Preface

A discussion of the ways in which Greek commentators in the (late) antiquity tried to explain the origin of Greek mathematics, as well as a historical survey of early cultural contacts between the Greeks and the Near East can be found in, for instance, van der Waerden, *Science Awakening 1* (1975 (1954)), 83 *ff*.

In the present book, a sequel to the author's *Unexpected Links Between Egyptian and Babylonian Mathematics*, Singapore: World Scientific (2005), Greek and Babylonian mathematical texts will be allowed to speak for themselves.

The following passage, of interest in the present connection, can be cited from the Preface to my *Unexpected Links*:

> "My observation that there seems to exist clear links between Egyptian and Babylonian mathematics is in conflict with the prevailing opinion in formerly published works on Egyptian mathematics, namely that practically no such links exist. However, in view of the dynamic character of the (writing of the) history of Mesopotamian mathematics, not least in the last couple of decades, it appeared to me to be *high time to take a renewed look at Egyptian mathematics against an up-to-date background in the history of Mesopotamian mathematics!* · · · ·
> The detailed comparison in this book of a large number of known Egyptian and Mesopotamian mathematical texts from all periods has led me to the conclusion that the level and extent of mathematical knowledge must have been comparable in Egypt and in Mesopotamia in the earlier part of the second millennium BCE, and that there are also unexpectedly close connections between demotic and "non-Euclidean" Greek-Egyptian mathematical texts from the Ptolemaic and Roman periods on one hand and Old or Late Babylonian mathematical texts on the other."

Also of relevance in the present connection are the following words from the summing-up in the last few lines of *Unexpected Links*:

> "The observation that Greek ostraca and papyri with Euclidean style mathematics existed side by side with demotic and Greek papyri with Babylonian style mathe-

> matics is important for the reason that this surprising circumstance is an indication that when the Greeks themselves claimed that they got their mathematics from Egypt, they can really have meant that they got their mathematical inspiration from Egyptian texts with mathematics of the Babylonian type. To make this thought much more explicit would be a natural continuation of the present investigation."

The following deliberation is in agreement with the cited passages:

The simplest way of explaining the many parallels found in this book between (certain parts of) Greek mathematics and Old or Late Babylonian mathematics is to assume that *in ancient Greece elementary education in mathematics for young students (not necessarily intending to become mathematicians) was conducted in terms of metric algebra in the Babylonian style*. Here "metric algebra" is a convenient name for the very special kind of mathematics, with *an elaborate combination of geometry, metrology, and linear or quadratic equations,* which is first documented in proto-Sumerian texts from the end of the fourth millennium BC, and which prevailed in Mesopotamia without much change to the Seleucid period close to the end of the first millennium BC. During the 2500 years of its existence already *before* the dawn of Greek mathematics, this kind of mathematics ought to have had ample opportunity to spread to more or less distant neighbors of Mesopotamia itself. That this hypothesis is correct in the case of Egypt was demonstrated in *Unexpected Links*. To show that it may be correct also in the case of ancient Greece is the object of the discussion below.

It is important to understand that one of the obstacles in the way for a better understanding of possible relations between Greek and Babylonian mathematics is the circumstance that Greek mathematics is documented mainly through copies of copies of important manuscripts with advanced mathematics, while Old Babylonian mathematics is documented mainly through clay tablets with relatively low level mathematics, written by mediocre scribe school students, and Late Babylonian/Seleucid mathematics is documented only through a small number of texts, for the simple reason that in the second half of the first millennium BC clay tablets had been replaced by more easily perishable materials as the preferred medium for writing. For these reasons, it is difficult to know what Greek mathematics at a lower level was like, and equally difficult to find out how advanced Old and Late Babylonian mathematics at a higher level may have been.

It is also important to understand that since the heated but inconclusive debate about Greek "geometric algebra" in the late 1970's, much has happened in the study of Babylonian mathematics. Thus many new mathematical cuneiform mathematical texts have been published since then, several of them with unexpected and astonishing revelations about the scope of Babylonian and pre-Babylonian mathematics, and many of the earlier published mathematical cuneiform texts have been explained in new, and much more satisfactory ways. Therefore, it is now obvious that the mentioned debate was conducted against a background of regrettably insufficient knowledge about the true nature of Babylonian mathematics.

