible choices of given pairs of parameters are d and $s/2$, d and $b/2$, p and $s/2$, p and $b/2$, $s/2$ and $b/2$. In *TMS* 1, by the way (Fig. 1.12.4), the given parameters are s and $d/2 + b/2$.

BM 85194, another OB mathematical recombination text contains two problems for circles, **## 21-22** (Høyrup, *LWS* (2002), 272):

BM 85194 ## 21-22, literal translation	explanation
 1 the arc,	The circumference is $a = 1\ 00$
2 that which I went down.	A chord is $p = 2$ below
The transversal (is) what?	The chord $s = ?$
You:	Do it like this:
2 square, 4 you see.	$2\ p = 2 \cdot 2 = 4$
4 from 20, the transversal, tear off,	$d = a/3 = 20$, $d - 2\ p = 20 - 4$
16 you see.	$= 16 = b$
20, the transversal, square, 6 40 you see.	sq. d = sq. $20 = 6\ 40$
16 square, 4 16 you see.	sq. b = sq. $16 = 2\ 24$
4 16 from 6 40 tear off, 2 24 you see.	sq. d – sq. $b = 6\ 40 - 2\ 24$
2 24 is what equalsided?	sqs. 2 24
12 equalsided, the transversal.	$= 12 = s$
Such is the doing.	Done
 If an arc 1 I curved,	$a = 1\ 00$
12 the transversal.	$s = 12$
That which I went down?	$p = ?$
You:	Do it like this:
20, the transversal, square, 6 40 you see.	$d = a/3 = 20$, sq. d = sq. $20 = 6\ 40$

12 square, 2 24.	sq. *s* = sq. 12 = 2 24
From 6 40 tear off, 4 16 you see.	sq. *d* – sq. *s* = 4 16 = sq. *b*
16 is what equalsided? 4 equalsided.	sqs. 16 = 4 (error for sqs. 4 16 = 16)
(In) half 4 break, 2 you see,	1/2 · 4 = 2 (cheating)
2 that which you went down.	*p* = 2 (the correct answer)
The doing.	Done

The stated problem in BM 85194 # 21 is closely related to the problem in MS 3049 § 1. The only difference is that in BM 85194 # 21 the circumference *a* = 3 *d* is given (with the usual Babylonian approximation Θ = appr. 3), while in MS 3049 § 1 the diameter *d* is given directly. The straightforward solution procedure in # 21 is based on a geometric construction like the one in Fig. 1.12.5, right. It is an interesting variant of the solution procedure based on the construction in Fig. 1.12.5, left. Note that because the sides of the triangle in the circle to the right are twice as long as the sides in the right triangle in the circle to the left, it is "obvious" that in the figure to the right *the triangle with its diagonal along the diameter is a right triangle*. (*Cf.* a similar remark in Høyrup, *LWS* (2002), 274.)

In BM 85194 # 22, the stated problem is to find *p* when the circumference *a* = 3 *d* and the chord *s* are given. The solution is corrupt, but leads nevertheless to the correct answer (known in advance from # 21).

A dressed up problem, closely, although indirectly, related to the circle problems discussed above is problem **# 9** in **BM 85196**, like BM 85194 an OB mathematical recombination text from the ancient city Sippar. This is the well known "pole-against-a-wall problem", discussed before by several authors, for instance, Friberg, *HM* 8 (1981), Muroi, *KK* 30 (1991), Høyrup, *LWS* (2002), 275, Melville, *HM* 34 (2004), 151.

BM 85196 # 9, literal translation	explanation
A pole. 30, a reed, at a wall is placed equally.	*c* = 30 (;30 ninda = 1 reed)
Above, 6 it went down,	*s* = 6
below, *what did it move away*?	*b* = ?
You:	Do it like this:
30 square, 15 you see.	sq. *c* = 15
6 from 30 tear off, *24 you see.*	*c* – *s* = 30 – 6 = 24
24 square, 9 36 you see.	sq. (*c* – *s*) = sq. 24 = 9 36
9 36 from *15 tear off*, 5 24 you see.	sq. *c* – sq. (*c* – *s*) = 15 – 9 36 = 5 24
5 24 what is it equalsided?	sqs. 5 24
18 it is equalsided.	= 18

18 on the ground it moved away.	= *b*
If 18 on the ground,	Conversely, given that *b* = 18
above, what did it go down?	*s* = ?
18 square, 5 24 you see.	sq. *b* = sq. 18 = 5 24
5 24 from 15 tear off, 9 36 you see.	sq. *c* – sq. *b* = 15 – 5 24 = 9 36
9 36, what its it equalsided?	sqs. 9 36
24 it is equalsided.	= 24 = *a*
24 from 30 tear off,	*c* – *a* = 30 – 24
6 you see, (what) it went down.	= 6 = *s*
The doing.	Done

In this dressed up problem, the stated question is as follows:

> A wooden pole with the length 1 reed = 1/2 ninda (about 3 meters) at first stands upright against a wall of the same height. Then it starts sliding so that its upper end moves straight down ;06 ninda. How much does its lower end move along the ground?

The situation is illustrated in Fig. 1.12.6, left, where it is assumed that a pole of length *c* at first was standing upright along a wall of height *c*. Its top then slid down the distance *s* and its foot slid out a corresponding distance *b*. The set task is to find *b* if *c* and *s* are given. The connection between this dressed up problem and straightforward circle problems of the kinds discussed above is demonstrated in Fig. 1.12.6, right.

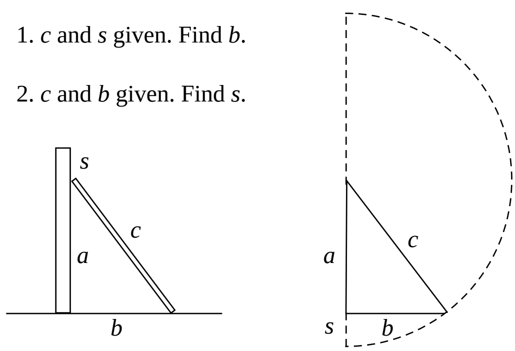


Fig. 1.12.6. BM 85196 # 9. A pole-against-a-wall problem.

The solution to the stated problem is obtained without effort by use of the diagonal rule. It is found that *b* = ;18 ninda. This result is then checked by a reversal of the problem.

The pole-against-a-wall problem in the form given to it in BM 85169 # 9 is in itself quite trivial and uninteresting. Yet it is important for a couple of reasons. One reason is that dressed up problems like this one are quite rare in OB mathematics. The other reason is that the problem type reappears in a Seleucid mathematical recombination text and in an Egyptian mathematical recombination text, both from the latter third of the first millennium BCE (See Friberg, *UL* (2005), Sec. 3.1 b)

As mentioned above, the corpus of known OB metric algebra problems for circles and chords is small, compared to the related corpus of known metric algebra problems for squares and rectangles. Yet this fact may in part be due to unlucky circumstances. Thus, it is clear that all known OB problems for circles and chords are isolated exercises in mathematical recombination texts. It is likely that there once existed one or several extensive and well organized OB mathematical theme texts with relatively large numbers of such problems, from which exercises like MS 3049 § 1 a-[f], BM 85194 ## 21-22, and BM 85169 # 9 were borrowed. Be that as it may, there appears to be a close relation between on one hand such OB problems for circles and chords, and on the other hand *El.* II.11 and II.14, and their hypothetical forerunners *El.* II.11* and II.14* (Sec. 1.7 above).

1.12 c. Old Babylonian problems for non-symmetric trapezoids

The only known OB predecessors to *El.* II.12 and II.13 (see Fig. 1.8.1) can be found in **VAT 7351**, a mathematical cuneiform text from the ancient city Uruk. That text is extensively discussed in Friberg, *UL* (2005), Sec. 3.7 c.[12] Here is the text of the last one of the four exercises in that text:

VAT 7531 # 4, literal translation

2 43 30 the long length, 1 56 30 the short length,
1 37 30 the upper (= left) front, 1 30 30 the lower (= right) front.
Its area, how much it is, find out,
then to 5 brothers equally divide it, and (each) soldier show him his stake.

