

Chapter 1

Cauchy Problem

1.1 Fundamental Theorems

1.1.1 *Statement of Cauchy Problem*

We study the system of ordinary differential equation

$$\frac{dx}{dt} = f(t, x), \quad (t, x) \in D \quad (1.1)$$

where D is a region in the space $\mathbb{R} \times \mathbb{R}^n$, and $f(t, x)$ is an n -vector¹ valued function defined on D . The variable x stands for an *unknown function* $x = x(t)$ valued in \mathbb{R}^n , and the integer $n (\geq 1)$ is called the order of the system (1.1). The region D is, in general, assumed to be open in the space $\mathbb{R} \times \mathbb{R}^n$.

If $x = \varphi(t)$ is a differentiable function in the interval $J \subset \mathbb{R}$, satisfying

$$\frac{d\varphi(t)}{dt} = f(t, \varphi(t)) \quad (t \in J),$$

then $x = \varphi(t)$ is called a *solution* of (1.1) on J . In geometry, a solution $x = \varphi(t)$ of (1.1) represents a curve Γ in D , called an *integral curve* of (1.1).

Initial-Value Problem Given a condition

$$x(t_0) = x_0, \quad (t_0, x_0) \in D, \quad (1.2)$$

find a solution $x = x(t)$ of (1.1) on some interval J , such that it satisfies the *initial condition* (1.2) (see [21]).

¹In this book, a vector will be always understood as a column-vector even if it is sometimes written in a row-form for simplicity.

The above initial-value problem is sometimes denoted by (1.1)+(1.2), or briefly by

$$(E) : \quad \frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,$$

where (t_0, x_0) is called an *initial point* in D . In literature, an initial-value problem is also called a *Cauchy problem*, for a memory of the contribution of Cauchy to the theory of differential equations.

In geometry, a solution $x = \varphi(t)$ of (E) stands for an integral curve of (1.1) passing through the initial point (t_0, x_0) in D .

On the other hand, it is easy to show that $x = x(t)$ is a solution of (E) on J if and only if $x = x(t)$ satisfies the *integral equation*

$$x = x_0 + \int_{t_0}^t f(s, x) ds, \quad \forall t \in J. \quad (1.3)$$

The initial-value problem (E) is mainly concerned with the *existence* and *uniqueness* of solution, and as well as the *dependence* of solution on initial condition (or parameter). We will consider these problems in the fundamental theorems mentioned below (see [78], [21], [56] and etc.).

1.1.2 Auxiliary Lemmas

Now, we introduce the following lemmas, which are the basic techniques in the analysis of differential equations.

Arzelà-Ascoli Lemma *Let X be a compact region in the space \mathbb{R}^p ($p \geq 1$). If*

$$f_n : \quad X \quad \rightarrow \quad \mathbb{R}^q \quad (n \in \mathbb{Z}^+)$$

is a sequence of uniformly bounded² and equi-continuous³ mappings, then there is at least a uniformly convergent subsequence of $\{f_n\}$.

² That is, there is a constant $B_0 > 0$, such that

$$|f_n(z)| < B_0 \quad (\forall z \in X), \quad \text{for all } n \in \mathbb{Z}^+.$$

³ That is, given any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, such that whenever $|u - v| < \delta$ ($u, v \in X$), we have

$$|f_n(u) - f_n(v)| < \varepsilon, \quad \text{for all } n \in \mathbb{Z}^+.$$

Proof. The proof is well-known in literature (see [21]). □

Gronwall Lemma *If $u(t)$ is a continuous function, satisfying the integral inequality*

$$0 \leq u(t) \leq C + K \int_{t_0}^t u(t) dt \quad (t_0 \leq t \leq t_1), \quad (1.4)$$

where $C \geq 0$ and $K > 0$ are constants, then

$$u(t) \leq C e^{K(t-t_0)}, \quad t_0 \leq t \leq t_1. \quad (1.5)$$

Proof. We apply a useful technique in the analysis of inequality. Letting

$$w(t) = \int_{t_0}^t u(t) dt \quad (t_0 \leq t \leq t_1),$$

and using (1.4), we obtain

$$\frac{dw(t)}{dt} - Kw(t) \leq C, \quad w(t_0) = 0.$$

It follows that

$$\frac{d}{dt} [w(t)e^{-Kt}] \leq Ce^{-Kt}.$$

Then, integrating from t_0 to $t(> t_0)$, we have

$$w(t) \leq \frac{C}{K} [e^{K(t-t_0)} - 1],$$

which together with (1.4) yields the desired inequality (1.5). □

1.1.3 Peano Theorem

Cauchy was the first person in history to prove the existence and uniqueness of solution of the initial-value problem (E), during the period of 1830's, under the assumption that $f(t, x)$ and $f'_x(t, x)$ are continuous in $(t, x) \in Q$, where Q is a local neighborhood at the initial point (t_0, x_0) . In 1886, Lipschitz improved the Cauchy theorem by using a simplified condition (i.e., the Lipschitz condition defined below) to replace the continuity of $f'_x(t, x)$. In 1890's, Peano proved the local existence of solution of (E) merely under the continuity condition of $f(t, x)$ on Q .

Now, we begin to consider the most elementary theorem; namely, the Peano Theorem. Without loss of generality, assume $f(t, x)$ is a continuous function defined in the rectangular region

$$Q := \{(t, x) \in D : |t - t_0| \leq a, \quad |x - x_0| \leq b\},$$

centered at the point (t_0, x_0) , with some constants $a > 0$ and $b > 0$. Then fix the constants

$$M > \sup_{(t,x) \in Q} |f(t, x)| \quad \text{and} \quad h = \min \left\{ a, \frac{b}{M} \right\}.$$

Now, we are ready to prove the *Peano Theorem*.

Theorem 1.1 *If the function $f(t, x)$ is continuous in $(t, x) \in Q$, then the Cauchy problem (E) has at least a solution $x = \varphi(t)$ on the interval $I = [t_0 - h, t_0 + h]$.*

Proof. We apply the method of Tonelli approximation (1910's) to the initial-value problem (E) as follows.

For a given integer $m \geq 1$, define

$$\phi_m(t) = \begin{cases} x_0, & \text{as } t \in [t_0 - h/m, t_0 + h/m], \\ x_0 + \int_{t_0}^{t - \frac{h}{m}} f(t, \phi_m(t)) dt, & \text{as } t \in [t_0 + h/m, t_0 + h], \\ x_0 + \int_{t_0}^{t + \frac{h}{m}} f(t, \phi_m(t)) dt, & \text{as } t \in [t_0 - h, t_0 - h/m]. \end{cases} \quad (1.6)$$

It can be seen from (1.6) that for $t \geq t_0$, the function $\phi_m(t)$ is first defined in the sub-interval $[t_0, t_0 + h/m]$; and next it is extended to the sub-interval $[t_0 + h/m, t_0 + 2h/m]$ via the integral; and in a similar manner, to the third sub-interval $[t_0 + 2h/m, t_0 + 3h/m]$; and finally to the last sub-interval $[t_0 + (m - 1)h/m, t_0 + h]$. Therefore, $\phi_m(t)$ is defined on the right sub-interval $[t_0, t_0 + h]$. Similarly, the function $\phi_m(t)$ is defined on the left sub-interval $[t_0 - h, t_0]$.

The above-defined sequence $\{\phi_m(t)\}$ is called the *Tonelli sequence* on I . It can be verified by using (1.6) that the Tonelli sequence satisfies

$$|\phi_m(t) - x_0| \leq Mh \quad (t \in I), \quad (1.7)$$

and

$$|\phi_m(t_2) - \phi_m(t_1)| \leq M|t_2 - t_1| \quad (t_1, t_2 \in I), \quad (1.8)$$

for $m \geq 1$. The property (1.7) states that the Tonelli sequence $\{\phi_m(t)\}$ is uniformly bounded on I , while the property (1.8) says that the Tonelli sequence $\{\phi_m(t)\}$ is equi-continuous on I .

Then, using the Arzelà-Ascoli lemma, we conclude that the sequence $\{\phi_m(t)\}$ has at least a uniformly convergent sub-sequence $\{\phi_{m_i}(t)\}$. Then,

$$\phi(t) = \lim_{m_i \rightarrow \infty} \phi_{m_i}(t) \quad (t \in I)$$

is a continuous function. Let $m = m_i$ in (1.6). It follows from the uniform convergence of $\{\phi_{m_i}(t)\}$ that

$$\phi(t) = x_0 + \int_{t_0}^t f(t, \phi(t)) dt \quad (t \in I).$$

This proves that $x = \phi(t)$ is a solution of (E) on I .

The proof of the Peano theorem is thus completed. \square

As an application of Peano theorem, we prove the following famous result in analysis.

Implicit-Function Theorem *Given the equation*

$$g(t, x) = 0, \tag{1.9}$$

where $g(t, x)$ is a C^1 -differentiable (n -vector valued) function in $(t, x) \in Q$, if it satisfies the condition

$$g(t_0, x_0) = 0 \quad \text{and} \quad \det [g'_x(t_0, x_0)] \neq 0, \tag{1.10}$$

then the equation (1.9) defines a unique differentiable implicit-function $x = \varphi(t)$ on $t_0 - \alpha \leq t \leq t_0 + \alpha$, satisfying the initial condition $x(t_0) = x_0$, where $\alpha > 0$ is some small constant.

Proof. Let us consider the auxiliary Cauchy problem

$$(E^*) : \quad \frac{dx}{dt} = -\{g'_x(t, x)\}^{-1} g'_t(t, x), \quad x(t_0) = x_0.$$

The assumption on $g(t, x)$ implies that the function

$$f(t, x) := -\{g'_x(t, x)\}^{-1} g'_t(t, x)$$

is continuous in the neighborhood of (t_0, x_0) . Hence, the Peano theorem asserts that the Cauchy problem (E^*) has at least a (differentiable) solution

$x = \varphi(t)$ on some local interval $|t - t_0| \leq \alpha$. Then we have

$$\begin{aligned} \frac{dg(t, \varphi(t))}{dt} &= g'_t(t, \varphi(t)) + g'_x(t, \varphi(t)) \frac{d\varphi(t)}{dt} \\ &= g'_t(t, \varphi(t)) + g'_x(t, \varphi(t))(-\{g'_x(t, x)\}^{-1}g'_t(t, x)) = 0, \end{aligned}$$

which yields

$$g(t, \varphi(t)) = C \quad (|t - t_0| \leq \alpha),$$

where C is an arbitrary constant. Then, using the initial condition given in (1.10), we obtain

$$C = g(t_0, \varphi(t_0)) = g(t_0, x_0) = 0.$$

Therefore, $x = \varphi(t)$ is a solution of the equation (1.28), satisfying the initial condition $x(t_0) = x_0$. This proves the existence of implicit-function.

For the uniqueness part, assume $x = \varphi_1(t)$ and $x = \varphi_2(t)$ are any implicit-functions of (1.9) in the interval $|t - t_0| \leq \alpha$, satisfying the initial condition $\varphi_1(t_0) = x_0 = \varphi_2(t_0)$.

Let $g_i(t, x)$ be the i -th component of $g(t, x)$, and let

$$h_i(\lambda) = g_i(t, \varphi_1(t) + \lambda(\varphi_2(t) - \varphi_1(t))).$$

It follows from the mean value theorem that

$$h_i(1) - h_i(0) = h'_i(\lambda_i), \quad 0 < \lambda_i < 1,$$

which yields

$$g_i(t, \varphi_2(t)) - g_i(t, \varphi_1(t)) = \frac{\partial g_i}{\partial x}(t, \tilde{\xi}_i)(\varphi_2(t) - \varphi_1(t)),$$

where $\tilde{\xi}_i$ is a vector given by

$$\tilde{\xi}_i = \varphi_1(t) + \lambda_i(\varphi_2(t) - \varphi_1(t)).$$

Hence, we have

$$0 = g(t, \varphi_1(t)) - g(t, \varphi_2(t)) = \mathbb{G}(\varphi_1(t) - \varphi_2(t)), \quad (1.11)$$

where \mathbb{G} is an $n \times n$ -matrix given by

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(t, \tilde{\xi}_1) & \cdots & \frac{\partial g_1}{\partial x_n}(t, \tilde{\xi}_1) \\ \vdots & \vdots & \vdots \\ \frac{\partial g_n}{\partial x_1}(t, \tilde{\xi}_n) & \cdots & \frac{\partial g_n}{\partial x_n}(t, \tilde{\xi}_n) \end{pmatrix}, \quad \text{for } |t - t_0| \leq \alpha.$$

It is noticed that $(t, \tilde{\xi}_i)$ tends to (t_0, x_0) whenever α approaches to 0. Hence, the matrix \mathbb{G} is thus sufficiently near to the non-singular matrix $g'_x(t_0, x_0)$. It follows that \mathbb{G} is invertible if α is small enough. Using (1.11) leads to

$$\varphi_1(t) = \varphi_2(t) \quad (|t - t_0| \leq \alpha).$$

We have thus proved the uniqueness of the implicit function.

