Chapter 0

Calculus in Terms of Images:

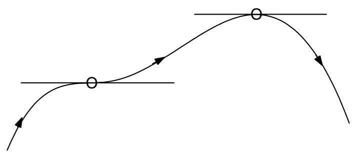
Hill-Climbing

This is an introduction of calculus for beginners and a review for students who have already known calculus. Here calculus is motivated by two measurements, slope and height, when climbing a hill. In other words, calculus is regarded as a curved trig beginning with the curved triangle (see Refs. 15–20), measuring the height with slopes.

A student (A) has a dialogue with a teacher (B) about differential calculus and integral calculus when they are climbing a smooth hill.

0.1. Hill Behavior and Slope

They climb up, and rest on a platform when they are tired. Then they climb up again.



smooth.

A: Is this platform a stationary point in differential calculus?

B: It is! Since the hillside is rising before the platform and rising again after the platform, this is called a stationary point.

They arrive at the second platform and climb down then.

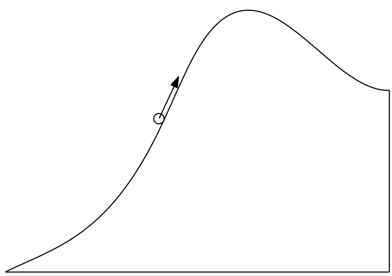
A: Is such a platform a local maximum in differential calculus?

B: It is! Since the hillside is rising before the platform but falling then, this is a top and called a local maximum.

A: So far we touch only some feels about rising, falling, platform or local maximum and local minimum, without mathematics.

B: Mathematics, or differential calculus, is invented to measure the feeling of climbing quantitatively (Bruter, 1973), with a parameter called the tangent slope (which was used in Section 1 of the Preface), corresponding to the sign > 0 (rising), < 0 (falling) or $= 0$ (platform), the gradient (steepness or gentleness) of rising or falling, and the turning point of bend.

A: The slope parameter is so fundamental for climbing.



B: The slopes describe how curve a function is. Generally speaking, the slope is invented to describe the changing world, e.g., economy states, raising or falling, fast or slow, or stable. For instance, Arnold uses the signs of slopes to observe the states of economy.[2]

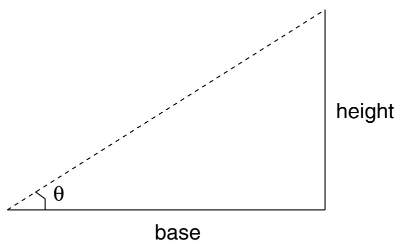
0.2. Hill Height and Slope: Unconstructive Tangent Formula

A: The slope is so fundamental to measure the behavior of a hill. Can the slope be also used to measure the height of a hill?

B: In the Preface, the tangent slope was used to measure the differential (=(starting slope) (base)). But how does one measure the horizontal base? It is almost impossible! So, the FT method is not practical. Indeed, nobody uses the FT to measure the hill height. The FT is to establish the ties between the height and slopes, just like Pythagoras theory establish the ties between the hypotenuse and the other two edges.

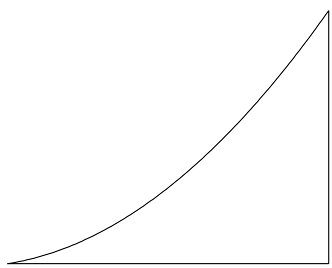
A: Do we have other ways to measure hill height?

B: It has been seen in the regular trig that the slope can be used to measure the height of a tree:



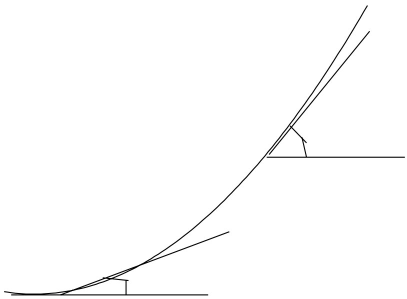
If a hill has a straight hillside (or hypotenuse), this is just a regular trig measurement (tangent formula).

A: But hills usually have a curved hillside (or curved hypotenuse).

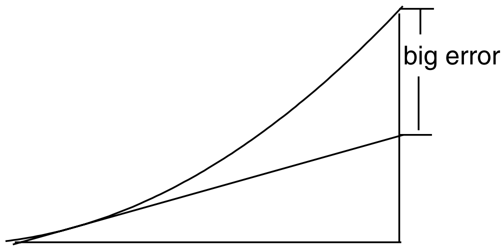


Can we still measure the height (unknown) of a hill a single slope and a single base?

B: But the slope of the hillside is changeable; different points would have different slopes: Which one do we assign?