More or less accidentally, the dedicated search in this book for parallels between Greek and Babylonian mathematics has, in addition, resulted in a rather extensive survey of certain important parts of Greek mathematics, as well as in new answers to a number of open problems in the history of Greek mathematics.

Here follows a brief survey of the contents of the book:

Chapter 1 is a continuation and more or less definite conclusion of the debate about what has been known as the "geometric algebra" in Euclid's *Elements* II. In this chapter it is shown that far from being Greek reformulations in geometric terms of Babylonian (non-geometric) algebra, the propositions in *Elements* II are *abstract, non-metric reformulations of a well defined set of basic equations or systems of equations in Babylonian metric algebra*, that is of *quadratic and linear equations or systems of equations for the lengths and areas of geometric figures*.

Strictly speaking, *Elements* II is not about "geometry" at all, in the literal sense of the word, which is 'land-measuring'.

Characteristically, as a consequence of the different Greek and Babylonian approaches to geometry, diagrams illustrating non-metric propositions in the *Elements* are what may be called "lettered diagrams", while diagrams illustrating Babylonian metric algebra problems are "metric algebra diagrams" with explicit indications of relevant lengths and areas.

As a whole, *Elements* II is a well organized "theme text" of the same kind as similarly well organized Babylonian mathematical theme texts.

Chapter 2 begins with a presentation of Euclid's proof of *El.* I.47, and of Pappus' proof of a generalization of *El.* I.47. Then follows a discussion of the OB (Old Babylonian) forerunner of *El.* I.47, “the OB diagonal rule (for rectangles)”. It is suggested that the rule may have been discovered accidentally in connection with the study of “rings of four rectangles (or four right triangles)”.[1] The argument is supported by the recent discovery of an OB “hand tablet” with a picture of a *ring of three trapezoids*. The hand tablet is published in the author’s *A Remarkable Collection of Babylonian Mathematical Texts*, New York: Springer (2007).

Chapter 3 is a confrontation of Greek rules for the generation of pairs of numbers (integers) such that the sum of their squares is also a square with OB rules for the generation of “diagonal triples”, rational sides of right triangles. The Greek rules are attributed to Euclid (lemma *El.* X.28/29), Pythagoras, and Plato, while the OB rule is manifested in a number of OB “igi-igi.bi problems”, as well as in the famous OB table text Plimpton 322.

Chapter 4 begins with a discussion of Euclid’s important lemma *El.* X.32/33, which says, essentially, that a right triangle is divided into two right sub-triangles similar to the whole triangle by the height against the hypothenuse. That this result was known also in Babylonian mathematics is demonstrated by an OB problem for a right triangle divided by use of a recursive procedure into a “chain of similar right sub-triangles”.

Chapter 5 contains a completely new approach to the study of the notoriously difficult tenth book of the *Elements*. It is shown that the theory of inexpressible straight lines in *El.* X is based on a number of fundamental lemmas and propositions such as the lemmas X.28/29, X.32/33, X.41/42, and the propositions X.17-18, X.30, X.33, X.54, X.57, X.60, all of which can best be explained by use of Babylonian metric algebra. As a matter of fact, a particularly great role is played in *El.* X by “quadratic-rectangular systems of equations of type B5”, by which is meant problems where both the sum of the squares of two unknowns and the product of the unknowns are given. Such problems appear as well in Babylonian mathematics.

Also discussed in this chapter is the relation between Euclid’s “parabolic application of areas” in *El.* I.44 and Babylonian “metric division”.

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1. Note that, since angles was a relatively unknown concept in OB mathematics, it is less anachronistic to speak of OB “right triangles” than of OB “right-angled triangles”.

Chapter 6 is devoted to a discussion of *Elements* IV, a well organized theme text concerned mainly with "figures within figures". It is shown, through a great number of examples, that figures within figures was a popular subject also in Babylonian mathematics.