The proof of the implicit-function theorem is thus completed. \square

1.1.4 Cauchy-Lipschitz Theorem

Let us introduce the Lipschitz condition. Assume the function $f(t, x)$ is defined on Q . If there is a constant $L > 0$, such that for any points (t, u) and (t, v) in Q we have

$$|f(t, u) - f(t, v)| \leq L|u - v|, \quad (1.12)$$

then we say that $f(t, x)$ satisfies the *Lipschitz Condition* on Q (with respect to x), and L is called the *Lipschitz constant*. It is obvious that the Lipschitz condition is merely the simple property of differentiability condition, but a Lipschitzian function may be not differentiable.

In literature, the following existence and uniqueness theorem of solution is called *Cauchy-Lipschitz theorem*.

Theorem 1.2 *If $f(t, x)$ is continuous in $(t, x) \in Q$ and satisfies the Lipschitz condition (1.12), then the initial-value problem (E) has one and only one solution $x = \varphi(t)$ on I .*

Proof. Since $f(t, x)$ is continuous in $(t, x) \in Q$, then Peano theorem guarantees the existence of solution of (E) on I . Hence, it suffices to prove the uniqueness of solution of (E).

Assume $x = \phi(t)$ and $x = \psi(t)$ are solutions of (E) on the interval $|t - t_0| \leq \alpha$ for some constant $\alpha > 0$. Then we have

$$\phi(t) = x_0 + \int_{t_0}^t f(t, \phi(t)) dt \quad (|t - t_0| \leq \alpha),$$

and

$$\psi(t) = x_0 + \int_{t_0}^t f(t, \psi(t)) dt \quad (|t - t_0| \leq \alpha).$$

It follows from the Lipschitz condition that

$$\begin{aligned} |\psi(t) - \phi(t)| &\leq \left| \int_{t_0}^t |f(t, \psi(t)) - f(t, \phi(t))| dt \right| \\ &\leq \left| \int_{t_0}^t L|\psi(t) - \phi(t)| dt \right|, \quad |t - t_0| \leq \alpha. \end{aligned}$$

If $t \geq t_0$, we have

$$|\psi(t) - \phi(t)| \leq \int_{t_0}^t L|\psi(t) - \phi(t)| dt \quad (t_0 \leq t \leq t_0 + \alpha).$$

It follows from the Gronwall lemma (with $u(t) = |\psi(t) - \phi(t)|$, $C = 0$ and $K = L$) that

$$|\psi(t) - \phi(t)| \leq 0 \quad (t_0 \leq t \leq t_0 + \alpha).$$

Hence

$$\psi(t) = \phi(t) \quad (t_0 \leq t \leq t_0 + \alpha).$$

If $t \leq t_0$, in a similar manner, we can prove

$$\psi(t) = \phi(t) \quad (t_0 - \alpha \leq t \leq t_0).$$

The uniqueness of solution of (E) is thus proved. So, the proof of Theorem 1.2 is complete. \square

It follows from the Cauchy-Lipschitz theorem that if the initial condition is given and the differential equation is sufficiently regular, the solution of the corresponding Cauchy problem is predictable. In this sense, the Cauchy problem is *well-posed* or *deterministic*.

1.1.5 Dependence of Solution on Parameter

1) Now, we consider the dependence of solution on initial condition.

As Poincaré pointed out, even if it were the case that the natural law of differential equations had no longer any secret for us, we would still only know the initial data approximately by measuring. Suppose the initial condition is measured approximately by

$$x(\tau) = \xi. \quad (1.2^*)$$

Naturally, there is an error between the approximate condition (1.2*) and the given condition (1.2). Hence, we have to consider the error between the solutions of the corresponding Cauchy problems

$$(E): \quad \frac{dx}{dt} = f(t, x), \quad x(\tau) = \xi,$$

and

$$(E_0): \quad \frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,$$

where (E) is the *perturbation* of the Cauchy problem (E₀). For convenience, let $p = (\tau, \xi)$ and $p_0 = (t_0, x_0)$ be the corresponding conditions. Hence, the problem concerns the *dependence* of solution of (E) on the initial data p .

Let

$$G = \{ (\tau, \xi) \in Q : |\tau - t_0| \leq h/4, |\xi - x_0| \leq b/2 \}.$$

It follows from Peano theorem that for each initial condition $p \in G$, the Cauchy problem (E) has at least a solution, existing on the interval $I_0 = [t_0 - h/4, t_0 + h/4]$. It is noticed that the solution of (E) may be unique or not. Anyway, let $x = \phi(t, p)$ be a solution of (E).

Lemma 1.1 *If the solution of (E₀) is unique on the interval I₀, then for any given constant $\varepsilon > 0$, there is a constant $\delta > 0$, such that*

$$|\phi(t, p) - \phi(t, p_0)| < \varepsilon, \quad \text{for } t \in I_0,$$

whenever $|p - p_0| < \delta$.

Proof. Assume the contrary. Then there exists $\varepsilon_0 > 0$, such that for any small constant $\delta_i > 0$, there is a point $(t_i, p_i) \in I_0 \times B_{\delta_i}(p_0)$, satisfying

$$|\phi(t_i, p_i) - \phi(t_i, p_0)| \geq \varepsilon_0. \quad (1.13)$$

Let $\delta_i \rightarrow 0$ when $i \rightarrow \infty$. It follows that $p_i \rightarrow p_0$ as $i \rightarrow \infty$, and $\{t_i\}$ has at least a cluster point $\hat{t} \in I_0$. Without loss of generality, assume $t_i \rightarrow \hat{t}$ as $i \rightarrow \infty$.

On the other hand, it can be easily proved that $\{\phi(t, p_i)\}$ is a uniformly bounded and equi-continuous sequence on the interval I_0 .

Using the Arzelà-Ascoli lemma, we conclude that there is a uniformly convergent subsequence $\phi(t, p_{i_k})$. Moreover, we can easily prove that

$$\psi(t) = \lim_{i_k \rightarrow \infty} \phi(t, p_{i_k}) \quad (t \in I_0)$$

is a solution of (E_0) . Then, the uniqueness of solution of (E_0) implies the identity: $\psi(t) = \phi(t, p_0)$, for all $t \in I_0$.

On the other hand, using (1.13) for its limit as $i \rightarrow \infty$, we obtain

$$|\psi(\hat{t}) - \phi(\hat{t}, p_0)| \geq \varepsilon_0,$$

which is in conflict with the identity: $\psi(t) = \phi(t, p_0)$.

We have thus proved Lemma 1.1 by contradiction. \square

The following theorem is a consequence of Lemma 1.1. It states the continuous dependence of the solution $x = \phi(t, p)$ of (E) on the initial condition $p \in G$.

Theorem 1.3 *Assume $f(t, x)$ is continuous in $(t, x) \in D$. If the solution $x = \varphi(t, p)$ of (E) is unique for each initial point $p \in G$, then $x = \varphi(t, p)$ is continuous in $(t, p) \in I_0 \times G$.*

2) Next we consider the dependence of solution on parameter.

It follows from the substitutions $t = s + \tau$ and $x = z + \xi$ that the initial-value problem (E) is transformed into the following one

$$(E_\mu) : \quad \frac{dz}{ds} = g(s, z; \mu), \quad z(0) = 0,$$

where $\mu = (\tau, \xi)$ is a parameter and $g(s, z; \mu) = f(s + \tau, z + \xi)$ is a continuous function of $(s, z; \mu)$.

Sometimes, the natural law of differential equation may indeed depend on certain parameter.

It leads us to consider a general form of Cauchy problem as follows

$$(E_\lambda) : \quad \frac{dx}{dt} = f(t, x; \lambda), \quad x(t_0) = x_0,$$

where λ is a parameter-vector in a compact region $\Lambda \subset \mathbb{R}^s$ and $f(t, x; \lambda)$ is a continuous function of $(t, x; \lambda) \in Q \times \Lambda$, which is valued in \mathbb{R}^n .

Similar to Cauchy problem (E), fix the constants

$$h^* = \min \left\{ a, \frac{b}{M^*} \right\} \quad \text{and} \quad M^* > \max_{(t,x;\lambda) \in Q \times \Lambda} |f(t, x; \lambda)|,$$

with the interval $I^* = [t_0 - h^*, t_0 + h^*]$.

Proposition 1.1 *If (E_λ) has a unique solution on I^* for any $\lambda \in \Lambda$, then its solution $x = \varphi(t, \lambda)$ is continuous in $(t, \lambda) \in I^* \times \Lambda$ (i.e., the solution of (E_λ) is continuous for the parameter $\lambda \in \Lambda$).*

Proof. Given $(\tau, \sigma) \in I^* \times \Lambda$, assume $(t_k, \lambda_k) \in I^* \times \Lambda$ is any sequence tending to (τ, σ) as $k \rightarrow \infty$.

It suffices to prove that for any given $\varepsilon > 0$, we have

$$|\varphi(t_k, \lambda_k) - \varphi(\tau, \sigma)| < \varepsilon, \quad \text{as } k \rightarrow \infty.$$

Assume the contrary. Then there are a constant $\varepsilon_0 > 0$ and a sequence (t_k, λ_k) tending to (τ, σ) , such that

$$|\varphi(t_k, \lambda_k) - \varphi(\tau, \sigma)| \geq \varepsilon_0, \quad \text{as } k \rightarrow \infty. \quad (1.14)$$

Since $x = \varphi(t, \lambda_k)$ is the solution of (E_{λ_k}) , we have

$$\varphi(t, \lambda_k) = x_0 + \int_{t_0}^t f(t, \varphi(t, \lambda_k); \lambda_k) dt, \quad t \in I^*, \quad (1.15)$$

which leads to the following conclusions:

- 1) $|\varphi(t, \lambda_k)| \leq |x_0| + M^* h^*$, for $t \in I^*$
(it means that $\varphi(t, \lambda_k)$ is uniformly bounded in I^*);
- 2) $|\varphi(t, \lambda_k) - \varphi(s, \lambda_k)| \leq M^* |t - s|$, for $t, s \in I^*$
(it implies that $\varphi(t, \lambda_k)$ is equi-continuous in I^*).

It follows from the Arzelà-Ascoli lemma that there is a uniformly convergent subsequence of $\varphi(t, \lambda_k)$ on the interval I^* . Without loss of generality, assume $\varphi(t, \lambda_k)$ is uniformly convergent on I^* , and let

$$\lim_{k \rightarrow \infty} \varphi(t, \lambda_k) = \psi(t), \quad t \in I^*,$$

which together with (1.15) yields

$$\psi(t) = x_0 + \int_{t_0}^t f(t, \psi(t); \sigma) dt, \quad t \in I^*.$$

Then the uniqueness of solution implies

$$\psi(t) = \varphi(t, \sigma), \quad t \in I^*. \quad (1.16)$$

On the other hand, it follows from (1.14) that

$$|\psi(\tau) - \varphi(\tau, \sigma)| \geq \varepsilon_0 > 0,$$

which is in conflict with (1.16).

Proposition 1.1 is thus proved by contradiction. \square

As remarked above, since (E) can be reduced to (E_μ) , Theorem 1.3 is a consequence of Proposition 1.1. Hence, (E) is well-posed whenever $f(t, x)$ is continuous in $(t, x) \in Q$ and the solution of (E) is unique.

Note that the following result is an important consequence of Lipschitz condition.

Corollary 1.1 *If $f(t, x; \lambda)$ is continuous in $(t, x; \lambda) \in Q \times \Lambda$ and satisfies the Lipschitz condition with respect to x , then the solution $x = \varphi(t, \lambda)$ of (E_λ) is continuous for the parameter $\lambda \in \Lambda$.*

3) Finally, we consider the differentiability of solution for parameter.