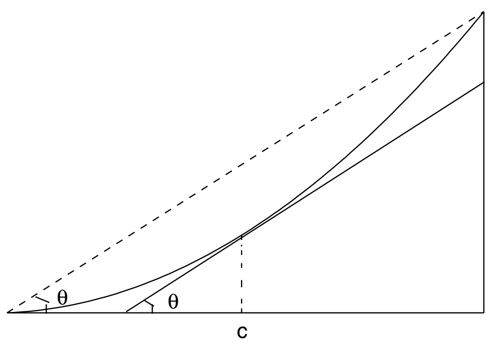


If we randomly select a point, e.g., again the starting point, through the slope at which to compute the height, a large error will occur.



A: Does there indeed exist such a slope to give the hill height?

B: Since the hill exists, with the foot and top



Curved and straight (secant) right triangles have the same height, base, and slope (θ).

there exist a secant line linking the foot and top and, intuitively, a particular tangent line on the smooth hillside parallel to this secant line, with the same slope. Hence, we get the true

$$\text{height} = (\text{secant slope})(\text{base}) = (\text{tangent slope at } c)(\text{base}),$$

a tangent formula for the curved right triangle, or called the Mean Value Theorem (MVT) in differential calculus. Thus if we want to link the height with a slope, we should use the MVT. It is geometrically evident, but a complete proof of this theorem is very long and is best left to advanced calculus.

A: Does the mean value c is knowable?

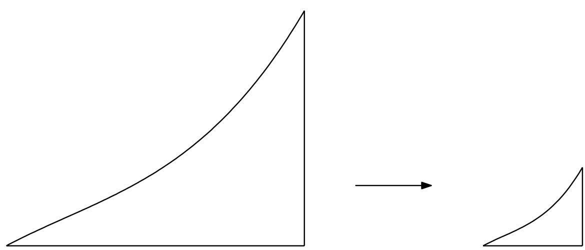
B: No. We are near-sighted. We don't see far away. We don't know where the value c is. Such a tangent formula is existing somewhere but we do not know where it is. It is not an algorithm. So the MVTs is an unconstructive tengent formular.

A: It is a pity that such a beautiful theorem is useless in practical sense!

B: Indeed, my friend Michael Livships hates the MVT, which controls a long and an unknown mean value!

0.3. Review for FT

B: Let us recall the inventing process of the FT, a typical Cartesian methodology. The first step is to shorten the whole unit into one segment. Let the curved hypotenuse be shortened into one segment.



The curved triangle is then shortened into a small one.

A: It seems that we have won nothing from the first step. Curved is still curved.

B: However, within the shortened curve segment the slopes at all points are nearly the same. Now choose an arbitrary point within the curve segment, e.g., the starting point; the slope at this point (called the starting slope) is the slope of the curve segment,

Curve (or curved triangle) ≈ tangent (or differential triangle).

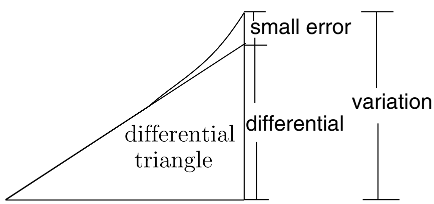
where within the curve segment

$$\text{height variation} \approx (\text{starting slope})(\text{base})$$

called a differential (measurable).

A: The quantity, differential, thus obtained is the height of the tangent line within the curved segment (or the height of the differential triangle), rather than the true variation (unmeasurable).

B: But the error should be small, even when compared with the small base.



$$\begin{aligned} \text{variation} &\approx \text{height of differential triangle} \\ (\text{by tangent formula}) &= (\text{slope})(\text{base}) \\ (\text{named by}) &= \text{differential}. \end{aligned}$$

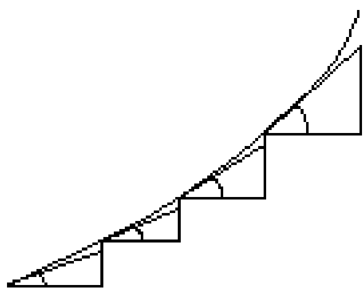
More accurately, the ratio of this error to the small base, the relative error, is still small. This is a key step (the second step), where by doing differential we have nearly gotten the height variation within one segment.

A: In the above step, within one segment the curve is replaced by the tangent line rather than the secant. Why?

