Preface

Standard calculus texts involve a substantial amount of *a priori* knowledge and intricate technique, requiring more than 100 pages for a complete proof. Students find it too difficult and too long to learn, and easily forgotten. An idealist would asks: "Can we use plan English (instead of mathematical symbols) and, at the same time, give rigorous (instead of intuitive) proofs of calculus theorems with just a few lines of arithmetic (or calculation), without *a priori* knowledge and tricky techniques? An "Unarmed" calculus does not exist but it only needs a definition property of the "differential." If this definition can be explained well, we would have arrived at the proofs for the fundamental theorem (FT) and Taylor theorem (TT).

$$\text{FT} = \text{a definition} + \text{few lines of arithmetic}$$
$$\text{TT} = \text{few lines of calculation from FT}$$

In the first part: Plain English Calculus, we concentrate on the explanation of the "differential", in an as elementary manner as possible. With just a few lines of proof, this will lead to calculus theorems.

Subsequently, the second part translate plain English into the function language to tally with the formal definition and theorems in standard texts. All together, the fundamental essence, including rigorous proofs of the FT and TT is done in less than 10 pages.

This is, of course, the statement of only one of the parties. In fact, almost all of the calculus experts frown upon this approach. Only few colleagues express admiration and use this book for students majoring in physical culture and liberal arts. Nevertheless, it is still useful as an introductory course for the physics and engineering cohort.

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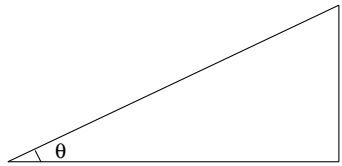
Part I: Plain English

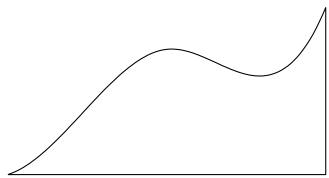
(with rigorous definition and proof)

Calculus' source and definition property of tangent line

One day I walked under an old tree. A tourist guide told sightseers that this tree grows up every year and a surveyor measures its height every year. I knew from middle school trig that the height can be measured in terms of one slope through a tangent formula:



without cutting the tree! A breakthrough is to change the tree to a hill where one faces a curved hypotenuse instead of the straight one:



From a special (straight) to the general (curve).

But, in calculus, the curved hypotenuse is divided into many short segments, each of which is replaced by a tangent line (a straight line closest to the curve near the starting node, if exists) with a height named by

$$\text{differential} = (\text{starting slop})\ (\text{base}).$$

Each differential, however, contains a measured error to the true height:

$$\text{True height} = \text{differential height} + \text{measured error}$$

(see Fig. 1). A problem arises: How small the measured error is? Can we guarantee it as small as possible? The measured error is nothing but a distance from the tangent (if exists) to the curve, much smaller than that from other secants (by the property of the tangent: A straight line closest to the curve near a node).

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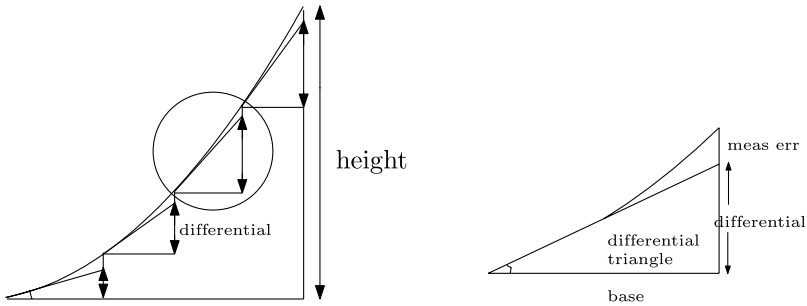

Figure 1. From the general to a special.

Since the latter (the distance from the secant to the curve) is almost the same as the small base:

$$\frac{\text{secant measured error}}{\text{base}} \approx \text{const}$$

(see the explanation at the end of this section), the former (the distance from the tangent to the curve) must be much smaller than the small base: Correctly speaking, the ratio of the tangent measured error to the small base,

$$\text{relative error} := \frac{\text{measured error}}{\text{base}}$$

is also small: as small as you want if reduce the base (denoted $\ll 1$ for simplicity),

i.e., the measured error of the tangent is not proportional to the small base but reducing faster than the base. This requirement is regard as the definition property

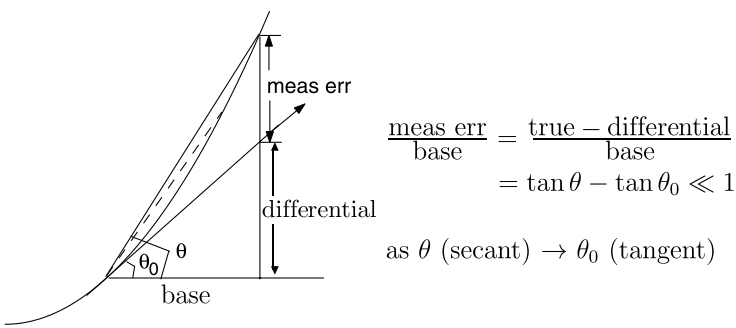

Figure 2. Tangent (θ_0) is the limit position of secants (θ).