Chapter 7 explains in terms of metric algebra the cutting of a straight line *in extreme and mean ratio* in *El.* VI.30, as well as the theory of the regular pentagon and the equilateral triangle in *El.* XIII.1-12. It is pointed out that the propositions *El.* XIII.1-11 can be interpreted as a "metric analysis" of the regular pentagon *relative to the radius of the circumscribed circle,* while a (hypothetical) corresponding Babylonian metric analysis of the regular pentagon necessarily would have operated *relative to the side of the pentagon*.

The relation of such a metric analysis of the regular pentagon (alternatively the regular octagon) to the theory of inexpressible straight lines in *El.* X is investigated.

The chapter ends with a survey of examples of regular polygons and related objects in Babylonian mathematics.

Chapter 8 is an account in terms of metric algebra of the construction of regular polyhedra inscribed in spheres in *El.* XIII.13-18. The account highlights the role played in some of these constructions by the diagonal rule in three dimensions.

Then follows the presentation of a Kassite (post-OB) text with the computation of the interior diagonal of a gate by use of the diagonal rule in three dimensions, and of another Kassite text with the computation of the weight of a colossal 'horn-figure' (icosahedron), constructed by use of 20 equilateral triangles with sides measuring 3 cubits and made of copper sheets 1 inch thick. Both texts are published in the author's *Remarkable Collection* (2007).

Chapter 9 begins with Euclid's demonstration in *El.* XII.3-7 of (essentially) the fact that every triangular prism can be cut into three triangular pyramids, each one of which has a volume equal to one third of the volume of the prism. Then follows a discussion of texts showing that OB mathematicians could compute correctly the volumes of various kinds of whole and truncated pyramids, as well as of whole and truncated cones. The manner of computation of the volume of a "ridge pyramid" in an OB mathematical text is compared with the dissections used in *El.* XII.3-7 and with similar dissections used by the famous Chinese mathematician Liu Hui in his commentary to problems in the Chinese mathematical classic *Nine Chapters*. It is pointed out that there are indications that also Babylonian mathematicians knew about similar dissections of prisms and pyramids.

Chapter 10 contains a detailed discussion in terms of Babylonian metric algebra of Euclid's parabolic, elliptic, and hyperbolic "application of areas" propositions *El.* I.43-44, *El.* VI.24-29 and *Data* 57-59, 84-85. In addition, a completely new explanation is given of Euclid's intriguing proposition *Data* 86, which is here shown to give the detailed solution to a complicated "quadratic-rectangular system of equations of type B6", related to the already mentioned quadratic-rectangular systems of equations of type B5 in the proofs of *El.* X.54 and X.57.

Chapter 11 begins with an account of some of the most interesting propositions in Euclid's lost book *On Divisions*, known mainly from an abstract published by a 10th century Persian geometer. Particular attention is given in this account to problems where triangles or trapezoids are divided by lines parallel to the base, and to an appealing proposition where the problem of dividing a triangle in two parts in a certain ratio by a line through a given point in the interior of the triangle is reduced to the problem of solving a certain quadratic equation.

Then follows a detailed discussion of numerous OB parallels in the form of problems for triangles or trapezoids divided in certain ratios by one or several transversals parallel to or orthogonal to the base. Among these problems are several of the most interesting and sophisticated of all known Babylonian mathematical problems. In particular, a completely new explanation is given here of an OB quite sophisticated "boundary value problem", where a trapezoid with known base and top is divided into a chain of three rational bisected sub-trapezoids.

The "confluent trapezoid bisections" in a couple of OB mathematical texts show that OB mathematicians knew how to combine a solution to an indeterminate quadratic equation of the form sq. s_{a} + sq. $s_{\mathrm{k}} = 2 \cdot$ sq. d with a solution to the indeterminate quadratic equation sq. a + sq. $b = 1$ in such a way that the result is a new solution to the first equation.

An interesting observation is that the famous "Bloom of Thymaridas" is a generalization of a system of equations connected with an OB method for the construction of solutions to trapezoid bisection problems.

Chapter 12 compares Hippocrates' *quadrature of lunes* with various Old and Late Babylonian computations of the areas and diameters of certain figures with curved boundaries, in particular certain double circle segments, but also "concave squares" and "concave triangles".