Properly speaking, VAT 7531 # 4 is an *assignment* rather than an exercise, since the question is not followed by a solution procedure and an answer. The object considered in the text is a quadrilateral field with the given lengths 2 43;30 and 1 56;30 (ninda), and the given fronts 1 37;30 and 1 30;30 (ninda). The field is to be divided equally among 5 brothers.

12. See now also the trapezoid diagonal problem in VAT 8393 in Appendix 1 below.

The form of the given field is not uniquely determined by the four sides. However, it must have been (silently) understood that the field should have the form of a trapezoid composed of a central rectangle and two flanking non-equal triangles (Fig. 1.12.7).

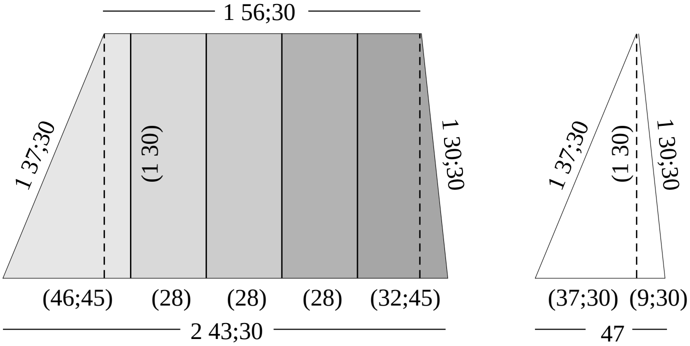


Fig. 1.12.7. VAT 7531 # 4. A trapezoidal field divided in five equal parts.

If the rectangle is removed, what remains is a non-symmetric (scalene) triangle with the sides 1 37;30, 1 30;30, and the base 47. Evidently, the OB author of this text was confident that his students knew how to compute the height of a non-symmetric (*scalene*) triangle! The way they would have done it was probably as follows: Let a, b, c be the sides of the triangle, and suppose that the height h against the side b divides b into the segments p and q, where p is greater than q. (See above, Fig. 1.8.1, right.) Then,

$p + q = b$ and
sq. c – sq. p = sq. a – sq. q (by the diagonal rule, since both are equal to sq. h).

This leads to a *quadratic-linear system of equations* for p and q:

$p + q = b$ and sq. p – sq. q = sq. c – sq. a.

This quadratic-linear system of equations can be solved by use of metric algebra. The solution can take several forms, for instance the following:

p = {sq. b + (sq. c – sq. a)}/(2 b), q = {sq. b – (sq. c – sq. a)}/(2 b).

With c, a, b = 1 37;30, 1 30;30, 47, these equations show that

p = 37;30, q = 9;30.

It is then easy to compute h = 1 30. The remaining part of the solution procedure for VAT 7531 # 4 is straightforward.

The result above shows that the triangle with the sides 1 37;30, 1 30;30, 47 is composed of two right triangles with the sides 1 37;30, 1 30, 37;30 = 7;30 · (13, 12, 5) and 1 30;30, 1 30, 9;30 = 30 · (3 01, 3 00, 19), glued together along a common side of length 1 30. This is clearly an OB predecessor to what is commonly known as "Heronic triangles". (*Cf.* the discussion of the pseudo-Heronic *Geometrica* 12 in Sec. 18.2 below.)

1.13. Late Babylonian Solutions to Metric Algebra Problems

1.13 a. Problems for rectangles and squares

The discussion above of *OB* forerunners to *Elements* II will be rounded off in this section with a discussion of solution procedures for metric algebra problems in **W 23291**, a *Late Babylonian* mathematical recombination text from Uruk, early in the second half of the first millennium BCE (Friberg, *BaM* 28 (1997)). W 23291 and the related text **W 23291-x** (Friberg, *et al.*, *BaM* 21 (1990)) are both concerned with the interesting topic of *a great variety of ways of measuring surface content*.

The first paragraph of W 23291 contains what looks like *the beginning of a well organized theme text with metric algebra problems*.

W 23291 § 1: Common seed measure and some basic problems in metric algebra

1 a The seed measure of a hundred-cubit-square. Metric squaring
1 b A rectangle of given front and seed measure. Metric division
1 c A square of given seed measure. Metric square side computation
1 d A rectangle of given side-sum and seed measure. Basic problem B1a
1 e A rectangle of given side-difference and seed measure. Basic problem B1b
1 f A square band of given width and seed measure. Basic problem B3b
1 g A circle of given seed measure divided into five circular bands of given width

In § 1 of W 23 291, the surface content of every square, rectangle, or other plane figure mentioned, is expressed in terms of "seed measure", by which is meant a capacity measure *proportional, in a certain ratio, to the area of the figure in question*. More precisely, the seed measure applied in § 1 is what may be called "common seed measure" (csm), the particular kind of seed measure characterized by the following igi.gub še.numun 'seed constant':

c_s = '20' = ;20 barig (= 1/3 barig) on each square of side 1 00 cubits (= 60 cubits).[13]

The barig (Akk. *parsiktu*) was the "basic unit" of Late Babylonian capacity measure, in the sense that *sexagesimal multiples* of the barig were used in computations involving capacity measures and in references to metrological constants like the seed constant.

In the present text, just as in the related text W 23291-x, a dual *ninda-and-cubit format* is used in many of the solution procedures. What this means is that the solution of a given problem is presented twice, first in a "ninda section" where the ninda (= 6 m.) is the basic unit of length measure, then in a parallel "cubit section" where the cubit (= 1/2 m.) is the basic unit. In the cubit sections, the seed constant for common seed measure is '20' = 1/3 barig./sq. (60 c.), as explained above. In the ninda sections it is, equivalently,

c_s = '48' = 48 barig on each square of side 1 00 ninda (= 60 ninda).

The equivalence of the two alternative expressions is obvious, since

1 00 cubits = 5 ninda, so that sq. (60 cubits) = sq. (5 ninda) = 25 sq.ninda.

Therefore,

1/3 barig / sq. (60 cubits) = 1/3 barig / 25 sq.ninda
= 12 · 12 · 1/3barig / sq. (60 ninda) = 48 barig / sq. (60 ninda).

The ninda section of a solution procedure is preceded by the phrase

šum-ma 5 *am-mat-ka* 'if 5 is your cubit'.

This phrase refers to the circumstance that when the ninda (= 12 cubits) is chosen as the basic unit of length measure, then 1 cubit is equal to 1/12 = 5/60 = ;05 of that basic unit. For a similar reason, the cubit sections are preceded by the phrase

šum-ma 1 *am-mat-ka* 'if 1 is your cubit'.

The seed measure of a hundred-cubit-square. Metric squaring

W 23291 § 1 a, literal translation explanation

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13. Note that the use of zeros and separators in the transliteration of numbers in a mathematical cuneiform text tends to destroy the inherent simplicity of the definitions of various Babylonian mathematical and metrological "constants". So, for example, the seed constant for common seed measure was not understood as ;00 00 20 barig/sq.cubit. Nor was it understood as ;20 barig/sq.(60 cubits). Instead it was almost certainly understood as '20' times the area, with the silent understanding that *when the sides of a rectangle amount to a few sixties of cubits, then the seed measure of the rectangle amounts to a few barig*.

1 hundred cubits length, 1 hundred cubits front.	A square with the side s = 100 c.
What shall the seed be?	C = ?
If 5 is your cubit:	**If you count with ninda:**
8 20 is 1 hundred cubits.	s = 100 c. = 100/12 n. = 8;20 n.
8 20 steps of 8 20 go,1 09 26 40.	A = sq. 8;20 n. = 1 09;26 40 sq. n.
1 09 26 40 steps of 48 go,	$C = c_s \cdot A$ = '48' $\cdot A$
55 33 20, 5 bán 3 1/3 sìla of seed.	= ;55 33 20 barig = 5 bán 3 1/3 s.
If 1 is your cubit:	**If you count with cubits:**
1 40 is 100 cubits.	s = 100 c. = 1 40 c.
1 40 steps of 1 40 go, 2 46 40.	A = sq. 1 40 c. = 2 46 40 sq. c.
2 46 40 steps of 20 go, 55 33 20,	$C = c_s \cdot A$ = '20' $\cdot A$ = ;55 33 20 barig
5 bán 3 1/3 sìla of seed.	= 5 bán 3 1/3 sìla.