Theorem 1.4 *If $f(t, x; \lambda)$ is continuous in $(t, x; \lambda) \in Q \times \Lambda$ and C^1 -differentiable with respect to (x, λ) , then the solution $x = \varphi(t, \lambda)$ of (E_λ) is existent on I^* and C^1 -differentiable for the parameter $\lambda \in \Lambda$.*

Proof. Let $\{\phi_m(t, \lambda)\}$ be the Tonelli sequence of (E_λ) , defined by

$$\phi_m(t, \lambda) = x_0, \quad \text{as } t \in [t_0 - h^*/m, t_0 + h^*/m];$$

$$\phi_m(t, \lambda) = x_0 + \int_{t_0}^{t-h^*/m} f(s, \phi_m(s, \lambda); \lambda) ds,$$

$$\text{as } t \in [t_0 + h^*/m, t_0 + h^*];$$

$$\phi_m(t, \lambda) = x_0 + \int_{t_0}^{t+h^*/m} f(s, \phi_m(s, \lambda); \lambda) ds,$$

$$\text{as } t \in [t_0 - h^*, t_0 - h^*/m],$$

$(m = 1, 2, \dots)$.

Since $f(t, x; \lambda)$ is C^1 differentiable with respect to (x, λ) , we have

$$\frac{\partial \phi_m(t, \lambda)}{\partial \lambda} = 0, \quad \text{as } t \in [t_0 - h^*/m, t_0 + h^*/m];$$

$$\frac{\partial \phi_m(t, \lambda)}{\partial \lambda} = \int_{t_0}^{t-h^*/m} \left[\frac{\partial f}{\partial x} \frac{\partial \phi_m(s, \lambda)}{\partial \lambda} + \frac{\partial f}{\partial \lambda} \right] ds,$$

$$\text{as } t \in [t_0 + h^*/m, t_0 + h^*];$$

$$\frac{\partial \phi_m(t, \lambda)}{\partial \lambda} = \int_{t_0}^{t+h^*/m} \left[\frac{\partial f}{\partial x} \frac{\partial \phi_m(s, \lambda)}{\partial \lambda} + \frac{\partial f}{\partial \lambda} \right] ds,$$

$$\text{as } t \in [t_0 - h^*, t_0 - h^*/m],$$

where

$$\frac{\partial f}{\partial x} = f'_x(s, \phi_m(s, \lambda); \lambda) \quad \text{and} \quad \frac{\partial f}{\partial \lambda} = f'_\lambda(s, \phi_m(s, \lambda); \lambda),$$

($m = 1, 2, \dots$).

It follows from the Gronwall lemma that

- (1) The sequence $\frac{\partial \phi_m(t, \lambda)}{\partial \lambda}$ is uniformly bounded in I^* ;
- (2) The sequence $\frac{\partial \phi_m(t, \lambda)}{\partial \lambda}$ is equi-continuous in I^* .

Using Arzelà-Ascoli lemma, assume without loss of generality that $\partial \phi_m(t, \lambda)/\partial \lambda$ is uniformly convergent on I^* . Let

$$\lim_{m \rightarrow \infty} \frac{\partial \phi_m(t, \lambda)}{\partial \lambda} = \zeta(t, \lambda).$$

It follows that

$$\zeta(t, \lambda) = \int_{t_0}^t [f'_x(t, \varphi(t, \lambda); \lambda)\zeta(t, \lambda) + f'_\lambda(t, \varphi(t, \lambda); \lambda)] dt.$$

Therefore, $z = \zeta(t, \lambda)$ is the solution of the Cauchy problem

$$\frac{dz}{dt} = f'_x(t, \varphi(t, \lambda); \lambda)\zeta(t, \lambda)z + f'_\lambda(t, \varphi(t, \lambda); \lambda), \quad z(t_0) = 0,$$

which together with Proposition 1.1 implies that $z = \zeta(t, \lambda)$ is continuous in (t, λ) . On the other hand, as shown above, $\phi_m(t, \lambda)$ uniformly converges to

$\varphi(t, \lambda)$ and $\frac{\partial \phi_m(t, \lambda)}{\partial \lambda}$ uniformly converges to $\zeta(t, \lambda)$ when $m \rightarrow \infty$. Hence, we have the partial derivative

$$\varphi'_\lambda(t, \lambda) = \zeta(t, \lambda),$$

which is continuous in $(t, \lambda) \in I^* \times \Lambda$.

On the other hand, it follows from $\varphi'_t(t, \lambda) = f(t, \varphi(t, \lambda); \lambda)$ that $\varphi(t, \lambda)$ is C^1 differentiable in $(t, \lambda) \in I^* \times \Lambda$.

The proof of Theorem 1.4 is thus complete. \square

Theorem 1.5 *If $f(t, x)$ is continuous in $(t, x) \in Q$ and C^1 differentiable with respect to x , then the solution $x = \varphi(t, \tau, \xi)$ of the Cauchy problem (E) is C^1 differentiable with respect to the initial condition $(\tau, \xi) \in G$.*

Proof. As mentioned in the Cauchy problem (E_μ) , this theorem seems to be a special case of Theorem 1.4. However, we have to make a remark as follows.

Indeed, the differentiability of $x = \varphi(t, \tau, \xi)$ with respect to ξ can be proved through Theorem 1.4 by using a substitution $x = z + \xi$, where the corresponding partial derivative $\partial \phi_m / \partial \xi$ is continuous.

However, it can be seen that the differentiability of $x = \varphi(t, \tau, \xi)$ with respect to τ cannot be proved in this way, since the continuity of the partial derivative $\partial \phi_m / \partial \tau$ depends on the continuity of $f'_t(t, x)$, which is in question since we do not assume even the existence of $f'_t(t, x)$. Therefore, we have to change the method of proof.

For this aim, let us modify the Tonelli sequence $\{\phi_m\}$ of (E) as follows:

$$\phi_m(t, \tau, \xi) = \xi + \int_\tau^t f(t, \xi) dt, \quad \text{as } t \in [\tau - h/4m, \tau + h/4m];$$

$$\phi_m(t, \tau, \xi) = \xi + \int_\tau^{\tau + \frac{h}{4m}} f(t, \xi) dt + \int_\tau^{t - \frac{h}{4m}} f\left(t + \frac{h}{4m}, \phi_m(t, \tau, \xi)\right) dt, \\ \text{as } t \in [\tau + h/4m, \tau + h/4];$$

$$\phi_m(t, \tau, \xi) = \xi + \int_\tau^{\tau - \frac{h}{4m}} f(t, \xi) dt + \int_\tau^{t + \frac{h}{4m}} f\left(t - \frac{h}{4m}, \phi_m(t, \tau, \xi)\right) dt, \\ \text{as } t \in [\tau - h/4, \tau - h/4m],$$

$(m = 1, 2, \dots)$.

We see that $\{\partial\phi_m/\partial\tau\}$ is continuous. Therefore, this sequence $\{\phi_m\}$ is available to prove Theorem 1.5 in a similar manner as before. \square

1.1.6 Carathéodory Theorem

In 1920's, Carathéodory generalized the initial-value problem

$$(E_0) : \quad \frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0$$

to the following sense (see [21]).

If $x = \varphi(t)$ is a function absolutely continuous in the interval J , such that it satisfies the differential equation

$$\varphi'(t) = f(t, \varphi(t)) \quad (\text{for almost all } t \in J),$$

and the initial condition $\varphi(t_0) = x_0$, then $x = \varphi(t)$ is called a *generalized solution* of (E_0) on J .

The following theorem is called the *existence theorem of Carathéodory*, which is widely applied in the modern theory of differential equations.

Theorem 1.6 *Let the function $f(t, x)$ be defined on Q . Assume $f(t, x)$ is measurable in t for each fixed x , and continuous in x for each fixed t . If there exists a Lebesgue-integrable function $m(t)$ on the interval $|t - t_0| \leq \alpha$, such that*

$$|f(t, x)| \leq m(t) \quad ((t, x) \in Q), \quad (1.17)$$

then there exists a generalized solution $x = \varphi(t)$ of (E_0) on a local interval $|t - t_0| \leq \alpha$ (for some constant $\alpha > 0$).

Proof. The proof of this Theorem is similar to that of Theorem 1.1 if we use the Lebesgue's integral instead of the Riemann integral therein. \square

1.2 Method of Euler Polygons

Roughly speaking, there are three elementary methods in the theory of differential equations for approximating the solution of the initial-value problem (E_0) . When the function $f(t, x)$ is lipschitzian for the variable x , the Picard's successive approximation is the most concise method of approximation. But if $f(t, x)$ does not satisfy the Lipschitz condition for x , the *Müller's example* expresses that the Picard method is not useful

(see [21]). However, both the methods of Euler's polygonal approximation and Tonelli's approximation can still work in that case. By the way, we point out that the Tonelli method is a unified method for proving all the fundamental theorems as we did above.

In the next subsection, we will simplify the proof due to Gardner [68] for the existence of primitive function in calculus through the method of Euler's polygonal approximation without use of integration.⁴

1.2.1 Existence of Solution off Integration

In particular, the Euler polygonal approximation in the planar case can be used to prove the Peano theorem without the helps of the Arzelà-Ascoli lemma and even the idea of integration (see [68]).

Now, we introduce the proof of Gardner with modifications.

1) Consider the Cauchy problem

$$(E_0) : \quad \frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,$$

of first order differential equation, where $f(t, x)$ is a continuous function on the planar rectangular region Q , centered at (t_0, x_0) . Let M and h be the positive constants and $I = [t_0 - h, t_0 + h]$ be the interval defined as above. For simplicity, we are confined in the case of right-hand interval $[t_0, t_0 + h]$. The case of left-hand interval $[t_0 - h, t_0]$ can be considered in a similar way.

Given an integer $m \geq 1$, divide the interval $[t_0, t_0 + h]$ by the points

$$t_0 < t_1 < \cdots < t_{m-1} < t_m (= t_0 + h), \quad (1.18)$$

with

$$\delta_m = \max_{1 \leq i \leq m} \{t_i - t_{i-1}\} \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

Let L_m be the Euler polygon of (E_0) with the cusps :

$$P_i = (t_i, x_i) \quad (i = 0, 1, \cdots, m),$$

where

$$x_{i+1} = x_i + f(t_i, x_i)(t_{i+1} - t_i), \quad i = 0, 1, \cdots, m - 1.$$

⁴It is already known in analysis that the existence of the primitive function without use of integration can be proved by the Weierstrass's polynomial approximation theorem.

Hence, the Euler polygon L_m is given by the formula

$$x = L_m(t) = x_i + f(t_i, x_i)(t - t_i) \quad (t \in [t_i, t_{i+1}]), \quad 0 \leq i \leq m - 1.$$

It follows from $|f(t, x)| \leq M$ that when $t_0 \leq t \leq t_0 + h$, we have

$$|L_m(t) - x_0| \leq \sum_{k=0}^{i-1} |f(t_k, x_k)|(t_{k+1} - t_k) \leq M.$$

It follows that the sequence of Euler's polygons $\{L_m(t)\}$ is uniformly bounded on $t_0 \leq t \leq t_0 + h$.

Denote by \underline{s}_m and \overline{s}_m the lower-limit and the upper-limit of a sequence s_m , respectively. It is well-known in textbook that

$$\underline{L_m(x) - L_m(y)} \leq \overline{L_m(x) - L_m(y)} \leq \overline{L_m(x) - L_m(y)}. \quad (1.19)$$

Let $\tau, s \in [t_0, t_0 + h]$ with $s \neq \tau$. It follows from (1.18) that $\tau \in [t_p, t_{p+1})$ and $s \in [t_q, t_{q+1})$ for some integers p, q satisfying $0 \leq p, q \leq m-1$. Without loss of generality, assume $p \leq q$.

2) Then we will prove the inequality:

$$\min_{p \leq i \leq q} f(t_i, x_i) \leq \frac{L_n(s) - L_n(\tau)}{s - \tau} \leq \min_{p \leq i \leq q} f(t_i, x_i). \quad (1.20)$$

In fact, let $A = (t, L_m(t))$ and $B = (s, L_m(s))$. Consider the auxiliary points

$$F_a = (t, M_a(s - t)) \quad \text{and} \quad F_b = (t, M_b(s - t)),$$

where

$$M_a = \max_{p \leq i \leq q} f(t_i, x_i) \quad \text{and} \quad M_b = \min_{p \leq i \leq q} f(t_i, x_i).$$

Then we have a triangle Δ having the vertices A, F_a and F_b .