Chapter 13 contains a discussion of a large number of examples of parallels to Babylonian metric algebra in Diophantus' *Arithmetica*. Thus, for instance, *Arithmetica* I is organized precisely like an OB theme text with equations or systems of equations for one or two unknowns. Particularly interesting here is the appearance of the word *plasmatikón*, the meaning of which has been debated. However, it is likely that when a problem is called *plasmatikón*, that means that it is 'representable', namely by a metric algebra diagram. It is also interesting that the *diorisms* appearing in certain problems are conditions for the existence of solutions which seem to have been derived from the study of such diagrams.

In *Arithmetica* II, some "basic examples" which are usually explained by reference to the "chord method", can just as well be explained by reference to metric algebra problems for triangles or trapezoids inscribed in circles, or by reference to trapezoids divided into parallel stripes. Similarly, the interesting and well known method of "approximation to limits" in *Ar*. "V".9, which can be explained by a variant of the chord method, can just as easily be explained with reference to the OB method of "confluent trapezoid bisections".

Diophantus' extremely interesting but obscure construction in *Ar*. III.19 of *a square number equal to a sum of two squares in four different ways* can with advantage be explained in terms of metric algebra with reference to a "birectangle" (a quadrilateral with two opposite right angles). This construction, too, seems to be intimately connected with the OB method of confluent trapezoid bisections and with the OB rule for the composition of a solution to an indeterminate quadratic equation of the form sq. s_{a} + sq. $s_{\mathrm{k}} = 2 \cdot$ sq. d with a solution to the indeterminate quadratic equation sq. a + sq. $b = 1$.

An indeterminate "price and number problem" which appears totally out of context in *Ar*. "V".30, is closely related to similar OB problems leading to systems of linear equations, but it is interesting also because it is solved by use of solutions to "quadratic inequalities" obtained through "completion of the square".

Arithmetica "VI" which is concerned with indeterminate equations for right triangles, has, like *Arithmetica* I, precisely the same form as an OB theme text. The construction problem *Ar*. "VI".16: *To find a right-angled triangle in which the bisector of an acute angle is rational*, appears in *Ar*. "VI" totally out of context, and is solved by what looks like metric algebra.

One of the few occurrences in *Babylonian* mathematics of indeterminate equations is particularly interesting because it equates (in a totally artificial way) the interest on a loan with a square, a cube, or a "cube-minus-1", the latter term meaning a "quasi-cube" of the form "cube n – square n". It is interesting to note in this connection that in *Ar*. "VI" all the undetermined right hand sides of equations are, likewise, either a square, a cube, a "quasi-square", or a "quasi-cube".

Heron's well known *area rule for triangles*, the Indian mathematician Brahmagupta's closely related *area rule for cyclic quadrilaterals*, and Ptolemy's and Brahmagupta's *diagonal rules for cyclic quadrilaterals* are treated together in **Chapter 14**. It is shown that all these rules can be derived in simple and straightforward ways by use of metric algebra, as long as no other cyclic quadrilaterals are considered than *triangles, rectangles, symmetric trapezoids, and birectangles or "cyclic orthodiagonals"*.

In **Chapter 15**, Theon of Smyrna's "side and diagonal numbers algorithm" is explained in terms of an "ascending chain of birectangles". It is shown that a similar construction works just as well when the equation sq. $d = 2$ is replaced by more general equations of the forms sq. p = sq. $q \cdot D - 1$ or sq. p = sq. $q \cdot D + 1$, where D = sq. d.

In this connection is discussed also a previously never clearly understood OB mathematical table text which may be related to an "ascending spiral chain of trapezoids". An OB "ascending and descending chain of trapezoids with fixed diagonals" is considered in Appendix 1.

Chapter 16 is devoted to a detailed discussion of two methods for the approximation of "square sides" (square roots) used in Heron's collected works. One method, which is essentially the same as a Babylonian "square side rule" is used in the great majority of cases. A second, more accurate method is explained here in terms of "third approximations", by which is meant approximations obtained through a kind of repeated composition of an initial approximation with itself, resulting in a "formal third power".