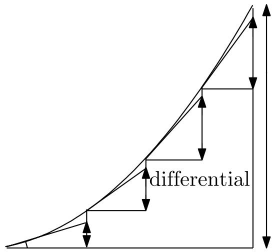
B: Standard textbooks used the secant to replace the curve (within a short segment) but then used the MVT to replace the secant slope with the tangent slope at an unknown mean value, appending unnecessary twists; so, why not use directly the tangent slope at the starting point (agreeing with Euler's constructive algorithm in Chap. 2) without the replacement. The latter produces an error (very small) but avoids the MVT (with a long proof and an unknown mean value). This is indeed a definition of the differential.

A: So far we have only solved the height variation within the initial segment, or just have completed a differential computation (or differentiation). How to extend the initial segment to the whole unit to complete the integral computation (or integration)?

B: The initial segment, together with the differential triangle, is extended segment by segment.



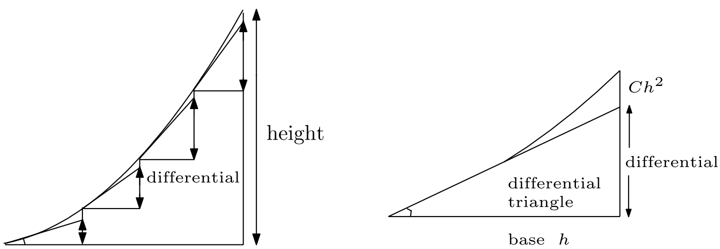
Add up the differential heights that we have obtained through all differential triangles within every segment (where we use all the slopes) and the resulting value will serve as the total true height of the curved triangle.



Total true height $\approx$ sum of differentials.

A: This value, however, is not the true height itself, the error consists of the accumulation of differential errors within every segment. Can we expect this value would still be near the true height of the hill, and the finer the better?

B: This is a bold and adventuristic expectation because the accumulation of differential errors, the total error, still may not be small, unless every differential error is so small (e.g., n^{-2} when base $= n^{-1}$) that even their accumulation (involving n terms) is still small (of n^{-1}). This can be achieved by the definition property of the tangent line mentioned in Section 1 of Preface.



A: When does close ($\approx$) become exact ($=$)?

B: Now it is possible to eliminate the total error by the infinitesimal method, where the total curved triangle is replaced by infinitely many infinitesimal differential triangles and the total true height is replaced by infinitely many infinitesimal differentials (and so all the slopes):

$$\begin{aligned}\text{total true height} &= \text{sum of (differentials} + O(n^{-2}))\\ &= \text{sum of differentials} + O(n^{-1});\end{aligned}$$

hence close (≈) becomes exact (=) for small base.

$$\text{total true height} = \text{integral of differentials}$$
$$\text{each differential} = (\text{starting slope})(\text{base}).$$

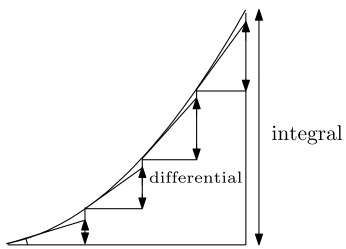
This is indeed a quick proof of the FT. More refined proof will require the use of the average relative error, see Section 2 of Preface. It is fundamentally constructive by differentials. It is not only a computed method for the curve height but gives the relationship between different quantities, e.g., between area and height. See Sec. 0.5.

A: I understand now. The FT is nothing but differentiation first and integration afterwards: shorten the curve, it becomes a differential or a tangent formula; draw the curve, it becomes an integral or the FT. In short,

$$\text{shortening} = \text{tangent formula}; \quad \text{drawing} = \text{FT}.$$

Essentially, integral calculus (university) is nothing more than a tangent formula (high school). When drawing, tangent formula (high school) becomes the FT (university).

B: To sum up, when you climb a hill you do a differential in each step; when you arrive at the top you complete an integral.



The FT has already been fished out.

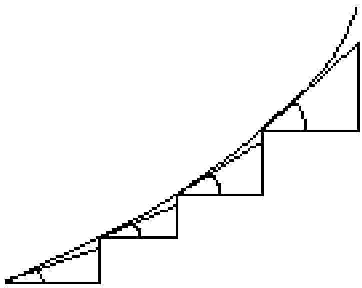
0.4. Hillside Length and Slope: Pythagoras Theorem

A: How do we measure the arclength? Can the length of the hillside be also measured by slopes?

B: If the hillside is straight, this is nothing but the Pythagoras formula,

$$\text{length} = \sqrt{1 + (\text{slope})^2}\ \text{ base}.$$

Within one segment, the length of the curved hillside (called arclength) can be approximately measured using the starting slope.