It is noticed that the slopes of the sides $\overline{AF_a}$ and $\overline{AF_b}$ are equal to M_a and M_b , respectively, ($M_b \leq M_a$).

Now, let us consider the sub-polygon $L_m[A, B]$ on L_m with the cusps

$$A, E_{p+1}, \dots, E_q, B.$$

Since the line-segments

$$\overline{AE_{p+1}}, \quad \overline{E_{p+1}E_{p+2}}, \quad \dots, \quad \overline{E_qB}, \quad (1.21)$$

have the slopes

$$f(t_p, x_p), \quad f(t_{p+1}, x_{p+1}), \quad \dots, \quad f(t_q, x_q),$$

respectively, we can prove by induction that each line-segment in (1.21) lies in the triangle Δ . Hence, we have

$$L_m[A, B] = \overline{AE_{p+1}} \cup \overline{E_{p+1}E_{p+2}} \cup \dots \cup \overline{E_qB} \subset \Delta,$$

which implies that the line-segment \overline{AB} is contained in Δ . It follows that the slope of \overline{AB} is bounded between the slope M_a of the side $\overline{AF_a}$ and the slope M_b of the side $\overline{AF_b}$. We have thus proved the inequality (1.20).

3) Finally, define the function

$$Y(t) := \limsup_{m \rightarrow \infty} L_m(t), \quad t_0 \leq t \leq t_0 + h.$$

Let $\tau, \tau + \sigma \in [t_0, t_0 + h]$ ($\sigma \neq 0$), and let $\xi = Y(\tau)$. Then we have

$$\frac{Y(\tau + \sigma) - Y(\tau)}{\sigma} = \frac{1}{\sigma} \left[\limsup_{m \rightarrow \infty} L_m(\tau + \sigma) - \limsup_{m \rightarrow \infty} L_m(\tau) \right].$$

It follows from (1.19) that

$$\begin{aligned} \liminf_{m \rightarrow \infty} \frac{L_m(\tau + \sigma) - L_m(\tau)}{\sigma} &\leq \frac{Y(\tau + \sigma) - Y(\tau)}{\sigma} \\ &\leq \limsup_{m \rightarrow \infty} \frac{L_m(\tau + \sigma) - L_m(\tau)}{\sigma}, \end{aligned}$$

which together with (1.20) yields

$$\min_{(t,x) \in Q_\sigma} \{f(t, x)\} \leq \frac{Y(\tau + \sigma) - Y(\tau)}{\sigma} \leq \max_{(t,x) \in Q_\sigma} \{f(t, x)\},$$

where

$$Q_\sigma = \{ (t, x) \mid |t - \tau| \leq |\sigma|, \quad |x - \xi| \leq M|\sigma| \}.$$

Then, letting $\sigma \rightarrow 0$, we get

$$Y'(\tau) = f(\tau, \xi) \quad (t_0 \leq \tau \leq \tau + h).$$

It follows from $\xi = Y(\tau)$ and $Y(t_0) = x_0$ that $x = Y(t)$ is a solution of the Cauchy problem (E_0) on the interval $[t_0, t_0 + h]$.

Therefore, we have proved the existence of solution of the planar Cauchy problem (E_0) without using the Arzelà-Ascoli lemma and even the idea of integration.

We can prove in a similar manner that

$$x = Z(t) = \liminf_{m \rightarrow \infty} L_m(t) \quad (t_0 \leq t \leq t_0 + h)$$

is a solution of the planar Cauchy problem (E_0) , with the property that

$$Z(t) \leq Y(t), \quad t_0 \leq t \leq t_0 + h.$$

Remark Since the solution $y = F(x)$ of the Cauchy problem

$$\frac{dy}{dx} = f(x), \quad y(0) = 0$$

is a primitive function of $f(x)$ (i.e., $F'(x) = f(x)$), we have proved, as a corollary of the above result, the existence of primitive functions for the continuous functions without using the idea of integration.

1.2.2 Maximal Solution and Minimal Solution

Consider a planar Cauchy problem

$$(E_0) : \quad \frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,$$

where $f(t, x)$ is a continuous function in the region

$$Q : \quad |t - t_0| \leq a, \quad |x - x_0| \leq b.$$

Now, let k be a positive integer. Assume $x = \phi_k(t)$ is a solution of the Cauchy problem

$$(E_{1/k}) : \quad \frac{dx}{dt} = f(t, x) + \frac{1}{k}, \quad x(t_0) = x_0,$$

and $x = \psi_k(t)$ is a solution of the Cauchy problem

$$(E_{-1/k}) : \quad \frac{dx}{dt} = f(t, x) - \frac{1}{k}, \quad x(t_0) = x_0.$$

Lemma 1.2 *If $x = \chi(t)$ is a solution of the Cauchy problem (E_0) , then*

$$\psi_k(t) \leq \chi(t) \leq \phi_k(t), \quad t_0 \leq t \leq t_0 + h. \quad (1.22)$$

The proof of (1.22) is trivial, and is thus omitted.

Moreover, it can be shown that the sequences $\{\phi_k(t)\}$ and $\{\psi_k(t)\}$ are uniformly bounded and equi-continuous in the interval $[t_0, t_0 + h]$. It follows from the Arzelà-Ascoli lemma that they have uniformly convergent subsequences, respectively. Without loss of generality, assume

$$\Phi(t) = \lim_{k \rightarrow \infty} \phi_k(t) \quad \text{and} \quad \Psi(t) = \lim_{k \rightarrow \infty} \psi_k(t),$$

for $t \in [t_0, t_0 + h]$. It follows from $(E_{1/k})$ and $(E_{-1/k})$ that $x = \Phi(t)$ and $x = \Psi(t)$ are solutions of the Cauchy problem (E_0) , and (1.22) implies

$$\Psi(t) \leq \chi(t) \leq \Phi(t), \quad t_0 \leq t \leq t_0 + h, \quad (1.23)$$

where $x = \chi(t)$ is any solution of (E_0) .

In this sense, $x = \Phi(t)$ is called the *maximal solution* and $x = \Psi(t)$ is called the *minimal solution* for the planar Cauchy problem (E_0) .

Hence, when $f(t, x)$ is continuous in $Q \subset \mathbb{R}^2$, there exist a maximal solution and a minimal solution of (E_0) on $[t_0, t_0 + h]$. The maximal solution equals to the minimal solution if and only if the solution of (E_0) is unique on $[t_0, t_0 + h]$.

A similar conclusion can be derived on the left-hand interval $[t_0 - h, t_0]$.

1.3 Local Behavior of Integral Curves

1.3.1 Integral Box

Let $f(t, x)$ be a continuous n -vector field in the region $D \subset \mathbb{R} \times \mathbb{R}^n$. It follows from the Peano theorem that there is at least a solution $x = \varphi(t, \tau, \xi)$ of (1.1) passing through each initial point $(\tau, \xi) \in D$.

Assume the solution $x = \varphi(t, \tau, \xi)$ passing through each point $(\tau, \xi) \in D$ is unique. Then it can be shown that the local behavior of the solutions $\{\varphi(t, \tau, \xi)\}$ is quite simple in geometry.

In fact, it follows from Theorem 1.3 that the solution $x = \varphi(t, \tau, \xi)$ of (E) is continuous in $(t, \tau, \xi) \in I_0 \times G$. In particular, for the fixed $\tau = t_0$, the solution $x = \psi(t, \xi) = \varphi(t, t_0, \xi)$ represents the integral curve Γ_ξ of (1.1) passing through the initial point (t_0, ξ) . It follows that the solution $x = \psi(t, \xi)$ is continuous in the box

$$B_0 = \{ (t, \xi) \in \mathbb{R} \times \mathbb{R}^n : |t - t_0| \leq h/4, \quad |\xi - x_0| \leq b/2 \},$$

with the property that

$$\psi(t, \xi_1) \neq \psi(t, \xi_2) \quad (\text{or } \Gamma_{\xi_1} \cap \Gamma_{\xi_2} = \emptyset) \quad \iff \quad \xi_1 \neq \xi_2.$$

Let

$$\mathcal{Y}_0 = \{ (t, x) \in Q : x = \psi(t, \xi), \quad |t - t_0| \leq h/4 \quad (|\xi - x_0| \leq b/2) \}.$$

Then

$$\mathcal{T} : \quad (t, \xi) \quad \mapsto \quad (t, \psi(t, \xi)) \quad (1.24)$$

is a topological transformation from B_0 onto \mathcal{Y}_0 , with the property that the straight line-segment

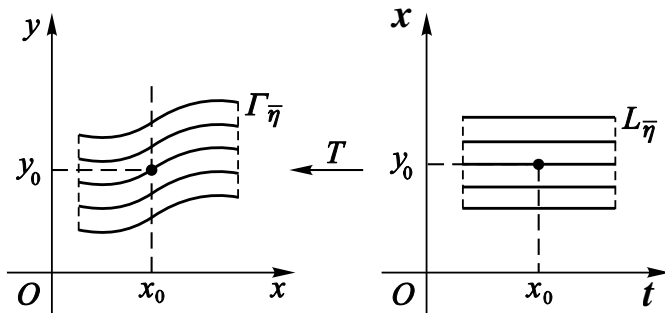
$$L_\xi = \{ (t, \xi) \in B_0 : |t - t_0| \leq h/4 \}$$

is transformed onto the integral curve

$$\Gamma_\xi = \{ x = \psi(t, \xi) : |t - t_0| \leq h/4 \}$$

(i.e., $\mathcal{T}(L_\xi) = \Gamma_\xi$). It follows that the family of integral curves $\{\Gamma_\xi\}$ in \mathcal{Y}_0 is topologically equivalent to that of parallel lines $\{L_\xi\}$ in B_0 , such that $L_\xi = \mathcal{T}^{-1}(\Gamma_\xi)$ for any $\Gamma_\xi \in \mathcal{Y}_0$.

The set \mathcal{Y}_0 of integral curves is called an *integral box* of the differential equation (1.1) across the initial point (t_0, x_0) (see the following Fig. 1).⁵



(Fig. 1)

We have thus proved the following result:

⁵Where the (t, x) -space is replaced by (x, y) -space.

Theorem 1.7 *If the solution $x = \varphi(t, \tau, \xi)$ of differential equation (1.1) satisfying the initial condition $x(\tau) = \xi$ is unique for each $(\tau, \xi) \in G$, then there is an integral box Υ_0 of (1.1) across the initial point (t_0, x_0) .*

1.3.2 Peano Broom

If the solution of the Cauchy problem (E) is not unique, there are at least two integral curves of (E) . Then

$$\Omega(E) = \{ \Gamma : \Gamma \text{ is an integral curve of } (E) \text{ in } Q \}$$

is called the *Peano broom* of (E) . Sometimes, for convenience, a Peano broom is also considered as the point-set occupied by the integral curves therein.

Example 1.1 Consider the planar Cauchy problem

$$(E_1) : \quad \frac{dy}{dx} = \sqrt{|y|} + b, \quad y(x_0) = 0, \quad (1.25)$$

where b is a constant. It can be shown that

- (1) If $b \neq 0$, then the solution of (E_1) is unique. Hence, (E_1) has no Peano broom;
- (2) If $b = 0$, then the solution of (E_1) is not unique. It is not hard to find the Peano broom

$$\Omega(E_1) = \begin{cases} 0 \leq y \leq \frac{1}{4}(x - x_0)^2, & 0 \leq x - x_0 < \infty; \\ -\frac{1}{4}(x - x_0)^2 \leq y \leq 0, & -\infty < x - x_0 \leq 0. \end{cases}$$

Hence, there is a Peano broom at each point x_0 on the line $y = 0$, which is a singular solution of the differential equation $dy/dx = \sqrt{|y|}$.

In fact, Peano broom exists at each point of a singular solution. We have trivial examples of differential equations, which have Peano brooms across all points of the singular solutions. But it is not trivial to find a differential equation having Peano brooms across all points in a region.

Example 1.2 In the paper [84], Lavrentief gave such an example

$$\frac{dy}{dx} = g^*(x, y), \quad (1.26)$$

where $g^*(x, y)$ is a continuous function defined on a closed square

$$Q = \{ (x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1 \},$$

such that the differential equation (1.26) has a Peano broom across each point in the square Q . It is said in literature (see [109] and [102]) that the differential equation (1.26) admits *Lavrentief Phenomenon* on Q .