Interestingly, the use of third approximations can explain not only Heron's accurate square side approximations, but also the well known and much debated Archimedian accurate estimates for sqs. 3, as well as the accurate square side approximations in Ptolemy's *Syntaxis* I.10.

The chapter ends with a discussion of Babylonian square side approximations and of examples of an elegant OB method of eliminating square factors from an area number before the computation of its square side.

In **Chapter 17** it is suggested that Theodorus of Cyrene's famous irrationality proof for square sides of non-square numbers, mentioned in *Thaetetus 147 C-D*, can have been carried out by use of a "descending chain of birectangles", of the same form as the *ascending* chain of birectangles used in Chapter15 for the explanation of Theon's side and diagonal numbers algorithm. The irrationality proof by use of such a descending chain of birectangles works only as long as a solution (in integers) is known to the equation sq. p = sq. $q \cdot D \pm 1$, where D is the given non-square number. If the pair p, q is a solution to an equation of this kind, it is convenient to call p/q an "optimal approximation" to sqs. D. As it turns out, it is easy to find such optimal approximations for all non-square numbers (integers) D from 2 to 17, when $D = 13$ by use of a "third approximation" of the kind discussed in Chapter 16, but not for $D = 19$. This circumstance may explain why Theodorus stopped his demonstration after reaching the case $D = 17$. (The case $D = 18$ can be neglected, since sqs. 18 = 3 · sqs. 2.)

There is an interesting connection between the explanation above of Theodorus' irrationality proof and Brahmagupta's well known observation that he could find a solution to the equation sq. p = sq. $q \cdot D + 1$ in every case when he already knew a solution to the equation sq. p = sq. $q \cdot D + r$, with $r = -1, \pm 2$, or ± 4. As a matter of fact, the method used by Brahmagupta in the non-trivial cases $r = \pm 4$ can be explained in terms of "formal third powers".

In **Chapter 18** it is observed that the Heronic *Metrica* is a typically Greek (Euclidean) mathematical hand book, while the "pseudo-Heronic" *Geometrica* is a compilation of various sources, some of them clearly influenced by Babylonian mathematics. The chapter contains, among other things, surprisingly simple new explanations of the solution procedures in *Geom*. 24.1-2 for a couple of indeterminate problems for the areas and perimeters of a pair of rectangles. Another interesting problem discussed in this chapter, with an obvious relation to a number of Babylonian mathematical problems, is concerned with the sides of a right triangle at a distance of 2 feet from a right triangle with given sides. The chapter is concluded with an explanation of an intricate *division of figures problem* in *Metrica* 3.4, which can be reduced to a rectangular-linear system of equations for two segments of one side of a triangle.

In **Appendix 1**, a new OB mathematical problem text of extraordinary interest is published jointly with J. Marzahn, curator of the collections of clay tablets at the Vorderasiatisches Museum, Berlin. The text begins with a diagram showing a chain of five trapezoids, all with the diagonal 3. The explicit computation of the various sides and transversals of this chain of trapezoids demonstrates that OB mathematicians were familiar with Ptolemy's diagonal rule in the case of symmetric trapezoids, and that they had found, in addition, an elegant rule for the construction of a linked pair of symmetric trapezoids with diagonals of the same length. The recursive procedure used for the computation of the sides and transversals in the chain of five trapezoids starts with the central trapezoid and continues with ascending and descending chains of trapezoids, much like the ascending and descending chains of birectangles discussed in Chs. 15 and 17 above.

The book ends with **Appendix 2**, which is a catalog of all plane and solid geometric figures appearing, in one way or another, in Mesopotamian mathematical texts. There are also an index of texts, an index of subjects, a bibliography, and a comparative set of Mesopotamian, Egyptian, and Greek timelines showing periods of documented mathematical activities.

The work with the manuscript for this book has been supported by
The Royal Society of Arts and Sciences in Gothenburg,
Wilhelm och Martina Lundgrens Vetenskapsfond 1, and
Gunvor och Josef Anérs Stiftelse.

September 1, 2006 Jöran Friberg