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of the tangent line (in the pointwise sense). In fact, for any secant with a slope $\tan\theta$ (with $\theta \neq \theta_0$):

$$\text{Secant relative error} = \frac{|(\tan\theta)(\text{base}) - \text{true}|}{\text{base}} \to |\tan\theta - \tan\theta_0| = \text{const} \neq 0.$$

Comparing with

$$\text{tangent relative error} \ll 1$$

we distinguish between the tangent and the secant.

To sum up, if the tangent exists,

$$\frac{\text{Secant measured error}}{\text{base}} \approx \text{const}$$
$$\frac{\text{Tangent measured error}}{\text{base}} \ll 1.$$

This is the uniquely fundamental concept of calculus in our book and the student must understand it thoroughly.

A rigorous enough proof of FT

From Fig. 1

$$\text{total true height} = \text{sum of differentials} + \text{total error}$$
$$\text{total error} = \text{sum of measured errors}$$

(over all segments). With a few lines of arithmetic we get an identity:

$$\begin{aligned}\text{total error} &= \text{sum of (relative errors) (bases)}\\ &= \text{sum of } \left(\frac{\text{bases}}{\text{sum of bases}} \text{ relative errors}\right)(\text{sum of bases})\\ &= (\text{average of relative errors}) \ (\text{sum of bases})\end{aligned}$$

(over all segments), proportional to the average:

$$\text{Total error} \approx \text{average of relative errors.}$$

If and only if the latter is small or, simply, the relative errors are uniformly small (this can be regarded simply as a definition property of the tangent line in the global

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sense), the total error $\ll 1$: as small as you want that can be eliminated, then we get an ideal equality:

$$\text{Total true height} = \text{integral of differentials.}$$

This is the FT of calculus. Thus the differential (the height of the differential triangle in Fig. 3) is an element of the FT (the total height of curved triangle), whereas the FT is a superposition (or integral) of differentials.

To sum up, if and only if the average is small or if the relative errors are uniformly small, we have the FT.

This completes a rigorous enough proof of the FT of a few lines arithmetic, with plain English and without mathematical symbols and tricky techniques.

Figure 1 is the single most natural figure to discover calculus with cartoon watching instead of words, and is often used for calculus popularization in China since the 1990s (Lin, 1997).

Second Part: Function Language

(with formal calculus)

In order to identify the above statement with standard texts we now have nothing to do, just a translation from plain English to the function language.

Derivative definition

The concept of the tangent slope or differential is determined by the derivative definition. Let $f = f(x)$ be a given function (producing a curve) defined on the interval $[a, b]$. Then, over each subinterval $[x, x+h]$, for all points $x+h$ near the node x, we have

True height: $f(x+h) - f(x)$

Tangent slope $\tan\theta_0$ at node x: Derivative $f'(x)$

Differential: $f'(x)h$

Measured error: $f(x+h) - f(x) - f'(x)h$

Relative error $\epsilon(x, h) := \frac{f(x+h)-f(x)}{h} - \frac{f'(x)h}{h}$

Each measured error = (relative error) (base):

$$f(x+h) - f(x) - f'(x)h = \epsilon(x, h)h$$

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If over each subinterval $[x, x+h]$, $|\epsilon(x,h)| \ll 1$ or simply

$$\epsilon(x,h) \ll 1$$

the derivative f' is uniquely defined pointwisely. The notation $\epsilon(x,h)$ will be explicit for concrete functions or the three test functions $f = x^n$, $\sin x$, e^x (hence their arithmetic and composites, and all elementary functions):

$$\epsilon(x,h) \leq Ch$$

(named by the Lipchitz assumption).