The problem stated and solved in § 1 a of W 23291 can be explained as follows: A square of side 100 cubits may be called a "hundred-cubit-square", or simply a "100-c.-square". As is shown by Late Babylonian metrological tables, notably *BE 20/1,* 30 (see Friberg, *GMS 3* (1993), 399), a "hundred-cubit" was occasionally used, in addition to the cubit and the ninda, as the *third* basic unit of Late Babylonian length measure. For this reason, it would be convenient to have at hand a *third* value of the seed constant for common seed measure, in addition to 48 barig./sq. (60 ninda) and ;20 barig./sq. (60 c), namely the common seed measure (csm) of a hundred-cubit-square. In W 23291 § 1 a this value is computed twice. In the ninda section, it is computed in the following way:

If, as in the present text, 1 n. = 12 c., then 1 c. = ;05 n. Therefore,

the *side* of the 100-cubit-square is s = 100 c. = 100 · ;05 n. = 8;20 n., so that

the *area* of the 100-cubit-square is A = sq. (100 c.) = sq. (8;20 n.) = 1 09;26 40 sq. n.

Note that all computations are carried out here in the traditional Babylonian way, that is by use of sexagesimal arithmetic. That is so in spite of the fact that in the statements of the problems in W 23291 § 1 linear measures are expressed as *decimal* multiples of the cubit!

Next, an application of the appropriate value of the seed constant proves that the *seed measure* of the 100.cubit-square is

C = 48 barig · 1 09;26 40 /sq. 1 00 = ;55 33 20 barig.

The final step of the computation is to convert this sexagesimal multiple of the barig into a *conventionally expressed capacity number*. This can be done, most conveniently, in the following way. (The computation is based on the fact that fractions of the barig when multiplied by a factor 6 yield multiples of the sub-unit bán, and that fractions of the bán when multiplied by another factor 6 yield multiples of the smaller sub-unit sìla.

C = ;55 33 20 barig = 6 · ;55 33 20 bán = 5;33 20 bán
= 5 bán + 6 · ;33 20 sìla = 5 bán 3;20 sìla = 5 bán 3 1/3 sìla.

In the *cubit section* of § 1 a, the computation of the common seed measure of a hundred-cubit-square proceeds in an entirely parallel way.

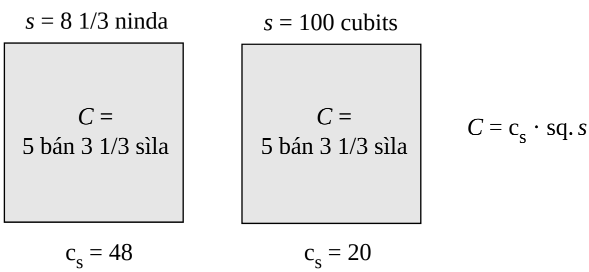


Fig. 1.13.1. W 23291 § 1 a. Metric squaring. The seed measure of a 100-cubit-square.

A rectangle of given front and seed measure. Metric division

W 23291 § 1 b, literal translation	explanation
1 hundred cubits front.	s = 100 cubits
The length, what shall it be long,	u = ?
so that there will be 1 gur of seed?	if, in addition, C = 1 gur = 5 barig
Since you do not know:	**Do it like this:**
The opposite of the front of the field raise,	Compute the reciprocal of the front
and steps of the opposite of	and multiply with the reciprocal
the seed constant you go,	of the seed constant
and the seed that was said to you go,	and multiply with the seed measure
the length you will see.	then you will se the length
If 5 is your cubit:	**If you count with ninda:**
8 20 is 1 hundred cubits.	s = 100 cubits = 8; 20 ninda
The opposite of 8 20, 7 12.	rec. s = rec. 8 20 = 7 12
7 12 steps of 1 15 go, 9.	rec. s · rec. c_s = 7 12 · 1 15 = 9
9 steps of 5 go, 45. 45 as much as	rec. s · rec. $c_s \cdot C$ = 9 · 5 = 45
the length of your field you will set.	u = 45
If 1 is your cubit:	**If you count with cubits:**
1 40 is 1 hundred cubits.	s = 1 40 cubits = 100 cubits
The opposite of 1 40, 36.	rec. s = rec. 1 40 = 36
36 steps of 3 go, 1 48.	rec. s · rec. c_s = 36 · 3 = 1 48

1 48 steps of 5 go, 9, that <for> 1 40 cubits as much as the length you will set.	rec. s · rec. $c_s \cdot C$ = 1 48 · 5 = 9 = u, when s = 1 40 cubits

The statement of the problem in the first three lines of § 1 b is followed by a *general computation rule* headed by the phrase mu nu zu-*ú* 'since you do not know'. It is easily checked that the two parallel solution procedures in the ninda and cubit sections of the paragraph are two *different but equivalent numerical implementations* of this general computation rule.

The computation in each of the two cases is straightforward. The ninda section, for instance, begins with the computation of the reciprocals of the given front (100 cubits) and (although not explicitly in the text) of the reciprocal of the seed constant '48'. Note that all computations are carried out in terms of *relative* (floating) sexagesimal numbers without any indication of their absolute size.

The answer is given in relative sexagesimal numbers as

u = '45' in the ninda section and u = '9' in the cubit section.

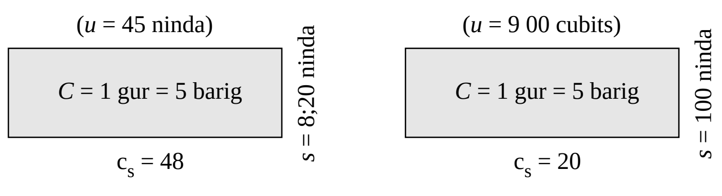
Since the length is always greater than the front in Babylonian mathematical texts dealing with rectangles, the obvious interpretation of this result in relative numbers is that the length u is equal to 45 ninda = 9 00 cubits. It is easy to verify that, with this value for u,

the area A = 45 ninda · 8;20 ninda = 6 15 sq. ninda = ;06 15 · sq. (60 ninda).

Therefore, as required,

the seed measure C = 48 · ;06 15 barig = 5 barig.

The result of the dual computation is summarized in Fig. 1.13.2 below.



$C = c_s \cdot u \cdot s \quad \cong \quad u = \text{rec. } c_s \cdot \text{rec. } s \cdot C$

Fig. 1.13.2. W 23291 § 1 b. Metric division.

A square of given seed measure. Metric square side computation

W 23291 § 1 c, literal translation explanation

The field, what each shall I make equalsided	Which are the equal sides (of a square)
so that 1 gur 2 bán will be the seed?	with seed measure 1 gur 2 bán?
Since you do not know:	**Do it like this:**
The seed that was said to you,	Take the mentioned seed measure
— what was said	multiply it with
steps of the seed constant,	the <reciprocal of> the seed constant
you go, the length.	compute the square side
If 5 is your cubit:	**If you count with ninda:**
5 20 steps of 1 15 go, 6 40,	C · res. c_s = 5 20 · 1 15 = 6 40
of which 20 each way take.	6 40 = sq. 20
20 ninda each way you make equalsided.	s = 20 <ninda> is the square side
If 1 is your cubit:	**If you count with cubits:**
5 20 steps of 3 go, 16,	C · res. c_s = 5 20 · 3 = 16
of which 4 each way take.	16 = sq. 4
2 hundred 40 cubits each	s = 4 00 = 240 <cubits>
you make equalsided.	is the square side

This exercise is quite straightforward. The given seed measure is

1 gur 2 bán = 5 1/3 barig = 5;20 <barig>,

and the computed square side is

20 ninda = 20 · 12 cubits = 240 cubits.