Example 1.3 Let

$$t = t, \quad x = r \cos \theta, \quad y = r \sin \theta$$

be the cylindrical coordinates in the space (t, x, y) . Consider a spatial Cauchy problem

$$(E_2) : \quad \begin{cases} \frac{dr}{dt} = 0, & \frac{d\theta}{dt} = \sqrt{|\sin \theta|}; \\ r(0) = 1, & \theta(0) = 0. \end{cases}$$

Since the integral

$$\int_0^c \frac{d\theta}{\sqrt{|\sin \theta|}} \quad (\text{for } c \neq 0)$$

is convergent, it follows from the Osgood's criterion (see [21]) that the solution of (E_2) is not unique. Hence, the Cauchy problem (E_2) has a Peano broom $\Omega(E_2)$ on the cylindrical surface Z_1 ($r = 1$). The intersection of $\Omega(E_2)$ and the plane T_c ($t = c > 0$) is an arc A_c on the circle $S_c = Z_1 \cap T_c$. If the condition

$$c < \int_0^{2\pi} \frac{d\theta}{\sqrt{|\sin \theta|}}$$

is satisfied, then the arc A_c is a proper sub-arc of S_c ; if

$$c \geq \int_0^{2\pi} \frac{d\theta}{\sqrt{|\sin \theta|}},$$

then the arc A_c covers the circle S_c . In any case, the Peano broom $\Omega(E_2)$ is a two-dimensional region on the cylindrical surface Z_1 .

This example shows that the Peano broom of differential equations in three-dimensional space \mathbb{R}^3 may contain no three-dimensional interior point. However, it can be seen from the Lemma 1.3 below that the Peano broom of differential equations in two-dimensional plane \mathbb{R}^2 contains at

least a two-dimensional interior point. Generally speaking, the structure of Peano brooms in higher dimensional space is rather complicated.

1.4 Peano Phenomenon

Consider the differential equation

$$\frac{dx}{dt} = f(t, x), \quad (1.27)$$

where the vector field $f(t, x)$ is continuous in the region $D \subset \mathbb{R} \times \mathbb{R}^n$. Since we consider the local behavior of the solutions of (1.27), there is no loss of generality to assume that the region D is a strip

$$S = \{ (t, x) : |t| \leq 1, \quad x \in \mathbb{R}^n \},$$

centered at the origin \mathbf{o} , and the vector field $f(t, x)$ has the property that

$$f(t, x) = 0, \quad (t, x) \in S \quad (\text{for } |x| \gg 1). \quad (1.28)$$

It can be seen that the solution $x = \varphi(t)$ of (1.27) passing through each point $p = (\tau, \xi) \in S$ (i.e., $\varphi(\tau) = \xi$) exists on the interval $I = [-1, 1]$.

It follows from Theorem 1.7 that the differential equation (1.27) has an integral box \mathcal{Y}_p across the point $p \in S$ if the solution $x = \varphi(t)$ of (1.27), which passes through each initial point $z \in S$ near the point p , is unique.

If the solution $x = \varphi(t)$ of (1.27) passing through the point $p \in S$ is not unique, then there is a Peano broom

$$\Omega_p = \{ (t, x) \in D : x = \varphi(t) \text{ is a solution of (1.27) passing through } p \}$$

across the point $p \in S$. In this case, the solution of (1.27) passing through the initial point p is not predicable. Really, *Peano phenomenon* happens to the Cauchy problem at p .

1.4.1 Density Theorem on Peano Phenomena

Now, consider the Cauchy problem

$$(E) : \quad \frac{dx}{dt} = f(t, x), \quad x(0) = 0,$$

where the continuous vector field $f(t, x)$ satisfies the property (1.28). It is obvious that the solution of (E) exists on the interval $I = [-1, 1]$.

The Cauchy problem (E) is called *regular* if it has just one solution on I , and *singular* if it has at least two distinct solutions on I .

Denote by \mathcal{F} the family of continuous functions $\{f(t, x)\}$ having the property (1.28). Define the norm

$$\|f\| = \sup_{(t,x) \in S} |f(t, x)|.$$

It is clear that \mathcal{F} is a linear functional space with the norm $\|\cdot\|$.

Consider the partition

$$\mathcal{F} = \mathcal{F}_U \cup \mathcal{F}_P,$$

where \mathcal{F}_U and \mathcal{F}_P are defined, respectively, by

- 1) $f \in \mathcal{F}_U$ if and only if (E) is regular;
- 2) $f \in \mathcal{F}_P$ if and only if (E) is singular.

If (E) is singular, then the Peano phenomenon happens to the Cauchy problem (E) . In this case, the solution of (E) is not predicable, and there is the *Peano broom*

$$\Omega(E) = \{(t, x) \in S : x = \phi(t) \ (t \in I) \text{ is a solution of } (E)\}$$

at the initial point \mathfrak{o} which describes the chaotic behavior of the Peano phenomenon.

There arises a natural problem:

How often does Peano phenomenon happen to (E) ? or What is the probability of Peano phenomena happening to (E) ?

In what follows, we will prove the following result, which estimates, in certain sense, the “lower bound” of the probability of Peano phenomena happening to differential equations.

Theorem 1.8 \mathcal{F}_P is dense in \mathcal{F} .

Proof. Let $f_0 \in \mathcal{F}$. Given $\varepsilon > 0$, using the approximation theorem, we have a C^∞ -differential function \hat{f} in \mathcal{F} , such that

$$\|\hat{f} - f_0\| < \frac{\varepsilon}{2}. \quad (1.29)$$

It is clear that the Cauchy problem

$$(\hat{E}) : \quad \frac{dx}{dt} = \hat{f}(t, x), \quad x(0) = \xi,$$

has a unique solution $x = \Phi(t, \xi)$. It is obvious that $\Phi(t, \xi)$ is C^∞ -differentiable in $(t, \xi) \in S$ and satisfies

$$\Phi(t, \xi) \equiv \xi, \quad \text{when } |\xi| \gg 1. \quad (1.30)$$

Denote by Γ_ξ the integral curve of (\hat{E}) ; that is,

$$\Gamma_\xi = \{ (t, x) \in S : x = \Phi(t, \xi), t \in I \}.$$

Consider the integral box

$$\mathcal{Y} = \{ \Gamma_\xi : \xi \in \mathbb{R}^n \},$$

which agrees with the strip S as a point set.

Then we have a topological mapping

$$\mathcal{T}_0 : (t, \xi) \mapsto (t, \Phi(t, \xi))$$

from S onto \mathcal{Y} , such that

$$\mathcal{T}_0(L_\xi) = \Gamma_\xi,$$

where

$$L_\xi = \{ (t, x) \in B : x = \xi, t \in I \}$$

is a horizontal straight line-segment in S .

It follows from the property of the mapping \mathcal{T}_0 that the family of integral curves $\{\Gamma_\xi\}$ in \mathcal{Y} is topologically equivalent to the family of parallel line-segments L_ξ in S .

Furthermore, \mathcal{T}_0 has a continuous tangent map

$$\mathcal{T}'_0 : S \times (\mathbb{R} \times \mathbb{R}^n) \rightarrow \mathcal{Y} \times (\mathbb{R} \times \mathbb{R}^n).$$

If the roles of the regions S and \mathcal{Y} are ignored for brevity, \mathcal{T}'_0 is a linear map

$$\mathcal{T}'_0 : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R}^n,$$

defined by

$$\mathcal{T}'_0 = \begin{pmatrix} 1 & 0_n \\ \hat{f}(t, \Phi(t, \xi)) & \Phi'_\xi(t, \xi) \end{pmatrix}$$

where 0_n denotes the n -row null vector, and $\Phi'_\xi(t, \xi)$ is the Jacobian matrix. It follows from (1.30) that $\Phi'_\xi(t, \xi)$ is the unit matrix for $(t, \xi) \in S$ when $|\xi| \gg 1$, and is thus bounded in S . Let

$$\|\Phi'_\xi\| < B_0, \quad (1.31)$$

where B_0 is a positive constant. It follows that \mathcal{T}'_0 is a linear continuous, invertible and bounded map, such that

$$\mathcal{T}'_0 \begin{pmatrix} 1 \\ \hat{0} \end{pmatrix} = \begin{pmatrix} 1 \\ \hat{f} \end{pmatrix} \quad \text{or} \quad (\mathcal{T}'_0)^{-1} \begin{pmatrix} 1 \\ \hat{f} \end{pmatrix} = \begin{pmatrix} 1 \\ \hat{0} \end{pmatrix},$$

where $\hat{0}$ is the n -column null vector.

Finally, let $\Lambda(u)$ be a continuous function in $u \in \mathbb{R}$, satisfying

$$\Lambda(u) = \begin{cases} \sigma\sqrt{|u|}, & |u| \leq 1; \\ 0, & |u| > 2, \end{cases}$$

with some small parameter $\sigma > 0$. It is clear that the Cauchy problem

$$\frac{du}{dt} = \Lambda(u), \quad u(0) = 0 \quad (u \in \mathbb{R})$$

is singular.

Then consider correspondingly the Cauchy problem

$$(E^*) : \quad \frac{dz}{dt} = g(z), \quad z(0) = 0 \quad ((t, z) \in S),$$

where

$$g(z) = \begin{pmatrix} \Lambda(z_1) \\ \vdots \\ \Lambda(z_n) \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n.$$

Take a sufficiently small constant $\sigma > 0$, such that

$$\|g\| < \frac{\varepsilon}{2B_0}. \quad (1.32)$$

It can be seen that (E^*) has a Peano broom $\Omega(E^*)$ in S . Let

$$\begin{pmatrix} 1 \\ \check{f} \end{pmatrix} = \mathcal{T}'_0 \begin{pmatrix} 1 \\ g \end{pmatrix}.$$

Then the Cauchy problem

$$(\check{E}) : \quad \frac{dx}{dt} = \check{f}(t, x), \quad x(0) = 0,$$

has the Peano broom $\Omega(\check{E}) = \mathcal{T}_0(\Omega(E^*))$. It implies

$$\check{f} \in \mathcal{F}_P. \quad (1.33)$$

On the other hand, we have

$$\begin{pmatrix} 1 \\ \check{f} \end{pmatrix} - \begin{pmatrix} 1 \\ \hat{f} \end{pmatrix} = \mathcal{T}'_0 \begin{pmatrix} 1 \\ g \end{pmatrix} - \mathcal{T}'_0 \begin{pmatrix} 1 \\ \hat{0} \end{pmatrix} = \mathcal{T}'_0 \begin{pmatrix} 0 \\ g \end{pmatrix},$$

which yields

$$\check{f} - \hat{f} = \Phi'_\xi g.$$

It follows from (1.31) and (1.32) that

$$\|\check{f} - \hat{f}\| = \|\Phi'_\xi g\| \leq B_0 \|g\| < \frac{1}{2}\varepsilon.$$

Finally, using (1.29), we have

$$\|\check{f} - f_0\| < \varepsilon,$$

which together with (1.33) implies that \mathcal{F}_P is dense in \mathcal{F} .

The proof of Theorem 1.8 is thus completed. \square

Theorem 1.8 shows that the Peano phenomena are dense among the differential equations of order n for any integer $n \geq 1$.

1.4.2 Scarcity Theorem on Planar Peano Phenomena

1) Now, consider a planar Cauchy problem

$$(E_0) : \quad \frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,$$

where $f(t, x)$ is a continuous and bounded function defined on the planar strip

$$S_2 = \{(t, x) \in \mathbb{R}^2 : |t| \leq 1, x \in \mathbb{R}^1\}.$$

Denote by \mathfrak{F} the family of continuous and bounded functions on S_2 , and let

$$\|f\| = \sup_{(t,x) \in S_2} |f(t,x)|,$$

be the norm of a function $f \in \mathfrak{F}$. It can be verified that \mathfrak{F} is a (complete) Banach space.

It is noted that for $f \in \mathfrak{F}$, each solution of the differential equation $dx/dt = f(t,x)$ exists on the interval $I = [-1, 1]$.

Assume the solution of (E_0) is not unique. Then (E_0) has a Peano broom $\Omega(E_0)$ across the initial point (t_0, x_0) .

