Analyticity in Two- and Three-Body Coulomb Scattering in Momentum Space

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Two- and three-charged particle systems are investigated in a mathematically rigorous approach in momentum space. A new method is proposed. The traditional screened Coulomb plus renormalization method is criticized in comparison with the new method. The new method is not only mathematically rigorous but also useful for numerical calculations as was the screened Coulomb potential method.

1. Introduction

The Rutherford scattering cross section has a singularity at forward angles. Therefore, the partial wave expansion converges only as a distribution.[1] On the other hand, the Fourier transformation of the Coulomb potential $V^C(r) = ZZ'e^2/r$ can only be obtained with the damping function $V^C(r) = \lim_{R\to\infty} V^R(r)$ with $V^R(r) = V^C(r)\xi(r,R)$ in which the Heaviside function $\xi(r,R) = \theta(r-R)$ or the exponential function $\xi(r,R) = \exp[-(r/R)^m]$ are popular choices for $\xi(r,R)$. By using such a potential, one can define the Lippmann-Schwinger equation,

$$T_l^C(p,p';\lambda;z) = V_l^C(p,p';\lambda) + \int_0^\infty V_l^C(p,p'';\lambda)G_0(p'';z)T_l^C(p'',p';\lambda;z)dp'', \quad (1)$$

where the limit of $\lambda \to 0$ leads to the so-called overlapping singularity in which the Green's function pole coincides with the logarithmic singularity of potential. We couldn't avoid such a singularity by any methods which were explored. Hence, our status quo is as follows: 1) The two-body Coulomb scattering problem can not be solved in momentum space. 2) Therefore, the three-body Faddeev equation can not be treated for the Coulomb problem in momentum space.[2] 3) However, nuclear reactions can not avoid Coulomb problems. 4) There exist historical techniques for treating Coulomb problems, but they suffer from a disease. Our final goal is to obtain a mathematically rigorous two-and three-body off-shell Coulomb amplitude and Coulomb-plus-nuclear scattering amplitude in momentum space.

2. Conventional Phase Shift Renormalization Methods

The traditional screening method is obtained by putting $V^{(R)} = V^S + V^R$, where V^S is the nuclear potential and V^R is the screened Coulomb potential. Therefore two-potential theory leads to,

$$T_l^{(R)} = T_l^{sR} + T_l^R = \varpi_l^R t_l^{sR} \omega_l^R + T_l^R, \tag{2}$$

$$t_l^{sR} = V_l^S + V_l^S G^R t_l^{sR}, \tag{3}$$

$$T_l^R = V_l^R + V_l^R G_0 T_l^R, \tag{4}$$

where the on-shell screened Coulomb amplitude is sandwiched by the Møller operators, using an approximation of $\varpi_l^R = 1 + T_l^R G_0 \approx e^{i\delta_l^R(k)}$, and $\omega_l^R = 1 + G_0 T_l^R \approx e^{i\delta_l^R(k)}$. Furthermore, they renormalize it by sandwiching it with the Gorshkov factor $e^{i\phi(k)}$.[3,4] Finally, the Coulomb scattering amplitude is defined by renormalizing $f_l^{(C)}(k) = -(\nu/2\pi)\, T_l^{(C)}(k)$, and

$$f_l^{(C)}(k) \to \lim_{R\to\infty} e^{i\phi(k,R)} f_l^{(R)}(k) e^{i\phi(k,R)} \equiv \lim_{R\to\infty} \tilde{f}^{(R)}(k) \tag{5}$$

with the renormalization phase $\phi(k,R) = \eta(k)[\ln 2kR - \gamma]$. [5] However, the r.h.s. of eq.(5) becomes

$$\tilde{f}^{(R)}(k) = e^{2i\phi(k,R)} \left(\frac{\exp[2i\{\delta_l^{sR}(k) + \delta_l^R(k)\}] - 1}{2ik} \right) \tag{6}$$

$$= e^{2i\{\phi(k) + \delta_l^R(k)\}} \left(\frac{e^{2i\delta_l^{sR}(k)} - 1}{2ik} \right) + e^{2i\phi(k,R)} \left(\frac{e^{2i\delta_l^R(k)} - 1}{2ik} \right) \tag{7}$$

$$\approx e^{2i\sigma_l(k)} \left(\frac{e^{2i\delta_l^{sR}(k)} - 1}{2ik} \right) + \frac{e^{2i\sigma_l(k)} - e^{2i\phi(k,R)}}{2ik}, \tag{8}$$

where the Coulomb phase shift is approximated by $\sigma_l(k) \approx \delta_l^R(k) + \phi(k,R)$. However, it is found that the second term of the numerator will oscillate rapidly due to $\phi(k,R) \to \infty$ when $R \to \infty$. Therefore, the r.h.s. of eq.(5) never converges to the Coulomb amplitude. The traditional renormalization amplitude doesn't converge to the Coulomb amplitude, but they replace it to conventionally converged function. [6–8]