It is interesting that the cubits are counted decimally in the answer.

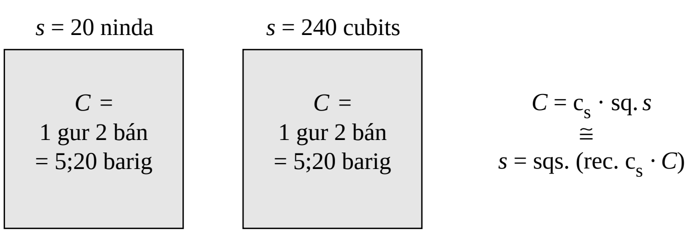


Fig. 1.13.3. W 23291 § 1 c. Metric square side computation.

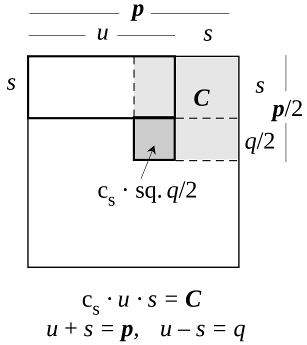
A rectangle of given side-sum and seed measure. Basic problem B1a

W 23291 § 1 d, literal translation	explanation
A field of 1 bán seed.	C = 1 bán (= ;10 barig)
Length and front heap, it is 1 30 cubits.	$u + s$ = 1 30 cubits (= 7;30 ninda)
The length, what shall it be,	u = ?
and the front what shall it be?	s = ?
Since you do not know:	**Do it like this:**

1/2 to the heap, 1 30 cubits, raise,	$1/2 \cdot$ the sum 1 30
45 cubits equalsided,	= 45 cubits, squared
to the constant of seed [raise] it.	times the seed constant c_s
1 bán of seed out of it lift,	subtract C = 1 bán
the opposite of the constant you raise to it,	times rec. c_s
<and bring (out)>	<Compute the square side>
to 45 cubits add on, the length,	add 45 cubits, you get the length
from 45 cubits lift, the front.	Subtract 45 cubits, you get the front
If 5 is your cubit:	**If you count with ninda:**
Length and front, 7 30, 1/2 of it, 3 45, take.	$(u + s)/2 = p/2 = 7\ 30/2 = 3\ 45$
3 45 steps of 3 45 you go, 14 03 45	sq. $p/2$ = sq. 3 45 = 14 03 45
and steps of 48 go, 11 15,	$c_s \cdot$ sq. $p/2 = 48 \cdot 14\ 03\ 45 = 11\ 15$
1 07 30 bán-measures.	= 1;07 30 bán
1 bán, 10, from 11 15 lift,	$c_s \cdot$ sq. $p/2 - C = 11\ 15 - 10$
1 15 the remainder.	= 1 15 (= $c_s \cdot$ (sq. $p/2 - A$))
1 15 steps of 1 15,	times 1 15 (= rec. c_s)
1 33 45, of which 1 15 each take.	= 1 33 45 = sq. 1 15 (= sq. $q/2$)
1 15 to 3 45 join,	$p/2 + q/2 = 3\ 45 + 1\ 15$
5 ninda, the length,	$= 5 = u$
1 15 from 3 45 lift,2 30, the front.	$p/2 - q/2 = 3\ 45 - 1\ 15 = 2\ 30 = s$
If 1 is your cubit:	**If you count with cubits:**
1 30 cubits, 1/2 of it, 45, take,	$p/2 = 1\ 30/2 = 45$
45, steps of 45 go, 33 45,	sq. $p/2$ = sq. 45 = 33 45
33 45 steps of 20 go,	$c_s \cdot$ sq. $p/2 = 20 \cdot 33\ 45$
the seed, 11 15.	= 11 15
1 bán, 10, from 11 15 lift,	$c_s \cdot$ sq. $p/2 - C = 11\ 15 - 10$
1 15 the remainder.	= 1 15 (= $c_s \cdot$ (sq. $p/2 - A$))
1 15 steps of 3 go,	times 3 (= rec. c_s)
3 45, of which 15 each — take.	= 3 45 = sq. 15 (= sq. $q/2$)
15 to 45 join	$p/2 + q/2 = 45 + 15$
1-sixty cubits, the length,	$= 1 \cdot 60$ cubits $= u$
from 45 lift, 30 cubits, the front.	$p/2 - q/2 = 30$ cubits $= s$

In the present exercise, § 1 d, just as in §§ 1 b and 1 c above, the question is followed by a *general computation rule* headed by the phrase mu nu zu-*ú* 'since you do not know'. It is interesting to note that in § 1 d the author of the text has not been quite successful in his formulation of a general computation rule, since he explicitly mentions the half-sum of the sides of the rectangle as '45 cubits' instead of just as '1/2 the heap'.

In the cuneiform text, there is no figure accompanying exercise § 1 d. Yet the wording of the solution procedure is such that there can be no doubt whatsoever that the author of the problem had in mind a *geometric* interpretation of the given problem and its solution. The most likely candidate for such an interpretation is based on a set-up like the one in Fig. 1.13.4 below, nearly identical with the set-up in Fig. 1.4.2 above, left, the suggested Babylonian style interpretation of the diagram in *El.* II.5. The only difference is the use of seed measure instead of area measure.



In the cubit section:

Given:
$\boldsymbol{p}$ = 1 30 cubits, $\boldsymbol{C}$ = 1 bán = ;10 barig
c_s = ;20 barig/ sq. (1 00 cubits)

Computed:
$c_s \cdot$ sq. $q/2$ = $c_s \cdot$ sq. $\boldsymbol{p/2}$ – $\boldsymbol{C}$ = ;01 15 barig
sq. $q/2$ = 3 45 sq. c. = sq. (15 cubits)
u = $\boldsymbol{p/2}$ + $q/2$ = 45 c. + 15 c. = 1 00 c.
s = $\boldsymbol{p/2}$ – $q/2$ = 45 c. – 15 c. = 30 c.

Fig. 1.13.4. W 23291 § 1 d. A rectangle of given side-sum and seed measure.

A rectangle of given side-difference and seed measure. Type B1b

The problem stated in **W 23291 § 1 e** is to find the length *u* and front *s* of a rectangle, if the seed measure of the rectangle is 1 bán 4 sìla, and if the length exceeds the front by 10 cubits. This is a routine variation of the problem in § 1 d, and it can be solved by an obvious modification of the solution procedure in that paragraph. Thus, if the given side-difference is called q = 10 cubits, then sq. $q/2$ = 25 sq. cubits, and $c_s \cdot$ sq. $q/2$ = ;08 20 barig, since c_s = ;20 barig/sq. (60 cubits). On the other hand, the given seed measure of the rectangle is C = 1 bán 4 sìla = ;16 40 barig, since (in this Late Babylonian text) 1 barig = 6 bán and 1 barig = 6 sìla.Therefore, $C + c_s \cdot$ sq. $q/2$ = ;16 48 20 barig = $c_s \cdot$ sq. $p/2$. Hence, sq. $p/2$ = 50 25 sq. cubits, and $p/2$ = 55 cubits. Thus, finally, u = (55 + 5) cubits = 1 00 cubits = 5 ninda, and s = (55 – 5) cubits = 50 cubits = 4 ninda 2 cubits.

The problem in W 23291 § 1 e is of course, except for the use of seed measure instead of area measure, *a basic rectangular-linear system of equations of type B1b*. It is, therefore, related to *El.* II.6.

A square band of given width and seed measure. Type B3b

The statement of the problem in **W 23291 § 1 f** and the associated general computation rule are both completely lost. Nevertheless, the fortunate circumstance that almost the whole cubit section of the solution procedure is preserved allows a reconstruction of most of the problem.