On the other hand, when the solution of (E_0) is not unique, there exist a unique maximal solution $\Phi(t)$ of (E_0) and a unique minimal solution $\Psi(t)$ of (E_0) on the interval I . Let

$$\Omega^* = \{(t,x) \in S : \Psi(t) \leq x \leq \Phi(t), t \in I\}.$$

Lemma 1.3 *If the solution of (E_0) is not unique, then the Peano broom*

$$\Omega(E_0) = \Omega^*.$$

Proof. Let

$$\Gamma : \quad x = \phi(t) \quad (t \in I),$$

be an integral curve of (E_0) . It follows from

$$\Psi(t) \leq \phi(t) \leq \Phi(t) \quad (t \in I)$$

that $\Gamma \subset \Omega^*$.

On the other hand, given a point

$$(\tau, \xi) \in \Omega^*,$$

the differential equation $dx/dt = f(t,x)$ has at least a solution $x = u(t)$ passing through the initial point (τ, ξ) (i.e., $u(\tau) = \xi$). Since $f(t,x)$ is bounded on S , $x = u(t)$ exists on the interval I .

If $x = u(t)$ is bounded by the minimal solution $\Psi(t)$ and the maximal solution $\Phi(t)$ on the interval I , then we have

$$\Psi(t) \leq u(t) \leq \Phi(t), \quad t \in I,$$

which implies $\Psi(t_0) = u(t_0) = \Phi(t_0) = x_0$. It follows that $x = u(t)$ is a solution of (E_0) , which passes through the point (τ, ξ) .

If $x = u(t)$ is not bounded by $\Psi(t)$ and $\Phi(t)$, then there exist a constant $\alpha \in [-1, \tau)$ and a constant $\beta \in (\tau, 1]$, such that

$$\Psi(t) \leq u(t) \leq \Phi(t), \quad t \in (\alpha, \beta),$$

satisfying

$$u(\alpha) = \Phi(\alpha) \text{ or } \Psi(\alpha) \quad \text{and} \quad u(\beta) = \Phi(\beta) \text{ or } \Psi(\beta).$$

For definiteness, assume

$$u(\alpha) = \Phi(\alpha) \quad \text{and} \quad u(\beta) = \Psi(\beta).$$

Then, let

$$w(t) = \begin{cases} \Phi(t), & \text{as } t \in [-1, \alpha], \\ u(t), & \text{as } t \in (\alpha, \beta), \\ \Psi(t), & \text{as } t \in [\beta, 1]. \end{cases}$$

It can be easily seen that $x = w(t)$ is a solution of (E_0) , which passes through the point (τ, ξ) .

Therefore, we have proved that the set Ω^* is covered by the integral curves of (E_0) .

Lemma 1.3 is thus proved. \square

Corollary 1.2 *Since $\Phi(t) \not\equiv \Psi(t)$ on I , the planar Peano broom $\Omega(E_0)$ contains at least an interior point (and thus at least a rational interior point $p^* = (t^*, x^*)$, where t^* and x^* are rational numbers).*

Remark 1.1 *It can be seen from the Example 1.3 that a spatial Peano broom may contains no spatial interior rational point.*

2) Now, let

$$\mathfrak{F} = \mathfrak{F}_U \cup \mathfrak{F}_P$$

be the partition defined by the planar Cauchy problem (E_0) in a similar manner as the partition of the space \mathcal{F} mentioned above.

It is known that the measure theory holds in a complete Banach space. Therefore, there is a normal measure μ in \mathfrak{F} , such that $\mu[\mathfrak{F}] = 1$ and $\mu[\mathcal{O}] > 0$ for any open set $\mathcal{O} \subset \mathfrak{F}$.

The following theorem asserts the scarcity of Peano phenomena among the planar differential equations (see [102]).

Theorem 1.9 *There is a normal measure ν in \mathfrak{F} , such that $\nu(\mathfrak{F}_P) = 0$.*

Proof. For brevity, we will give a sketch of the proof. Let

$$\mathfrak{F}_0 = \{g \in \mathfrak{F} : g(0, 0) = 0\}.$$

Then \mathfrak{F}_0 is a linear subspace of \mathfrak{F} . It follows that, for given $f \in \mathfrak{F}$, we have

$$f(t, x) = f_0(t, x) + \lambda,$$

where $f_0 \in \mathfrak{F}_0$ and $\lambda = f(0, 0) \in \mathbb{R}^1$. Hence,

$$\mathfrak{F} = \mathfrak{F}_0 \times \mathbb{R}^1. \quad (1.34)$$

Since \mathfrak{F}_0 is a closed subset in \mathfrak{F} , it is measurable.

On one hand, for any $g \in \mathfrak{F}_0$, we have

$$\{g\} \times \mathbb{R} = (\{g\} \times \Lambda_U(g)) \cup (\{g\} \times \Lambda_P(g)) \quad (1.35)$$

where the sets $\Lambda_U(g)$ and $\Lambda_P(g)$ are defined as follows:

$$\Lambda_U(g) = \{\lambda \in \mathbb{R} : g(t, x) + \lambda \in \mathfrak{F}_U\};$$

$$\Lambda_P(g) = \{\lambda \in \mathbb{R} : g(t, x) + \lambda \in \mathfrak{F}_P\}.$$

On other hand, for any positive integer n , let

$$\mathfrak{F}_P^{(n)} = \left\{ f \in \mathfrak{F}_P : \max_{t \in I} |\Phi(t) - \Psi(t)| \geq \frac{1}{n} \right\}.$$

It is clear that $\mathfrak{F}_P^{(n)}$ is a closed set and is thus measurable. Since

$$\mathfrak{F}_P = \bigcup_{n \geq 1} \mathfrak{F}_P^{(n)},$$

\mathfrak{F}_P is also a measurable set.

Let μ_0 be the restricted measure of μ in the subspace \mathfrak{F}_0 , and let σ be the Lebesgue measure in \mathbb{R}^1 . Then, $\nu(\cdot) = \mu_0(\cdot) \times \sigma(\cdot)$ is a well-defined measure in the product space $\mathfrak{F}_0 \times \mathbb{R}^1 (= \mathfrak{F})$.

Besides, the initial-value problem (E_0) can be put in the form:

$$(E)_\lambda : \quad \frac{dx}{dt} = f^*(t, x) + \lambda, \quad x(t_0) = x_0,$$

where $f^* \in \mathfrak{F}_0$.

Lemma 1.4 *If $\lambda_1 \neq \lambda_2$, the Peano broom $\Omega((E)_{\lambda_1})$ does not intersect the Peano broom $\Omega((E)_{\lambda_2})$ except at the initial point (t_0, x_0) .*

Proof. We claim that the solution of $(E)_{\lambda_1}$ does not intersect the solution of $(E)_{\lambda_2}$ except at the initial point (t_0, x_0) .

For definiteness, let

$$\lambda_1 < \lambda_2, \quad (1.36)$$

and let $x = x_1(t)$ and $x = x_2(t)$ be the solutions of $(E)_{\lambda_1}$ and $(E)_{\lambda_2}$, respectively. It is noticed that

$$x_1(t_0) = x_2(t_0) = x_0.$$

It follows that there is a constant $\alpha > 0$, such that

$$x_2(t) > x_1(t), \quad t_0 < t \leq t_0 + \alpha.$$

It suffices to prove that $t_0 + \alpha \geq 1$.

Otherwise, we have $t_1 = t_0 + \alpha < 1$ ($t_1 > t_0 \geq -1$), such that

$$x_2(t) > x_1(t) \quad (t_0 < t < t_1) \quad \text{and} \quad x_2(t_1) = x_1(t_1),$$

which implies $x'_1(t_1) \geq x'_2(t_1)$. That is,

$$f^*(t_1, x_1(t_1)) + \lambda_1 \geq f^*(t_1, x_2(t_1)) + \lambda_2.$$

It follows from $f^*(t_1, x_1(t_1)) = f^*(t_1, x_2(t_1))$ that

$$\lambda_1 \geq \lambda_2,$$

which contradicts the assumption (1.36).

Therefore, we have

$$x_2(t) > x_1(t) \quad (t_0 < t \leq 1).$$

Similarly, we can prove

$$x_2(t) < x_1(t) \quad (-1 \leq t < t_0).$$

The proof of Lemma 1.4 is thus completed. \square

Now, for given $f \in \mathfrak{F}_P$, we have $f = g + \lambda$ with $\lambda \in A_P(g)$. Using Corollary 1.2, we choose a fixed rational point p_λ^* in the interior of $\Omega((E)_\lambda)$. It follows from Lemma 1.4 that the Peano brooms $\{\Omega((E)_\lambda)\}$ corresponds, one by one, to the rational points $\{p_\lambda^*\}$. Hence, $A_P(g)$ is a countable set. It follows that $\sigma(A_P(g)) = 0$.

Finally, using Fubini theorem with (1.34) and (1.35) in mind, we get

$$\nu(\mathfrak{F}_P) = \int_{\mathfrak{F}_0} \int_{\Lambda_P(g)} d\lambda d\mu_0 = \int_{\mathfrak{F}_0} \sigma(\Lambda_P(g)) d\mu_0 = \int_{\mathfrak{F}_0} 0 d\mu_0 = 0,$$

which proves Theorem 1.9. \square

Theorem 1.9 states that Peano phenomena are scarce among planar differential equations (i.e., differential equations of first order). Indeed, in this sense, it estimates the “upper bound” of the probability of planar Peano phenomena happening to the Cauchy problem (E_2) for $f \in \mathfrak{F}$.

However, we do not know whether or not the Peano phenomena are scarce among differential equations of higher-order too. Anyway, it can be seen from the proof of Theorem 1.9 and the Remark 1.1 that it is not trivial to prove the scarcity of Peano phenomena in higher dimensional space.

1.5 Convergence Theorem on Difference Methods

1.5.1 Classical Difference Methods

Consider the Cauchy problem

$$(E) : \quad \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

where $f(x, y)$ is a continuous n -vector valued function of (x, y) in a closed rectangular region $Q \subset \mathbb{R}^1 \times \mathbb{R}^n$. On the one hand, it is well-known in the theory that the Cauchy problem (E) has at least a solution $y = \varphi(x)$ on the interval $I = [x_0 - h, x_0 + h]$. On the other hand, it is a problem in the application to compute the solution. In literature, there is a large number of approximation methods in numerically solutions of differential equations (see [70] for example). The difference methods are widely used in numerical analysis. Since our treatment is admittedly elementary, we are only concerned with the convergence problem of difference methods.

Suppose that the integral curve

$$\Gamma : \quad y = \varphi(x), \quad x_0 - h \leq x \leq x_0 + h,$$

is interpolated by a finite number of points

$$S_m : \quad (x_i, y_i), \quad i = 0, \pm 1, \dots, \pm m,$$

such that $y_i = y(x_i)$, where

$$x_0 - h = x_{-m} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_m = x_0 + h.$$

Let

$$\delta_m = \max_{0 \leq i \leq m-1} (x_{i+1} - x_i).$$

Assume S_m converges⁶ to Γ_0 as $\delta_m \rightarrow 0$.

The main idea of difference methods is to find out such a series S_m through an approximation

$$\frac{\Delta y_i}{\Delta x_i} = F(x_i, y_i, y_{i+1}, \delta), \quad 0 \leq i \leq m-1, \quad (1.37)$$

of the Cauchy problem (E), where

$$\frac{\Delta y_i}{\Delta x_i} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \approx \frac{dy}{dx}$$

and

$$F(x_i, y_i, \Delta x_i) \approx f(x, y). \quad (1.38)$$

The *difference scheme* is given by the approximation (1.38), which is usually determined by the first few terms of the Taylor expansion

$$\begin{aligned} F(x_i, y_i, \Delta x_i) &= f(x_i + \Delta x_i, y_i + \Delta y_i) \\ &= f(x_i, y_i) + [f'_x(x_i, y_i)\Delta x_i + f'_y(x_i, y_i)\Delta y_i] \\ &\quad + \frac{1}{2!}[f''_{x_i x_i}(x_i, y_i)\Delta^2 x_i + 2f''_{x_i y_i}\Delta x_i \Delta y_i + f''_{y_i y_i}(x_i, y_i)\Delta^2 y_i] + \cdots. \end{aligned}$$

Based on the knowledge of the convergent series from calculus, we know that the accuracy of our approximation improves with the number of terms retained. On the other hand, the description of the higher-order terms gets more and more complicated and the associated calculations more profuse. The basic idea of the Runge-Kutta methods is to preserve the order of a Taylor approximation (in the sense of the error involved) while eliminating the necessity of calculating the various partial derivatives of $f(x, y)$ that are involved. The alternative proposed by these methods involves evaluating

⁶In fact, the point sequence S_m is understood here as a polygonal approximation with vertices (x_i, y_i) ($i = 0, \pm 1, \dots, \pm m$).

the function $f(x, y)$ at certain judicious points rather than evaluating the specific partial derivatives.