3. New Method

3.1. *Two-body Coulomb amplitude*

Let us separate the Coulomb potential into a short range screened Coulomb potential (SCP) V^R and the auxiliary potential (AP) V^ϕ,

$$V^C = V^R + (V^C - V^R) = V^R + V^\phi. \tag{9}$$

We obtain the *LS* equation in terms of V^ϕ,

$$T^\phi = V^\phi + V^\phi G_0 T^\phi \equiv V^\phi \omega^\phi \equiv \varpi^\phi V^\phi, \tag{10}$$

with $\omega^\phi \equiv 1 + G_0 T^\phi$, and $\overline{\omega}^\phi \equiv 1 + T^\phi G_0$, respectively. If we could solve the auxiliary equation (10), then the screened Coulomb t-matrix $t^{R\phi}$ would be given by two-potential theory,

$$t^{R\phi} = V^R + V^R G^\phi t^{R\phi} \equiv V^R \omega^R \equiv \overline{\omega}^R V^R, \tag{11}$$

with $\omega^R \equiv 1 + G^\phi t^{R\phi}$, and $\overline{\omega}^R \equiv 1 + t^{R\phi} G^\phi$, and the resolvent

$$G^\phi = \frac{1}{z - H_0 - V^\phi} = G_0 + G_0 T^\phi G_0. \tag{12}$$

Therefore, the full off-shell Coulomb t-matrix would be defined by

$$T^C \Leftarrow T^{R\phi} + T^\phi = \overline{\omega}^\phi t^{R\phi} \omega^\phi + T^\phi. \tag{13}$$

We would like to solve eq.(10); however, the AP contains the Coulomb potential. Then, we have still the long range problem in the equation. The author (S.O.) has found a new theorem:
1) If $T^\phi(k, k; z) = 0$ exists, then $T^\phi(p, k; z) = T^\phi(k, p'; z) = 0$ will be satisfied.
2) If theorem (1) is valid, then the off-shell $T^\phi(p, p'; z)$ can be obtained by the K-matrix equation.
Consequently, it is found that the kernel of the K-matrix equation has no overlapping singularity.[9,10] Therefore, once we find the range $R = R_{cl}(k)$, we can obtain the off-shell $T^\phi(p, p'; z)$. The AP on-shell t-matrix vanishes in two variables, k and R, for a fixed l. Therefore, one can solve the off-shell K-matrix for all energies for different ranges. The next problem is how to obtain the proper range $R_{cl}(k)$. One method is to obtain it from the differential equation, which was shown in Ref [9], and another method is to solve (10) directly. Finally, it is found that the range $R_{c0}(k)$ is approximately given by $120/\sqrt{z}$ for the proton-proton scattering.

3.2. *Two-body Nuclear amplitudes*

Let us consider a two-body nuclear charged particle system. The potential is given by

$$V^{(C)} = V^S + V^C = (V^S + V^R) + (V^C - V^R) = V^{(R)} + V^\phi \tag{14}$$

where V^S is a short range nuclear potential. By analogy with eq.(10) and eq.(11), the t-matrix is given as,

$$T^{(R)} = V^{(R)} + V^{(R)} G^\phi T^{(R)} = \overline{\omega}^R t^{sR} \omega^R + t^{R\phi}, \tag{15}$$

$$t^{sR} = V^S + V^S G^{R\phi} t^{sR}, \tag{16}$$

where $t^{R\phi}$ is given by (11). Then we easily deduce $T^{(C)}$ by using eqs.(10), (11), (13), and (15)

$$\begin{aligned} T^{(C)} &= \overline{\omega}^\phi T^{(R)} \omega^\phi + T^\phi = \overline{\omega}^\phi (\overline{\omega}^R t^{sR} \omega^R + t^{R\phi}) \omega^\phi + T^\phi \\ &= \overline{\omega}^\phi \overline{\omega}^R t^{sR} \omega^R \omega^\phi + \overline{\omega}^\phi t^{R\phi} \omega^\phi + T^\phi = \overline{\omega}^C t^{sR} \omega^C + T^C, \end{aligned} \tag{17}$$

where it is proved that the Coulomb Møller operators are given by $\omega^C = \omega^R\omega^\phi = (1+G^\phi t^{R\phi})(1+G_0T^\phi) = (1+G_0T^C)$ and $\overline{\omega}^C = \overline{\omega}^\phi\overline{\omega}^R = (1+T^\phi G_0)(1+t^{R\phi}G^\phi) = (1+T^C G_0)$. Furthermore, we can prove that $t^{sR} = t^{sC}$ by using the resolvent; that is,

$$G^{R\phi} = \frac{1}{z - H_0 - V^R - V^\phi} = \frac{1}{z - H_0 - V^C} = G_0 + G_0T^C G_0 \equiv G^C. \quad (18)$$