W 23291 § 1 f, literal translation	explanation
....	....
If 1 is your cubit:	**If you count with cubits:**
10 is 10 cubits.	s = 10 cubits = 10
The opposite of 10 raise, 6.	rec. s = rec. 10 = 6
6 steps of 3 you go, 18.	rec. $s \cdot$ rec. c_s = 6 · 3 = 18
18 steps of 10, 1 bán, raise, 3.	rec. $s \cdot$ rec. $c_s \cdot C$ = 18 · 10 = 3
3, its 4th raise, 45.	rec. $s \cdot$ rec. $c_s \cdot C/4$ = 3/4 = 45 = u
10 from 45 lift, 35 the remainder.	$u - s$ = 45 – 10 = 35
35 cubits equalsided.	q = 35
[...]	[$q + 2\,s$ = 35 + 20 = 55 = p]

The form of this solution procedure, and the position of W 23291 § 1 f in the text, between the better preserved § 1 e and § 1 g, makes it fairly certain that the problem stated in § 1 f was to find the sides of the squares bounding a square band, when the width (10 cubits) and the seed measure (1 bán) of the square band are given. The way in which a solution to this problem could be found is illustrated in Fig. 1.13.5 below.

The problem treated in W 23291 § 1f can be formulated as follows. Let the square band be interpreted as the difference of two parallel and concentric squares with the sides p and q, respectively. Then p and q can be computed as the solutions to the following *subtractive quadratic-linear system of equations of type B3b*:

$c_s \cdot$ (sq. p – sq. q) = C = 1 bán, ($p - q$)/2 = s = 10 cubits, p, q = ?

In modern terminology, all that is required to solve a problem of this type is an application of the algebraic “conjugate rule”

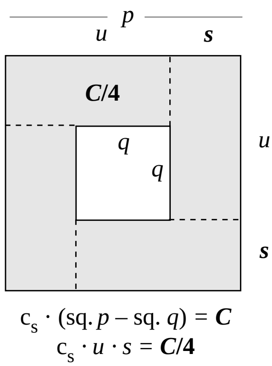
sq. p – sq. $q = (p + q) \cdot (p - q)$,

followed by a straightforward division. A *metric* counterpart of this algebraic conjugate rule can be based on the observation that a square band can be constructed in two ways, either as the space between two parallel (and, if so desired, concentric) squares with the sides p, q or as a ring of four rectangles with the sides *u*, *s*. In the present text, where surface content is measured in terms of seed measure, the resulting "metric conjugate rule" takes the following form:

$C = c_S \cdot (\text{sq. } p - \text{sq. } q) = 4 \cdot c_S \cdot u \cdot s = 4 \cdot c_S \cdot (p + q)/2 \cdot (p - q)/2.$

Accordingly, the recorded solution procedure in W 23291 § 1 f corresponds to the solution formula

$u = (p + q)/2 = 1/c_S \cdot 1/s \cdot S/4, \quad p = u + s, \quad q = u - s.$



In the cubit section.

Given:
$s = (p - q)/2 = 10$ cubits, $C = 1$ bán = ;10 barig
c_s = ;20 barig/sq. (1 00 cubits)

Computed:
$1/s \cdot 1/c_s \cdot C/4 = 45$ cubits $= u = (p + q)/2$
$u - s = 35$ cubits $= q$
$(q + 2s = 55$ cubits $= p)$

Fig. 1.13.5. W 23291 § 1 f. A square band of given width and seed measure.

The metric algebra problem in W 23291 § 1 f is obviously closely related to *El.* II. 8, which can be seen if, for instance, Fig. 1.13.5 is compared with Fig. 1.5.2. Note however, that Euclid chose to operate with *non-concentric* parallel squares, and that, as a consequence of this choice, in *El* II.8 the difference between the two squares takes the form of a square corner (a *gnomon*) rather than that of a square band.

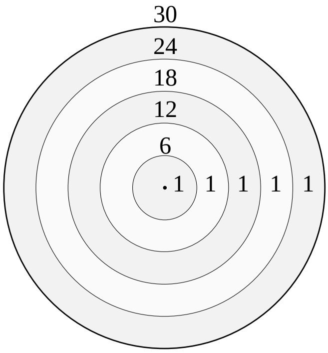
1.13 b. Problems for circles

A circle of given seed measure divided into five bands of equal width

W 23291 § 1 g, literal translation	explanation
A field of 1 barig seed I curved. Steps 4, 1 each, the decrease came down. What each are the arcs I curved, from the outermost arc to the innermost arc?	A circle of seed measure 1 barig and four inner circles with 1 ninda's distance What are the arcs (circumferences) of all the circles from the first to the last?

Since you do not know:	**Do it like this:**
1 steps of 6 go, 6.	$1 \cdot 6 = 6$
6 from [···] 30 lift, 24 the remainder,	$30 - 6 = 24$
the second arc.	the second arc.
6 from 24 lift, 18 the remainder,	$24 - 6 = 18$
18, the third arc.	the third arc
6 from 18 lift, 12 the remainder	$18 - 6 = 12$
12, the fourth arc.	the fourth arc
6 from 12 lift, 6 the remainder,	$12 - 6 = 6$
6, the fifth arc.	the fifth arc
6 is the innermost arc field	6 is arc of the innermost circle
he will take off.	since $6 - 6 = 0$

This exercise is only loosely related to the six preceding metric algebra problems. Note, in particular, that there is no general computation rule, and no separate ninda and cubit sections. (The basic unit of length measure is the ninda.) It is also likely that essential parts of the problem have been omitted both at the beginning and at the end of the problem.



$C = c_s \cdot ;05 \cdot$ sq.a,
c_s = 48 barig/sq. (1 00 ninda)
≅
a = sqs. ($1/c_s \cdot 12 \cdot C$)
$d = ;20 \cdot a$

Example:
C = 1 barig
≅
a = 30 ninda, d = 10 ninda

Fig. 1.13.6. W 23291 § 1 g. A circle of given seed measure. Five circular bands.

Thus, after it has been stated that the seed measure of the given circle is 1 barig, the arc a and the diameter d of the circle must have been computed, but this is not done explicitly in the text. To find the arc of a circle when the seed measure of the circle is given is a problem of the same type as the one in W 23291 § 1 c, to find the side of a square of given seed measure. The omitted computation should have had the following form:

$c_s \cdot ;05 \cdot$ sq.$a = C$ = 1 barig, c_s = 48 barig/ sq. (60 ninda) (1/4Θ= appr. ;05)
a = sqs. ($1/c_s \cdot 12\, C$) = sqs. (15 00 sq. ninda) = 30 ninda,

d = ;20 · a = 10 ninda   (1/Θ= appr. ;20)

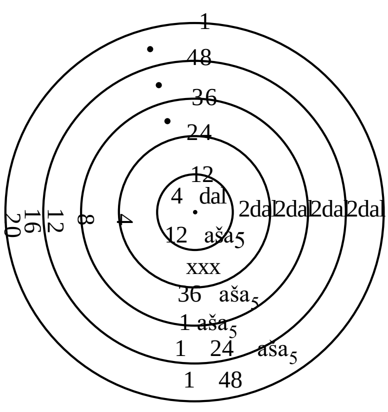
The preserved part of the solution procedure can be explained as follows: If the width of each one of the circular bands is 1 ninda, then the diameter of each band is equal to the diameter of the preceding band minus 2 ninda, and the arc of each band is equal to the arc of the preceding band minus 6 ninda (counting with Θ= appr. 3). This gives the arcs listed in the text, 30, 24, 18, 12, and 6 ninda.

The computation of the seed measure of each one of the five circular bands has, for some reason, been omitted from the text of § 1g.

In the closely related Late Babylonian mathematical recombination text **W 23291-x** (Friberg, *et al.*, *BaM* 21 (1990)) there are, among other things, parallels to four of the seven exercises in W 23291 § 1. It is interesting to compare the parallel exercises with each other, for the reason that the exercises in W 23291-x resemble OB mathematical exercises more than what the corresponding exercises in W 23291 do.