Example 1.4 The Euler polygonal approximation is in fact a simple difference method given by

$$y_{i+1} = y_i + F(x_i, y_i, \Delta x_i) \Delta x_i, \quad (1.39)$$

where

$$F(x_i, y_i, \Delta x_i) = f(x_i, y_i).$$

Example 1.5 The Runge-Kutta approximation of order 3 is given by the following collection of formulas:

$$y_{i+1} = y_i + F(x_i, y_i, \Delta x_i) \Delta x_i, \quad (1.40)$$

where

$$F(x_i, y_i, \Delta x_i) = \frac{u_i + 4v_i + w_i}{6},$$

and

$$\begin{cases} u_i = f(x_i, y_i), \\ v_i = f(x_i + \frac{1}{2}\Delta x_i, y_i + \frac{1}{2}u_i), \\ w_i = f(x_i + \Delta x_i, y_i + 2v_i - u_i). \end{cases}$$

Example 1.6 The Runge-Kutta approximation of order 4 is given by the following collection of formulas:

$$y_{i+1} = y_i + F(x_i, y_i, \Delta x_i) \Delta x_i, \quad (1.41)$$

where

$$F(x_i, y_i, \Delta x_i) = \frac{a_i + 2b_i + 2c_i + d_i}{6},$$

and

$$\begin{cases} a_i = f(x_i, y_i), \\ b_i = f(x_i + \frac{1}{2}\Delta x_i, y_i + \frac{1}{2}a_i), \\ c_i = f(x_i + \frac{1}{2}\Delta x_i, y_i + \frac{1}{2}b_i), \\ d_i = f(x_i + \Delta x_i, y_i + c_i). \end{cases}$$

1.5.2 Generalized Difference Schemes

In 1971, W. Gear generalized the classical difference methods into a unified form as follows.

Let us approximate the solution of (E) by the difference equation:

$$\left\{ \begin{array}{l} y_{i+1} - y_i = (x_{i+1} - x_i)F(x_i, y_i, y_{i+1}, \Delta x_i) \quad (\Delta x_i \neq 0), \\ x_{i+1} = x_i + \Delta x_i, \quad i = 0, \pm 1, \pm 2, \dots, \pm(m-1), \\ \text{with } x_{\pm m} = x_0 \pm h, \end{array} \right. \quad (1.42)$$

where the function $F(x, y, z, \lambda)$ is continuous on the region

$$D^* : |x - x_0| \leq a, \quad |y - y_0| \leq b, \quad |z - z_0| \leq b, \quad |\lambda| \leq c$$

and satisfies the consistency condition:

$$F(x, y, y, 0) = f(x, y). \quad (1.43)$$

The scheme (1.42) satisfying the consistency condition 1.43 is called the *Gear's difference equation of (E)*, which unifies the classical difference schemes.⁷

In what follows, the consistency condition (1.43) will be tacitly assumed when we consider the scheme (1.42).

If the finite sequence

$$S_m : \left\{ \begin{array}{l} (x_i, y_i) \in Q, \quad i = 0, \pm 1, \dots, \pm(m-1), \\ \text{with } \Delta x_i = x_{i+1} - x_i \neq 0, \quad x_{\pm m} = x_0 \pm h, \end{array} \right.$$

satisfies (1.42), then S_m is called a solution of the difference equation (1.42).

The following result (due to Gear) is a general convergence theorem of difference methods.

Theorem 1.10 *If $F(x, y, z, \lambda)$ is a continuous function in the region D^* and satisfies the Lipschitz condition (with respect to y and z):*

$$|F(x, y_1, z_1, \lambda) - F(x, y_2, z_2, \lambda)| \leq L(|y_1 - y_2| + |z_1 - z_2|), \quad (1.44)$$

⁷Compared with the Runge-Kutta methods, the Gear's scheme is concerned with a more accurate approximation of $f(x, y)$ (for example, the difference computations of $f'_y(x, y)$).

then the solution of (1.42) converges to the solution of the initial-value problem (E) as the maximal steps

$$\delta_m = \max_{0 \leq |i| \leq |m-1|} |\Delta x_i| \rightarrow 0 \quad (\text{if } m \rightarrow \infty).$$

This theorem proves the convergence of the approximate solutions of Gear's difference equations. It can be seen that if $F(x, y, z, \lambda)$ satisfies the Lipschitz condition (1.44), then $f(x, y)$ satisfies the Lipschitz condition with respect to y . But, the converse is uncertain. Therefore, a natural question arises:

Does the solution of the Gear's differential equation (1.42) converge to the solution of (E) only if $f(x, y)$ is Lipschitzian with respect to y ?

The application of difference methods is really dependent on the answer of this question. The following theorem gives an affirmative answer to the question (see [28]).

Theorem 1.11 *If the solution $y = \Phi_0(x)$ of Cauchy problem (E) is unique, then the approximate solution of the difference equation (1.42) is convergent to $y = \Phi_0(x)$ as $\delta_m \rightarrow 0$.*

We will give the proof in the following several steps.

1.5.3 Preparatory Works

Take a constant

$$M \geq \max_{(x, y, z, \lambda) \in D^*} |F(x, y, z, \lambda)|.$$

Let

$$\alpha = \min \left(a, \frac{b}{M+1} \right), \quad \delta = \frac{\alpha}{m},$$

where m is a positive integer, satisfying $m > \alpha/c$.

Without destroying the generalization, we only consider the right-hand solution $[y_0, y_1, \dots, y_m]$ for the points $x_{i+1} = x_i + \delta$, $i = 0, 1, \dots, m-1$. The general case can be discussed in a similar manner.

Lemma 1.5 *The difference equation (1.42) has a right-hand solution $[y_0, y_1, \dots, y_m]$, satisfying*

$$(x_k, y_k) \in Q \quad (k = 0, 1, \dots, m). \quad (1.45)$$

Proof. It is clear that $(x_0, y_0) \in Q$. Now, assume $(x_j, y_j) \in Q$ for $j = 0, 1, \dots, k$, ($k < m$), and

$$|y_j - y_0| \leq \frac{j b}{m} \quad (j = 0, 1, \dots, k). \quad (1.46)$$

Then, we want to find the point y_{k+1} of (1.42) when $i = k$. It needs to prove the existence of the implicit solution $z = y_{k+1}$ of the equation

$$z = y_k + F(x_k, y_k, z, \delta)\delta.$$

For this aim, let

$$G(z) = y_k + F(x_k, y_k, z, \delta)\delta.$$

It is clear that $y = G(z)$ is continuous on the closed ball $B_b(y_0)$ (i.e., $|z - y_0| \leq b$), and satisfies the condition

$$\begin{aligned} |G(z) - y_0| &\leq |y_k - y_0| + \delta|F(x_k, y_k, z, \delta)| \\ &\leq \frac{k b}{m} + \frac{\alpha}{m} M \leq b. \end{aligned} \quad (1.47)$$

Therefore, $y = G(z)$ is a continuous mapping from the closed ball $B_b(y_0)$ into itself. Using the Brouwer fixed-point theorem (see the subsequent chapter of fixed point theorems), we assert that the mapping $y = G(z)$ has at least a fixed-point, denoted by y_{k+1} , in $B_b(y_0)$. It follows that

$$y_{k+1} = y_k + F(x_k, y_k, y_{k+1}, \delta)\delta, \quad |y_{k+1} - y_0| \leq b.$$

Then, it follows from the method of induction that (1.42) has a right-hand solution $[y_0, y_1, \dots, y_m]$, satisfying (1.45).

In a similar way, we can prove that (1.42) has a left-hand solution.

Lemma 1.5 is thus proved. \square

Now, for a right-hand solution $[y_0, y_1, \dots, y_k, y_{k+1}, \dots, y_m]$ of (1.42), construct an interpolation as follows. Let

$$\Delta y_k = y_k - y_{k-1}, \quad w_k = \frac{1}{\delta} \Delta y_k,$$

then

$$w_k = F(x_{k-1}, y_{k-1}, y_k, \delta), \quad (k = 1, 2, \dots, m);$$

in particular, let

$$w_0 = F(x_0, y_0, y_0, \delta).$$

Assume

$$\Delta w_{k+1} = w_{k+1} - w_k \quad (k = 0, 1, \dots, m-1).$$

Then, on the interval

$$J_k = [x_0 + k\delta, x_0 + (k+1)\delta],$$

construct the following function

$$g_k(x) = y_k + (x - x_k)[w_{k+1} - (x - x_{k+1})^2\delta^{-2}\Delta w_{k+1}]$$

($k = 0, 1, \dots, m-1$).

Lemma 1.6 For each k ($0 \leq k \leq m-1$), the function $g_k(x)$ has the following properties :

1. $g_k(x_k) = y_k, \quad g_k(x_{k+1}) = y_{k+1};$
2. $g'_k(x_k) = w_k, \quad g'_k(x_{k+1}) = w_{k+1};$
3. $|g'_k(x) - w_{k+1}| \leq |\Delta w_{k+1}|.$

Proof. Property 1 is obvious. It follows from the definition of $g_k(x)$ that

$$g'_k(x) = w_{k+1} - (x - x_{k+1})(3x - x_{k+1} - 2x_k)\delta^{-2}\Delta w_{k+1},$$

which implies

$$g'_k(x_k) = w_{k+1} - \Delta w_{k+1} = w_k, \quad g'_k(x_{k+1}) = w_{k+1}.$$

Property 2 is thus proved.

Finally, denoting by $(y)_i$ the i -th component of the vector y , we have

$$(g_k(x))_i = (y_k)_i + (x - x_k)[(w_{k+1})_i - (x - x_{k+1})^2\delta^{-2}(\Delta w_{k+1})_i].$$

Because the function

$$(g_k(x))''_i = -(6x - 4x_{k+1} - 2x_k)\delta^{-2}(\Delta w_{k+1})_i,$$

changes its sign once only in the interval J_k , the parabola

$$u = (g_k(x))'_i = (w_{k+1})_i - (\Delta w_{k+1})_i(x - x_{k+1})(3x - x_{k+1} - 2x_k)\delta^{-2}$$

assumes its maximal and minimal value only possibly at the end-points of the interval J_k or at the point of vertex

$$x = \hat{x} = \frac{2x_{k+1} + x_k}{3},$$

corresponding respectively to the values:

- (1) $(g'_k(x_k))_i = (w_k)_i = (w_{k+1})_i - (\Delta w_{k+1})_i$,
- (2) $(g'_k(x_{k+1}))_i = (w_{k+1})_i$,
- (3) $(g'_k(\hat{x}))_i = (w_{k+1})_i + \frac{1}{3}(\Delta w_{k+1})_i$.

It follows that $|(g'_k(x))_i - (w_{k+1})_i|$ has the maximal $|(\Delta w_{k+1})_i|$. We have thus

$$|g'_k(x) - w_{k+1}| \leq |\Delta w_{k+1}|,$$

which proves Property 3.

The proof of Lemma 1.6 is thus completed. \square

Then, on the interval $J = [x_0, x_0 + \alpha]$, define the Spline function $Z_m(x)$, such that $Z_m(x) = g_k(x)$ for $x \in J_k$, ($0 \leq k \leq m-1$).

Lemma 1.7 *The Spline function $Z_m(x)$ has the following properties:*

- (1) $Z_m(x) \in C^1(J)$;
- (2) $Z_m(x_k) = y_k$, $Z'_m(x_k) = w_k$, and

$$|Z'_m(x) - w_{k+1}| \leq |\Delta w_{k+1}| \quad (x \in J_k);$$

- (3) $(x, Z_m(x)) \in Q$, if $|\Delta w_{k+1}| \leq 1$.

Proof. Properties 1 and 2 are immediately derived from the definition of $Z_m(x)$ and Lemma 1.6.