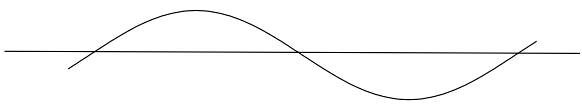


$$\text{arcdifferential} = \sqrt{1 + (\text{starting slope})^2}\ \text{ base}$$

(i.e., Pythagoras' formula holds on a short segment). Adding up gives

whole arclength = integral of infinitely many infinitesimal arcdifferentials.

Arnold gives a remarkable example showing that the length of a sine curve increases by 20% only than that of the cross axle, e.g., a 5 m axle corresponds to a 6 m sine curve.[2]

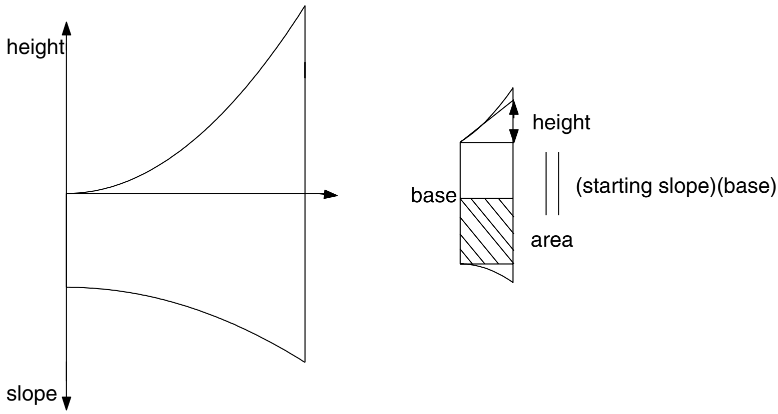


Thanks to the arclength formula which tells us quantificationally how long the curved road is! This is a real problem which needs the slop's help.

0.5. Area and Slope

A: The slope parameter prevails everywhere in trigonometry and calculus. Can we use the slope to measure areas enclosed by some curves?

B: Since the slopes are the fundamental quantity of a curve let us record them and get a slope curve, the inverted image of the original (height) curve. It is amazing that the area enclosed by the slope curve is equal to the height of the original curve, as seen in the following height–slope figure:



So, such an area can be computed by the height of the original curve:

area enclosed by the slope curve = height of the original curve.

A: This figure also appeared in a paper from the United States in 2003 with comments: "it is probably the single most important figure in this book. It's a picture worth a thousand symbols and equations, encapsulating the essence of integration in a single snapshot ... If you understand only half of what I've just written, you're way ahead of most students of calculus."

0.6. Explaining All of Calculus in a Single Figure

B: It was my goal to explain all of calculus in a single figure. Such a single figure first appeared in Lin's Calculus Cartoon, Chinese Dailies, 1997 (see also Lin's English print),[15] and appeared in the cover of Lin's *Calculus Cartoon*, 1999,[16] and then reappeared in the text, *Calculus for Dummies*, 2003.[23]

A: Besides solving the problems from the geometry measurement, can calculus be used for other fields? Literature students would think they read novels without knowing calculus.

0.7. Calculus and Novels

B: The calculus spirit, differentiation first and integration afterwards, seeps into all fields, even into Tolstoy's "War and Peace."[26] In fact, Tolstoy said in his book: "only by taking an infinitesimally small unit for observation (the differential of history, that is, the individual tendencies of men) and attaining to the art of integrating them (that is, finding the sum of these infinitesimals) can we hope to arrive at the laws of history." How do you understand what he said if you do not understand the calculus spirit? You will know that the story was so but not why it was so.

A: Are there more examples in humanities?

B: A car engineer proposed a plan to vitalize car industry in China, named "differentiation first and integration afterwards." This is a general principle in the management science. That is to divide an unmeasurable system into many measurable units (differentials), and that small errors that arise in subsystems will not cause any large error output of the whole system. Or, details (differentials) are the most important, if you do well for details, so do for the whole system.

A: Such arguments are far-fetched.

B: The calculus language is more credible for natural sciences, in which a derivative is a rate (velocity), and perhaps the rate (velocity) concept is easier to be understood than the slope for a beginner. But for natural sciences we need differential equations to express their laws and to solve them. See Chap. 3.

A: What is a differential equation? We only know algebra equations in high schools.

B: The simplest differential equation is the FT, knowing slopes and solving for the height. The differential equation approach is a revolution in natural sciences, where Newton used it first and then all scientists (including Maxwell and Einstein) follow him.

A: So far, calculus is dramatic, developing acts by acts. What is its future?

B: A big subject is about multivariable differential equations and their numerical methods. Our dialog is a drop in the mathematical bucket.

A: I cannot accept so many materials. We must stop now.

B: Thank you, my daughter, for your role A and your enquisition from start to finish.