Here is, first, the text of the parallel in W 23291-x to W 23291 § 1 g:

A circle of given circumference divided into five bands of equal width



a_1 = 100 ninda	d_1 = 20 ninda
a_2 = 48 ninda	d_2 = 16 ninda
a_3 = 36 ninda	d_3 = 12 ninda
a_4 = 24 ninda	d_4 = 8 ninda
a_5 = 12 ninda	d_5 = 4 ninda

A_1 = 1 48 šar
A_2 = 1 24 šar
A_3 = 1 00 šar
A_4 = 36 šar
A_5 = 12 šar

Fig. 1.13.7. The diagram associated with W 23291-x § 2.

W 23291-x § 2, literal translation	explanation
1 (= the first) arc-field 1(60) ninda I curved.	A circle of arc 1 00 ninda
Steps, 4, 2 ninda each	4 inner circles with a distance of

as decrease I made come up.	2 ninda between each pair
What each are the fields?	The areas between the circles = ?
(Fig.)	(Fig.)
54 steps of 2, 1 48,	54 · 2 = 1 48
1(iku) 8 šar is the outermost decrease.	= 1 iku 8 šar (the area of band 1)
42 steps of 2, 1 24,	42 · 2 = 1 24
1/2(iku) 34 šar is the 2nd decrease.	= 1/2 iku 34 šar (the area of band 2)
30 steps of 2, 1,	30 · 2 = 1 00
1/2(iku) 10 šar is the 3rd decrease.	= 1/2 iku 10 šar (the area of band 3)
18 steps of 2, 36,	18 · 2 = 36
36 šar is the 4th decrease.	= 36 šar (the area of band 4)
12 steps of 12, 2 24,	12 · 12 = 2 24
2 24 steps of 5 go, 12,	2 24 · 5 = 12
12 šar is the 5th and innermost decrease.	= 12 šar (area of the innermost circle)
Heap them, all of them are 3(iku).	Check: 3 iku = the total area

In this exercise, four circular bands, all of width 2 ninda, are broken off from a circle of given arc length 1 00 n. The exercise is illustrated by a diagram, exhibiting the arcs of the five circles bounding the circular bands, the diameters of those circles, and the *area measures* of the four circular bands and the innermost circular core. In the solution procedure, only the computation of the area measures of the circular bands is expressly indicated. The use of traditional area measure as well as the absence of a general computation rule and of separate ninda and cubit sections are some conspicuous features of the first three exercises on W 23291-x, including this one. It is likely that these initial exercises were copied with only superficial changes from some OB mathematical text. The parallel text W 23 291 § 1 g, on the other hand, may be viewed as a Late Babylonian *revised edition* of a text like W 23291-x § 2, with the OB area measure replaced by the Late Babylonian seed measure.

In this connection it may be noted that it is likely that the purpose of computations with the ninda as the basic length unit in the ninda sections of Late Babylonian mathematical exercises was to make students familiar with the *Old Babylonian* way of counting, so that they would be able to understand Old Babylonian mathematical texts. In Late Babylonian non-mathematical texts, the *cubit* is always the basic length measure.

The other parallels to W 23291 § 1 in W 23291-x are the exercises in § 4 of the latter text. They are reproduced here, in literal translation.

W 23291-x § 4 a-d

§ 4 a. Rules for the computation of areas of rectangles and square sides

Reeds, such that	Reed measure (surface content) when
1-ninda-reed length, 1-ninda-reed front	length and front both = 1 ninda
is 1 šar.	makes 1 šar.
If 5 is your cubit:	**If you count with ninda:**
The line steps of ditto and steps of 1 go.	The area of a square = sq. $s \cdot 1$
Steps of 1, each take.	The side of a square is sqs. $(1 \cdot A)$
If 1 is your cubit:	**If you count with cubits:**
The line steps of ditto and steps of 25 go.	The area of a square = sq. $s \cdot 25$
Steps of 2 24, each take.	The side of a square is sqs. $(2\ 24 \cdot A)$

§ 4 b. Example of metric squaring

1 *ṣuppān* the length,	Length and front both equal to
and 1 *ṣuppān* the front.	1 *ṣuppān*
What are the šar?	What is that in šar?
If 5 is your cubit:	**If you count with ninda:**
5 is the *ṣuppān*.	5 (ninda) = 1 *ṣuppān*
5 steps of 5 go, 25, 25 šar.	$5 \cdot 5 \cdot 1 = 25$ (sq.ninda) = 25 šar
If 1 is your cubit:	**If you count with cubits:**
1 is the *ṣuppān*.	1 (· 60 cubits) = 1 *ṣuppān*
1 steps of 1 go, 1,	$1 \cdot 1 = 1$ (· sq. (60 cubits))
1 steps of 25 go, 25, 25 šar.	$1 \cdot 1 \cdot 25 = 25 = 25$šar

§ 4 c. Example of metric square side computation

[···] of 25šar.	A *square* of 25 šar
The equalside shall be what?	What is the square side?
If 5 is your cubit:	**If you count with ninda:**
Each of 25 take.	The square side of 25 (sq. ninda)
<a *ṣuppān* is the equalside>.	= 5 (ninda) <= 1 *ṣuppān*>
If 1 is your cubit:	**If you count with cubits:**
25 steps of 2 24 go,	$25 \cdot 2\ 24$
1, of which each take,	= 1 (· sq. (60 cubits)), the square side
a *ṣuppān* is the equalside.	= 1 (· 60 cubits) = 1 *ṣuppān*

§ 4 d. Example of metric division

The front is 4 (ninda).	$s = 4$ (ninda).
The length, what shall it be long,	$u = ?$
so that it is 20 šar?	if, in addition, $A = u \cdot s = 20$ šar
If 5 is your cubit:	**If you count with ninda:**
The 4th-part, 15,	$1/s = 1/4 = ;15$

15 steps of 20 go, 5,	$1/s \cdot A = 15 \cdot 20 = 5$ (ninda)
a *ṣuppān*, it is long.	$u = 5$ (ninda) = 1 *ṣuppān*
If 1 is your cubit:	**If you count with cubits:**
The 48th-part, 1 15,	(s = 48 cubits), $1/s = 1/48$ = ;01 15
1 15 steps of 2 24 go, 3.	1 15 · 2 24 = 3
3 steps of 20 go, 1.	3 · 20 = 1 (· 60 cubits)
< a *ṣuppān*, it is long.>	<u = 1 (· 60 cubits) = 1 *ṣuppān*>

It is obvious that W 23291-x § 4 is another example of (the beginning of) a theme text with metric algebra problems, just like W 23291 § 1. The brief and idiomatic style of the text makes the literal translation quite hard to read, so that the explanation in the right column is indispensable.

Anyway, this is what is going on here: In § 4 a, a rule is first formulated for the computation of areas of *rectangles* in terms of the unit šar = 1 square-ninda. When lengths are expressed in terms of ninda, the rule is simply that $A = 1 \cdot u \cdot s$. However, when lengths are expressed in terms of cubits, the rule takes the form A = ;00 25 $\cdot u \cdot s$, for the reason that

1 sq. cubit = 1 sq. (;05 ninda) = ; 00 25 sq. ninda = ;00 25 šar.

In § 4 a, a rule is formulated also for the computation of the "square side" of a given area. When lengths are expressed in terms of ninda, the rule is simply that the length of the square side is s = sqs. $(1 \cdot A)$, a length number such that sq. $s = A$. However, when lengths are counted in cubits, the rule is that the square side is s = sqs. $(2\ 24 \cdot A)$, for the reason that

1 šar = 1 sq. ninda = 1 sq. (12 cubits) = 2 24 sq. cubits.

In § 4 b-d, examples of the most basic metric algebra problems are worked through. The computations are quite simple although they are somewhat complicated by repeated references to the OB length unit

1 *ṣuppān* = 5 ninda = 1 00 cubits.

A Seleucid pole-against-a-wall problem

The OB pole-against-a-wall problem in BM 85196 # 9 (see above, Fig. 1.12.6) has a counterpart in **BM 34568 # 12** (Høyrup, *LWS* (2002), 391 ff), an isolated exercise in a large mathematical recombination text from the Seleucid period in Mesopotamia (the last third of the 1st millennium BCE).

