## Chapter 1

# Introduction

We can find few fundamental physical models amenable to exact treatment. Approximation methods like perturbation theory are necessary and are part of our physics culture.

Among the important approximation methods for quantum field theories (qft's) are strong coupling methods based on lattice discretization of the underlying spacetime or perhaps its time-slice. They are among the rare effective approaches for the study of confinement in QCD and for non-perturbative regularization of qft's. They enjoyed much popularity in their early days and have retained their good reputation for addressing certain fundamental problems.

One feature of naive lattice discretizations however can be criticized. They do not retain the symmetries of the exact theory except in some rough sense. A related feature is that topology and differential geometry of the underlying manifolds are treated only indirectly, by limiting the couplings to "nearest neighbors". Thus lattice points are generally manipulated like a trivial topological set, with a point being both open and closed. The upshot is that these models have no rigorous representation of topological defects and lumps like vortices, solitons and monopoles. The complexities in the ingenious lattice representations of the QCD $\theta$-term [12] illustrate such limitations. There do exist radical attempts to overcome these limitations using partially ordered sets [13], but their potentials are yet to be adequately studied.

As mentioned in the preface, a new approach to discretization, under the name of "fuzzy physics" inspired by noncommutative geometry (NCG), is being developed for a while now. The key remark here is that when the underlying spacetime or spatial cut can be treated as a phase space and quantized, with a parameter $\hat{\hbar}$ assuming the role of $\hbar$, the emergent quantum space is fuzzy, and the number of independent states per ("classical") unit volume becomes finite. We have known this result after Planck and Bose introduced such an ultraviolet cut-off and quantum physics later justified it. A "fuzzified" manifold is expected to be ultraviolet finite, and if the parent manifold is compact too, supports only finitely many independent states. The continuum limit is the semi-classical $\hat{\hbar} \to 0$ limit. This unconventional discretization of classical topology is not at all equivalent to the naive one, and we shall see that it does significantly overcome the previous criticisms.

There are other reasons also to pay attention to fuzzy spaces, be they spacetimes or spatial slices. There is much interest among string theorists in matrix models and in describing $D$-branes using matrices. Fuzzy spaces lead to matrix models too and their ability to reflect topology better than elsewhere should therefore evoke our curiosity. They let us devise new sorts of discrete models and are interesting from that perspective. In addition, as mentioned in the preface, it has now been discovered that when open strings end on $D$-branes which are symplectic manifolds, then the branes can become fuzzy. In this way one comes across fuzzy tori, $\mathbb{C}P^N$ and many such spaces in string physics.

The central idea behind fuzzy spaces is discretization by quantization. It does not always work. An obvious limitation is that the parent manifold has to be even dimensional. If it is not, it has no chance of being a phase space. But that is not all. Successful use of fuzzy spaces for qft's requires good fuzzy versions of the Laplacian, Dirac equation, chirality operator and so forth, and their incorporation can make the entire enterprise complicated. The torus $T^2$ is compact, admits a symplectic structure and on quantization becomes a fuzzy, or a non-commutative torus. It supports a finite number of states if the symplectic form satisfies the Dirac quantization condition. But it is impossible to introduce suitable derivations without escalating the formalism to infinite dimensions.

But we do find a family of classical manifolds elegantly escaping these limitations. They are the co-adjoint orbits of Lie groups. For semi-simple Lie groups, they are the same as adjoint orbits. It is a theorem that these orbits are symplectic. They can often be quantized when the symplectic forms satisfy the Dirac quantization condition. The resultant fuzzy spaces are described by linear operators on irreducible representations (IRR's) of the group. For compact orbits, the latter are finite-dimensional. In addition, the elements of the Lie algebra define natural derivations, and that helps to find the Laplacian and the Dirac operator. We can even define chirality with no fermion doubling and represent monopoles and instantons. (See chapters 5, 6 and 8). These orbits therefore are altogether well-adapted for qft's.

Let us give examples of these orbits:

- $S^2 \simeq \mathbb{C}P^1$: This is the orbit of $SU(2)$ through the Pauli matrix $\sigma_3$ or any of its multiples $\lambda\,\sigma_3$ $(\lambda \neq 0)$. It is the set $\{\lambda\, g\, \sigma_3\, g^{-1} : g \in SU(2)\}$. The symplectic form is $j\, d\,(\cos)\,\theta \wedge d\phi$ with $\theta, \phi$ being the usual $S^2$ coordinates. Quantization gives the spin $j$ $SU(2)$ representations.
- $\mathbb{C}P^2$: $\mathbb{C}P^2$ is of particular interest being of dimension 4. It is the orbit of $SU(3)$ through the hypercharge $Y = 1/3$ diag$(1, 1, -2)$ (or its non-zero multiples):

$$\mathbb{C}P^2 : \{g\, Y\, g^{-1} \;:\; g \in SU(3)\}. \tag{1.1}$$

  The associated representations are symmetric products of 3's or $\bar{3}$'s.

  In a similar way $\mathbb{C}P^N$ are adjoint orbits of $SU(N+1)$ for any $N \geq 3$. They too can be quantized and give rise to fuzzy spaces.
- $SU(3)/[U(1) \times U(1)]$: This 6-dimensional manifold is the orbit of $SU(3)$ through $\lambda_3 = \text{diag}(1, -1, 0)$ and its non-zero multiples. These orbits give all the IRR's containing a zero hypercharge state.

In this book, we focus on the fuzzy spaces emerging from quantizing $S^2$. They are called the fuzzy spheres $S_F^2$ and depend on the integer or half integer $j$ labelling the irreducible representations of $SU(2)$. Physics on $S_F^2$ is treated in detail. Scalar and gauge fields, the Dirac operator, instantons, index theory, and the so-called UV-IR mixing [14–20] are all covered. Supersymmetry can be elegantly discretized in the approach of fuzzy physics by replacing the Lie algebra $su(2)$ of $SU(2)$ by the superalgebras $osp(2,1)$ and $osp(2,2)$. Fuzzy supersymmetry is also discussed in these lectures including its instanton and index theories. We also briefly discuss the fuzzy spaces associated with $\mathbb{C}P^N$ $(N \geq 2)$. These spaces, especially $\mathbb{C}P^2$, are of physical interest. We refer to the literature [29–33] for their more exhaustive treatment.

Fuzzy physics draws from many techniques and notions developed in the context of noncommutative geometry. There are excellent books and reviews on this vast subject some of which we include in the bibliography [3, 34–38].