Then, it follows from

$$|Z'_m(x) - w_{k+1}| \leq |\Delta w_{k+1}| \leq 1$$

that

$$|Z'_m(x)| \leq |w_{k+1}| + 1 \leq M + 1.$$

Using the Lagrange mean-value inequality for vector-function, we get

$$\begin{aligned} |Z_m(x) - y_0| &= |Z_m(x) - Z_m(x_0)| \\ &\leq \sup_{\xi \in J} |Z'_m(\xi)| \cdot |x - x_0| \leq (M + 1)\alpha \leq b, \end{aligned}$$

which implies the Property 3.

The proof of Lemma 1.7 is thus completed. \square

Now, consider the remainder

$$R_m(x) := Z'_m(x) - F(x, Z_m(x), Z_m(x + \delta), \delta),$$

for $x \in [x_0, x_0 + (m - 1)\delta]$; and

$$R_m(x) := R_m(x_0 + (m - 1)\delta),$$

for $x \in (x_0 + (m - 1)\delta, x_0 + \alpha]$.

Lemma 1.8 *The remainder $R_m(x)$ is continuous on the interval J , and converges uniformly to 0 as $m \rightarrow \infty$.*

Proof. Let $\varepsilon > 0$. Since the function F is continuous on a closed bounded region D , there is a constant $\sigma > 0$, such that if

$$|x - x^*| < \sigma, \quad |y - y^*| < \sigma, \quad |z - z^*| < \sigma,$$

the inequality

$$|F(x, y, z, \delta) - F(x^*, y^*, z^*, \delta)| < \frac{\varepsilon}{2}$$

is valid on D .

On the other hand, using (1.42), we conclude that if

$$m > N = \min \left\{ n \in \mathbb{Z} : n > \frac{(M + 1)\alpha}{\sigma} \right\},$$

then

$$|\Delta y_{k+1}| = |\delta F(x_k, y_k, y_{k+1}, \delta)| \leq \frac{\alpha}{m}(M + 1) < \sigma,$$

for $k = 1, \dots, m - 1$, and thus

$$\delta = \frac{\alpha}{m} \leq \frac{\alpha}{m}(M + 1) < \sigma.$$

It follows that if $m > N$, we have

$$|\Delta w_1| = |F(x_0, y_0, y_1, \delta) - F(x_0, y_0, y_0, \delta)| < \frac{\varepsilon}{2},$$

and

$$|\Delta w_{k+1}| = |F(x_k, y_k, y_{k+1}, \delta) - F(x_{k-1}, y_{k-1}, y_k, \delta)| < \frac{\varepsilon}{2},$$

where $k = 1, \dots, m - 1$.

Now, for any given $x \in J$, choose the maximal x_k , such that $x_k \leq x$. We get $|x - x_k| < \delta < \sigma$, and then

$$|Z_m(x) - y_k| = |Z_m(x) - Z_m(x_k)| \leq (M + 1)\delta < \sigma$$

and

$$|Z_m(x + \delta) - y_{k+1}| = |Z_m(x + \delta) - Z_m(x_k + \delta)| \leq (M + 1)\delta < \sigma.$$

It follows that if $m > N$ and $x_0 \leq x \leq x_0 + (m - 1)\delta$, we have

$$\begin{aligned} |R_m(x)| &= |Z'_m(x) - F(x, Z_n(x), Z_m(x + \delta), \delta)| \\ &\leq |Z'_m(x) - F(x_k, y_k, y_{k+1}, \delta)| \\ &\quad + |F(x_k, y_k, y_{k+1}, \delta) - F(x, Z_m(x), Z_m(x + \delta), \delta)| \\ &< |Z'_n(x) - w_{k+1}| + \frac{\varepsilon}{2} \leq |\Delta w_{k+1}| + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

On the other hand, when $x \in (x_0 + (m - 1)\delta, x_0 + \alpha]$, we have

$$|R_m(x)| = |R_m(x_0 + (m - 1)\delta)| < \varepsilon.$$

It follows that when $m > N$, the inequality $|R_m(x)| < \varepsilon$ holds on the interval J .

Lemma 1.8 is thus proved. \square

It is noticed that although $\{R_m(x)\}$ converges uniformly to 0, we can not conclude that $\{Z_m(x)\}$ converges uniformly.

1.5.4 Proof of Convergence

Now, we need to prove the uniform convergence of the sequence $\{Z_m(x)\}$. From Lemma 1.7, we conclude that

$$|Z_m(x)| \leq |y_0| + b, \quad x \in J.$$

It means that $\{Z_m(x)\}$ is uniformly bounded. Moreover, it follows from

$$|Z'_m(x)| \leq M + 1$$

that $Z_m(x)$ is equicontinuous. It follows from the Ascoli Lemma that there is a uniformly convergent subsequence $\{Z_{m_j}(x)\}$. Letting

$$\zeta(x) = \lim_{j \rightarrow \infty} Z_{m_j}(x),$$

we have

$$F(x, \zeta(x), \zeta(x), 0) = \lim_{j \rightarrow \infty} F\left(x, Z_{m_j}(x), Z_{m_j}\left(x + \frac{\alpha}{m_j}\right), \frac{\alpha}{m_j}\right).$$

It follows from the consistency condition that

$$F(x, \zeta(x), \zeta(x), 0) = f(x, \zeta(x)).$$

Then, using the definition of the remainder $R_m(x)$, we get

$$Z_{m_j}(x) = y_0 + \int_{x_0}^x \left[F\left(x, Z_{m_j}(x), Z_{m_j}\left(x + \frac{\alpha}{m_j}\right), \frac{\alpha}{m_j}\right) + R_{m_j}(x) \right] dx,$$

and letting $m_j \rightarrow \infty$ yields

$$\zeta(x) = y_0 + \int_{x_0}^x f(x, \zeta(x)) dx.$$

Therefore, $y = \zeta(x)$ is a solution of (E).

It is noticed that the Peano Existence Theorem (i.e., Theorem 1.1) can be proved by all the difference methods.

Now we are in a position to complete the proof of Theorem 1.11.

Because the solution of (E) is unique, we conclude that $y = \zeta(x)$ is equal to the unique solution $y = \Phi_0(x)$ of (E).

Claim: $y = Z_m(x)$ uniformly converges to $y = \Phi_0(x)$ of (E).

Assume the contrary. Then $\{Z_m(x)\}$ does not uniformly converge to $\Phi_0(x)$. It follows that there is a constant $\varepsilon_0 > 0$, such that for any positive integer s , there exists $x_s \in J$ satisfying

$$|Z_s(x_s) - \Phi_0(x_s)| \geq \varepsilon_0 > 0. \quad (1.48)$$

Without destroying the truth, assume $\{x_s\}$ is a convergent sequence. Let $x_s \rightarrow \hat{x}$. Then, similar to the above discussion, we can choose a uniformly convergent subsequence $\{Z_{s_j}(x)\}$ of the sequence $\{Z_s(x)\}$, which uniformly converges to a solution of (E). Since (E) has a unique solution $y = \Phi_0(x)$, we get

$$\lim_{j \rightarrow \infty} Z_{s_j}(x) = \Phi_0(x),$$

which is in conflict with (1.48).

We have thus proved Theorem 1.11. □

1.5.5 Necessary Condition of Convergence

Theorem 1.11 shows that the uniqueness of solutions of (E) is a sufficient condition for the convergence of Gear's difference approximation.

Now, we claim: *The uniqueness of solutions of (E) is also a necessary condition for the convergence of Gear's difference approximation.*

Proof. In fact, assume the contrary. Let $y = \Phi(x)$ and $y = \psi(x)$ be two different solutions of (E) . Then, we claim that there are Gear's difference equations admitting of divergent solutions.

Without loss of generality, we assume

$$\Phi(x) \neq \psi(x), \quad x \in J = [x_0, x_0 + \alpha].$$

Now, construct a polygon $y = \Phi_m(x)$ passing through vertex-points $(x_i, \Phi(x_i))$ ($i = 0, 1, \dots, m$). Similarly, construct a polygon $y = \psi_m(x)$ passing the vertex-points $(x_i, \psi(x_i))$ ($i = 0, 1, \dots, m$).

It is clear that the polygonal approximations $y = \phi_m(x)$ and $y = \psi_m(x)$ tend to the solutions $y = \phi(x)$ and $y = \psi(x)$ of (E) , respectively. Consider the positive constants

$$\sigma_m = \max_{x \in J} |\phi(x) - \phi_m(x)| + \frac{1}{m}$$

and

$$\tau_m = \max_{x \in J} |\psi(x) - \psi_m(x)| + \frac{1}{m}.$$

It follows that $\sigma_m \rightarrow 0$ and $\tau_m \rightarrow 0$ as $m \rightarrow \infty$.

Consider the tube-regions

$$T_{\sigma_m}(\phi) : |y - \phi(x)| \leq \sigma_m, \quad x \in J$$

and

$$T_{\tau_m}(\psi) : |y - \psi(x)| \leq \tau_m, \quad x \in J,$$

centered at the integral curves $y = \phi(x)$ and $y = \psi(x)$ ($x \in J$), respectively.

It can be seen that

$$\phi_m(x) \in T_{\sigma_m}(\phi) \quad \text{and} \quad \psi_m(x) \in T_{\tau_m}(\psi), \quad (x \in J).$$

For the positive integer m , construct a continuous vector-function $F_m(x, y)$ on Q , such that:

1) When m is odd, let

$$F_m(x, y) = \begin{cases} f(x, y), & \text{as } (x, y) \in Q \setminus T_{\sigma_m}(\phi); \\ \frac{\phi_m(x_{i+1}) - \phi_m(x_i)}{x_{i+1} - x_i}, & \text{as } (x, y) = (x_i, \phi(x_i)), \end{cases}$$

for $i = 0, 1, \dots, m-1$;

2) When m is even, let

$$F_m(x, y) = \begin{cases} f(x, y), & \text{as } (x, y) \in Q \setminus T_{\tau_m}(\psi); \\ \frac{\psi_m(x_{i+1}) - \psi_m(x_i)}{x_{i+1} - x_i}, & \text{as } (x, y) = (x_i, \psi(x_i)), \end{cases}$$

for $i = 0, 1, \dots, m-1$.

Now, construct a continuous vector-function $F(x, y, \lambda)$ on Q with a parameter λ ($|\lambda| \leq \alpha$) as follows:

For $\lambda \leq 0$, let $F(x, y, \lambda) = f(x, y)$. In particular, we have

$$F(x, y, 0) = f(x, y).$$

For $\lambda > 0$, there is a unique positive integer m , such that

$$\frac{\alpha}{m+1} < \lambda \leq \frac{\alpha}{m},$$

then define

$$F(x, y, \lambda) = \frac{m(m+1)}{\alpha} \left[\left(\frac{\alpha}{m} - \lambda \right) F_{m+1}(x, y) + \left(\lambda - \frac{\alpha}{m+1} \right) F_m(x, y) \right].$$

It is easy to verify that $F(x, y, \lambda)$ is continuous and satisfies

$$F\left(x, y, \frac{\alpha}{m}\right) = F_m(x, y).$$

Then, the initial-value problem (E) admits of a difference equation

$$y_{i+1} - y_i = \frac{\alpha}{m} F\left(x_i, y_i, \frac{\alpha}{m}\right), \quad (i = 0, 1, \dots, m-1), \quad (1.49)$$

which belongs to the type (1.42), and has a solution $[y_0, y_1, \dots, y_m]$ satisfying:

$$y_k = \begin{cases} \phi_m(x_k), & \text{as } m \text{ is odd;} \\ \psi_m(x_k), & \text{as } m \text{ is even,} \end{cases}$$

where $k = 0, 1, \dots, m - 1$. It can be seen that

$$(x_k, \phi(x_k)) \in T_{\sigma_m}(\phi), \quad (x_k, \psi(x_k)) \in T_{\tau_m}(\psi)$$

($k = 0, 1, \dots, m - 1$). Then, we have

- 1) If m is odd and tends to $+\infty$, then $(x_m, y_m) \rightarrow (x, \phi(x))$;
- 2) If m is even and tends to $+\infty$, then $(x_m, y_m) \rightarrow (x, \psi(x))$.

The condition $\phi(x) \neq \psi(x)$ implies that the approximate solutions $[y_0, y_1, \dots, y_m]$ do not converge.

We have thus proved that when the solution of (E) is not unique, there exists a Gear's difference equation, such that the approximate solutions do not converge.

In this sense, the uniqueness of solutions of (E) is also a necessary condition for the convergence of the Gear's approximate solutions. \square