The question in this exercise can be rephrased as:

A reed of unknown length at first stands upright against a wall of the same height.
Then it starts sliding so that its upper end moves straight down 3 cubits.
At the same time, its lower end moves away from the wall 9 cubits.
What is the length of the reed, how far up the wall does the reed reach?

With the notations in Fig. 1.12.6 above, the question takes the form

s = 3 cubits, b = 9 cubits. c = ?, a = ?

The obvious way of solving this problem would be to proceed as follows:

sq. c – sq. a = sq. b = sq. 9, $c - a = s = 3$.

This is *a subtractive quadratic-linear system of equations of type B3b*. In BM 34568 # 12, the solution to this problem is given in the form

c = (sq. b + sq. s)/2 · 1/s = (sq. 9 + sq. 3)/2 · 1/3 = 45 · 1/3 = 15,
sq. a = sq. c – sq. b = sq. 15 – sq. 9 = 2 24, a = sqs. 2 24 = 12.

One way in which the solution can have been obtained in this form is illustrated in Fig. 1.13.8 below. The problem is then interpreted as a problem for a "semichord" in a semicircle. If the semichord, of length b, divides the diameter of the semicircle in two parts of lengths u and s, then

$(u + s)/2 = c$ (the radius), $(u - s)/2 = a$, and consequently
$u \cdot s$ = sq. {$(u + s)/2$} – sq. {$(u - s)/2$} = sq. b.

(*Cf.* the proof of *El.* II.14 and Fig. 1.7.2, right.) It follows that

$c \cdot s = (u + s)/2 \cdot s = (s \cdot u$ + sq. $s)/2$ = (sq. b + sq. s)/2, so that c = (sq. b + sq. s)/2 · 1/s.

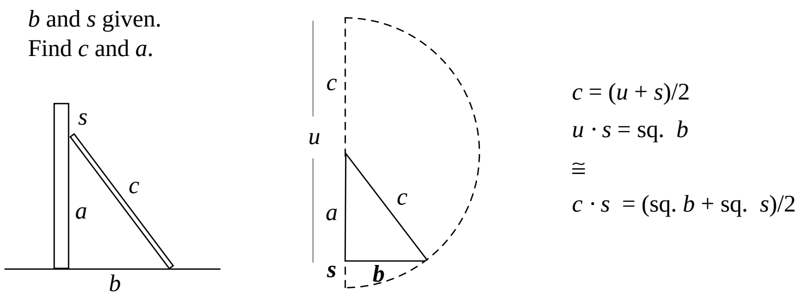


Fig. 1.13.8. BM 34568 # 12. A Seleucid pole-against-a-wall exercise.

Note that the same geometric configuration, with various permutations of the given parameters, is behind the three pole-against-a-wall problems in BM 34568 # 12 (b and s given) and BM 85196 # 9 (c and s, or c and b given), as well as behind the proposed forerunners *El.* II.11* and *El.* II 14* to *El.* II.11 and *El.* II 14 (*b* and *a*, or *c* and *b* given; see Fig. 1.7.2).

It is interesting that further examples of the pole-against-a-wall problem appear in § 8 g-h of *P.Cairo J. E. 89127-30, 89137-43*, an Egyptian demotic mathematical text from the third century BCE (Parker, *DMP* (1972) ## 30-31; Friberg, *UL* (2005), Sec. 3.1 b). The solution method is the same in the demotic text as in BM 34568 # 12.

The problem type reappears in § 1of *Liber Mahameleth*, a Latin manuscript based on Islamic sources, compiled in Spain in the 12th century by a Christian traveller (Sesiano, *Cent.* 30 (1987)). Here is the first part of the text of the third exercise in § 1 (my translation):

> Another example. If a ladder, I don't know how long, standing against a wall of the same height and moved 6 cubits from the foot of the wall descends from the top of the wall two cubits, then how much is the length?
> You do it like this: Multiply 6 with itself, and 2 with itself, and subtract the smaller product from the larger, and 32 will remain. Of which the half, which is 16, divide by 2 cubits, and 8 will come out. To which add 2 cubits, and it makes 10, and so much is the height of the ladder or the wall.

In this exercise, b = 6 cubits, s = 2 cubits, and the solution is given as

$$c = (\text{sq. } b - \text{sq. } s)/\ 2s + s = (\text{sq. } 6 - \text{sq. } 2)/\ 2 \cdot 2 + 2 = 8 + 2 = 10.$$

This is clearly *not* the same solution method as the one in BM 34568 # 12. (*Cf.* also Tropfke, *GE 4* (1940), Sec. 4.2.3.1.1.)

Seleucid parallels to *El.* II.14* (systems of equations of type B1a)

AO 6484 is another large Seleucid mathematical recombination text of mixed content. In that text, § 7 is a series of four "igi-igi.bi problems" (Friberg, *RC* (2007), Appendix 7). The most interesting of those problems is § 7 a, because of the extreme values of the given data in that exercise.

AO 6484 § 7 a, literal translation	explanation
igi and igi.bi 2 00 00 33 20.	igi + igi.bi = p = 2 00 00 33 20
igi and igi.bi *how much* …	igi and igi.bi = ?
· 30 go, then 1 00 00 16 40.	$p/2$ = 1 00 00 16 40
1 00 00 16 40 · *1 00 00 16 40 go,*	sq. $p/2$ = sq. 1 00 00 16 40
then 1 00 00 33 20 04 57 46 40.	= 1 00 00 33 20 04 57 46 40
1 from inside (it) remove,	sq. $p/2 - 1$
then remains 33 <20> 04 37 46 40.	= 33 <20> 04 37 46 40
What · *what may I go,*	sqs. 33 <20> 04 37 46 40

then 33 <20> 04 37 46 40?	= ?
44 43 20 · 44 43 20 go,	sq. 44 43 20
then 33 <20> 04 37 4*6 40.*	= 33 <20> 04 37 46 40
44 43 20 to 1 00 00 16 40 repeat,	1 00 00 16 40 + 44 43 20
then1 00 45, the igi.	= 1 00 45 = igi
44 04 43 20 from 1 00 00 16 40 rem*ove,*	1 00 00 16 40 – 44 43 20
then 59 15 33 20, the igi.bi.	= 59 15 33 20 = igi.bi

In this exercise, the terms igi and igi.bi denote a "reciprocal pair" of sexagesimal numbers, by which is meant any pair of (positive) sexagesimal numbers such that their product is equal to '1' (any power of 60). Therefore, the question in the exercise can be interpreted as *a rectangular-linear system of equations of type B1a* of the following *special form*:

igi · igi.bi = 1, igi + igi.bi = 2 00 00 33 20.

Presumably the Seleucid mathematicians used some kind of geometric model to help them find a solution procedure, just like their OB predecessors had done. Two candidates for such a model are shown in Fig. 1.13.9 below. The one to the left is the "square-difference model" related to *El.* II.5 (Fig. 1.4.2, left). The one to the right is the "semi-chord model", related to *El.* II.14* (Fig. 1.7.2, right).

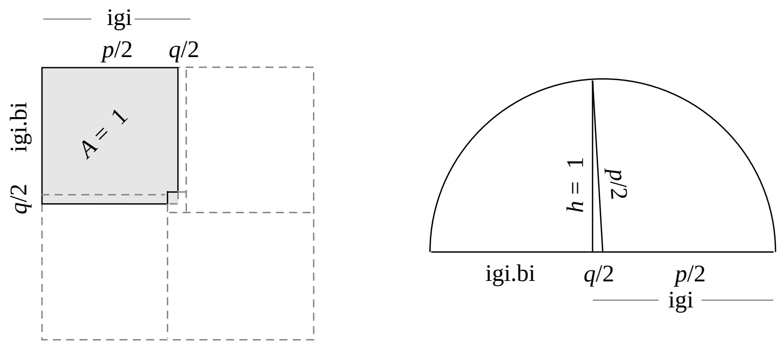


Fig. 1.13.9. Two possible geometric models for the solution procedure in AO 6484 § 7 a.