One line rigorous proof of FT

The standard proof of the FT uses a lot of *a priori* knowledge (the inf-sup theorem in the real number theory, the extreme value theorem of continuous functions, the mean value theorem, ...) and tricky techniques. Students learn it and then forget it. Instead, we use (Lin, 1977–2005) only the differential definition itself and its superposition to obtain the FT without more *a priori* knowledge and tricky techniques. In fact, based on the height measurement or the differential version on each subinterval $[x, x+h]$, one can calculate the total true height of f on $[a,b]$ through a superposition involving h^{-1} terms and get the total error:

$$\begin{aligned} f(b) - f(a) - \text{sum of } f'(x)h &= \text{sum of } \epsilon(x,h)h \\ &= (b-a)\left[\frac{h}{b-a}\text{sum of } \epsilon(x,h)\right], \end{aligned}$$

proportional to the average. If and only if the average is small or if the relative errors are uniformly small, or simply the upper relative error

$$\epsilon(h) = \text{upper } \epsilon(x,h) \ll 1$$

(which, combining with the pointwise definition of the derivative in the last section, can be regarded simply as the global definition of the derivative).

Few lines direct proof of TT (as a corollary of FT)

The standard proof of the TT uses indirect "integration by parts", spending much thinking and time. Students, however, prefer a direct (mechanical) calculation

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through the FT itself as follows:

Constant approximation

$$f(x+h) - f(x) = \int_x^{x+h} f'(s)\mathrm{d}s :\approx h \underset{[x,x+h]}{\text{upper}} |f'|,$$

Linear approximation

$$\begin{aligned} f(x+h) - f(x) - hf'(x) &= \int_x^{x+h} [f'(s_2) - f'(x)]\mathrm{d}s_2 \\ &= \int_x^{x+h} \int_x^{s_2} f'(s_1)\mathrm{d}s_1\mathrm{d}s_2 \\ &:\approx \underset{[x,x+h]}{\text{upper}} |f'| \int_x^{x+h} \int_x^{s_2} \mathrm{d}s_1\mathrm{d}s_2 \\ &= \frac{h^2}{2} \underset{[x,x+h]}{\text{upper}} |f''|, \end{aligned}$$

Quadratic approximation

$$\begin{aligned} f(x+h) \; - \; & f(x) - hf'(x) - \frac{h^2}{2} f''(x) \\ &= \int_x^{x+h} \int_x^{s_3} [f''(s_2) - f''(x)]\mathrm{d}s_2\mathrm{d}s_3 \\ &= \int_x^{x+h} \int_x^{s_3} \int_x^{s_2} f'''(s_1)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3 \\ &:\approx \underset{[x,x+h]}{\text{upper}} |f'''| \int_x^{x+h} \int_x^{s_3} \int_x^{s_2} \mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3 \\ &= \underset{[x,x+h]}{\text{upper}} |f'''| \frac{1}{3!} \left(\int_x^{x+h} \mathrm{d}s \right)^3 = \frac{h^3}{3!} = \underset{[x,x+h]}{\text{upper}} |f'''|, \end{aligned}$$

and so on, where we use the fact that if $|g| \leq C$ (constant) then $|\int g(s)\mathrm{d}s| \leq C \int \mathrm{d}s$ (easily verifiable). This completes the direct proof of the FT—I believe it is the simplest.

History

Such an approach, using the regular pointwise definition of the derivative, accompanying with the uniform assumption (or even the concrete Lipschitz assumption), to give rigorous proofs of the FT and TT of calculus (including abstract calculus) with a few lines of calculations, has appeared previously in my early booklets (since 1990's). When I'm writing this new one, a friend, Michael Livshits, told me that the uniform Lipschitz assumption, have been used by himself (2004) and Hermann

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Karcher (2002). Then I found in the internet another book ≪Applied Calculus ≫ (by Karl Heinz Dovermann, July 1999) with a story about the pointwise Lipschitz assumption: "This approach, which should be to easy to follow for anyone with a background in analysis, has been used previously in teaching calculus. The author learned about it when he was teaching assistant for a course taught by Dr. Bernd Schmidt in Bonn about 20 years ago. There this approach was taken for the same reason, to find a less technical and efficient approach to the derivative. Dr. Schmidt followed suggestions which were promoted and carried out by Professor H. Karcher as innovations for a reformed high school as well as undergraduate curriculum. Professor Karcher had learned calculus this way from his teacher, Heinz Schwarze. There are German language college level textbooks by Kutting and Moller and a high school level book by Muller which use this approach." On the other hand, the general uniform definition of the derivative has have been used since 1940's in many books (e.g., Vainberg (1946,1956), Ljusternik–Sobolev (1965), Lax, etc. (1976), Lin (1999, 2002), Karcher (2002), Livshits (2004), Zhang, 2006).

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Qun Lin

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