Assume that the given number 2 00 00 33 20 in AO 6484 § 7 a (written with a special sign for internal zeros) can be interpreted as, for instance, 2;00 00 33 20 (2 plus a very small fractional part). Then the successive steps of the solution procedure in the text can be explained as follows:

p/2 = 2;00 33 20 / 2 = 1;00 16 40

sq. *p*/2 = 1;00 00 33 20 04 57 46 40
sq. *q*/2 = sq. *p*/2 – 1 = ;00 00 33 20 04 37 46 40
q/2 = ;00 44 43 20
igi = *p*/2 + *q*/2 = 1;00 16 40 + ;00 44 43 20 = 1;00 45 (= 81/80)
igi.bi = *p*/2 – *q*/2 = 1;00 16 40 – ;00 44 43 20 =;00 59 15 33 20 (= 80/81)

The curious choice of data is best explained by the semi-chord model. Apparently, the purpose of the exercise was to show that an *extremely thin right triangle* can be constructed by letting the sides of the triangle be

c, *b*, *a* = *p*/2, 1, *q*/2 = (igi + igi.bi)/2, 1, (igi – igi.bi)/2,

where igi and igi.bi are the sides of a *nearly square rectangle* with the area 1. (*Cf.* Friberg, *RC* (2007), Appendix 8, Fig. A8.5.))

1.14. Old Akkadian Square Expansion and Square Contraction Rules

It is known (see Friberg, *CDLJ* (2005:2), Figs. 8 and 10) that already mathematicians in the Old Akkadian period in Mesopotamia (ca. 2340-2200 BCE) may have been familiar with the "square expansion rule"

sq. (*u* + *s*) = sq. *u* + sq. *s* + 2 *u* · *s*,

and with the closely related "square contraction rule"

sq. (*u* – *s*) = sq. *u* + sq. *s* – 2 *u* · *s*.

These rules are clearly the *Old Akkadian forerunners to El. II.4 and II.7*. (Compare Fig. 1.14.1 below with Fig. 1.3.2 above.)

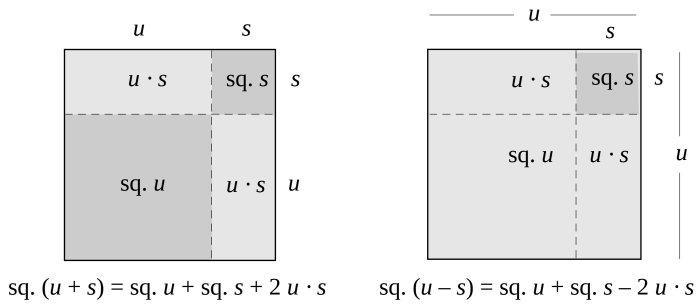


Fig. 1.14.1. The Old Akkadian square expansion and square contraction rules.

Thus, for instance, in the Old Akkadian mathematical exercise ***DPA* 36** (Friberg, *op. cit.*, Fig. 7), the area is given of a square with the side

11 ninda 1 1/2 1/4 seed-cubit = 10 ninda + 1/8 · 10 ninda + 1/4 seed-cubit.

(1 n. = 6 seed-cubits.) The area was probably computed by use of a repeated application of the square expansion rule, as follows:

1. sq. (10 n. + 1/8 · 10 n.) = sq. 10 n. + 2 · 1/8 · sq. 10 n. + sq. 1/8 · sq. 10 n.
2. sq. (10 n. + 1/8 · 10 n. + 1/4 s.c.)
   = sq. (10 n. + 1/8 · 10 n.) + 2 · (10 n. + 1/8 · 10 n.) · 1/4 s.c. + sq. 1/4 s.c.

For the details of the computation, which is quite complicated because of the involvement of various Old Akkadian units for length and area measure, the reader is referred to Friberg, *op. cit.*

Similarly, in the Old Akkadian mathematical exercise ***DPA* 37**, for instance (Fig. 5.3.2 below), the area is given of a square with the side

1 šár ninda 5 géš ninda – 1 seed-cubit (1 šár = 60 · 60, 1 géš = 60).

The area was probably computed by use of an application of the square expansion rule, followed by an application of the square contraction rule. For the complicated details of the computation, see Friberg, *op. cit.*

It is known through a number of examples that the mentioned rules were applied in various situations also by OB mathematicians.

1.15. The Long History of Metric Algebra in Mesopotamia

The oldest known examples of metric algebra are applications of a “field expansion procedure” in proto-cuneiform texts from the end of the 4th millennium BCE (Friberg, *AfO* 44/45 (1997/98); *UL* (2005), Fig. 2.1.15.) The aim of the field expansion procedure seems to have been to find rectangles of given area with the lengths of the sides of the rectangle in a given ratio.

Next in time, in the small corpus of known mathematical texts from the Old Akkadian (Sargonic) period, c. 2340-2200 BCE, there are several known, quite elaborate examples of *metric squaring* (such as the ones mentioned in Sec. 1.14 above) and *metric division*, possibly also an even more elaborate example of the *metric computation of a side of a square with given area*. Moreover, although the known examples of Old Akkadian metric squaring and metric division problems are written only one or two at a time on small clay tablets, they appear to have been excerpted from systematically arranged theme texts. (*Cf.* Friberg, *CDLJ* (2005:2).)

In the large corpus of OB mathematical texts, metric algebra is, as is well known, one of the most popular subjects. The extensive discussion in

Secs. 1.10-1.12 above shows that there are several known examples of *well organized OB theme texts with metric algebra problems,* in particular metric algebra problems for one, two, or several squares.

In Sec. 1.13 it was shown that examples exist also of *well organized Late Babylonian/Seleucid theme texts with metric algebra problems,* resembling such OB theme texts. Several features suggest that those Late Babylonian/Seleucid texts were written in direct imitation of OB models.

Thus, for instance, the problem for concentric circles in W 23291-x § 2 is indistinguishable, at least in translation, from an OB mathematical text. It measures length in ninda and surface content in square ninda (šar), although in Late Babylonian cuneiform texts lengths are normally measured in cubits or reeds (= 7 cubits) and surface content in either seed measure or "reed measure" (the length of a rectangle with the given surface content and with one side equal to precisely 1 reed).

Also the fragment of a theme text in W 23291 § 4 measures surface content in šar, expressly defined as 1 square ninda. It shows its dependence on an OB archetype by having a separate ninda section, and in the cubit section the cubit is 1/12 of a ninda, which implies that the cubit is 1/6 of a reed, as in OB texts, not 1/7.

The problem for concentric circles in W 23291 § 4 g is more removed from its OB archetype by measuring surface content in terms of seed measure, but it still measures lengths in ninda. The metric algebra problems in W 23291 § 1 b-f also measure surface content in terms of seed measure and have separate ninda and cubit sections, with the cubit equal to 1/12 ninda in the cubit sections.

Summing up, it is now possible to conclude that metric algebra problems were studied systematically in Mesopotamian scribe schools during a time span of at least 2000 years, from the Old Akkadian to the Late Babylonian period. The investigation has also shown that, at least in some respects, *Late Babylonian mathematics was directly influenced by OB mathematics, actually in the same way that OB mathematics must have been inspired by Old Akkadian mathematics*. This is not an unexpected conclusion, and it is supported but other facts not mentioned here. Still, it is remarkable, since the terminology used in Late Babylonian mathematical texts is in many ways different from the terminology used in corresponding OB mathematical texts.

Thus, when *Elements* II, or more likely a lost Greek forerunner to *Elements* II was written in imitation of some oriental archetype, it was only the last link in an extremely long chain of theme texts with metric algebra problems. The heated debate over the question whether some of the propositions in *Elements* II were Greek *geometric reformulations* of Babylonian *algebra* can now be laid to rest. In reality, *Elements* II appears instead to have been a direct *translation* into non-metric and non-numerical "geometric algebra" of key results from Babylonian *metric algebra*. It is noteworthy that, in spite of this translation, Greek geometric algebra still relied on *the same geometric models* as Babylonian metric algebra.