Here it should be stressed that eq.(18) contains an "important proof" of the equivalency of the screened Coulomb with the pure Coulomb Green's function; *i.e.* $G^{R\phi} = G^C$ at a "finite given range R". Thus, if we miss V^ϕ, then we have to take $R \to \infty$ to reach G^C. Therefore, all other calculations should be done with an infinite range. If the results converge at a finite range, such as in traditional methods, their results are inconsistent [6–8]. Then eq.(16) becomes,

$$\begin{aligned} t^{sR} &= V^S + V^S G^C t^{sR} = V^S + V^S\{1 + G_0T^C\}G_0t^{sR} \\ &= V^S + \mathcal{K}^{SC}G_0t^{sR} \equiv t^{sC}. \end{aligned} \quad (19)$$

Therefore eq.(17) gives

$$T^{(C)} = \overline{\omega}^C t^{sC}\omega^C + T^C. \quad (20)$$

3.3. *Three-body Nuclear amplitudes*

Our "regulation" for the two-body Coulomb t-matrix in eq.(13) and (17) is obviously written for the three-body transition t-matrix $T^{(C)}$ for a nuclear system in which the full potential is given by

$$V^{(C)} = V^S + W^0 + V^C = (V^S + W^0 + V^R) + (V^C - V^R) = V^{(R)} + V^\phi, \quad (21)$$

where V^S, W^0, V^C are a nuclear force, a short range three-body force, and the Coulomb force, respectively. Here we introduce the three-body Jacobi-coordinate channels: α, β, γ or 1, 2, and 3. The two-body potentials V are given by V_α, V_β, and V_γ, while the three-body force W^0 could be denoted by $W^0_{\alpha\beta}$. Therefore, eq.(5) indicates

$$\begin{aligned} V^{(C)}_{\alpha\beta} &= V^S_\alpha\delta_{\alpha\beta} + W^0_{\alpha\beta} + V^C_\alpha\delta_{\alpha\beta} = (V^S_\alpha\delta_{\alpha\beta} + W^0_{\alpha\beta} + V^R_\alpha\delta_{\alpha\beta}) + (V^C_\alpha - V^R_\alpha)\delta_{\alpha\beta} \\ &= V^{(R)}_{\alpha\beta} + V^\phi_\alpha\delta_{\alpha\beta}. \end{aligned} \quad (22)$$

Hereafter we suppress indices for simplicity except when necessary. Then the formal equation for such a three-body t-matrix can be represented by,

$$T^{(C)} = V^{(C)} + V^{(C)}G_0T^{(C)} = (V^{(R)} + V^\phi) + (V^{(R)} + V^\phi)G_0T^{(C)}. \quad (23)$$

Therefore, the three-body t-matrix can also be decomposed using two-potential theory,

$$\begin{aligned} T^{(C)} &= \overline{\omega}^\phi T^{(R)}\omega^\phi + T^\phi = \overline{\omega}^\phi(\overline{\omega}^R T^{sR}\omega^R + T^R)\omega^\phi + T^\phi \\ &= \overline{\omega}^\phi(\overline{\omega}^R(\overline{\Omega}^0 T\Omega^0 + T^0)\omega^R + T^R)\omega^\phi + T^\phi \\ &= \overline{\omega}^C\overline{\Omega}^0 T\Omega^0\omega^C + \overline{\omega}^C T^0\omega^C + T^C, \end{aligned} \quad (24)$$

where it has an onion-like structure. These t-matrices are labeled by the three-body Jacobi channels α, β, γ.

Finally, the three-body channel representation of eqs.(24) , (24) with α, β, γ, becomes as follows [11,12]

$$T^{(C)} = \sum_{\alpha,\beta=1}^{3} \sum_{\eta,\zeta=1}^{3} \sum_{\gamma,\delta=1}^{3} (\overline{\omega}^{C}_{\alpha\eta}(\overline{\Omega}^{0}_{\eta\gamma} T_{\gamma\delta} \Omega^{0}_{\delta\zeta} + T^{0}_{\eta\zeta})\omega^{C}_{\zeta\beta} + T^{C}_{\alpha\beta}). \tag{25}$$

4. Conclusion

We have presented a new method to solve nuclear charged particle systems in momentum space. One does not solve the two-body Lippmann-Schwinger equation but treats it by the mathematically rigorous two-potential theory. First of all, we introduced the Auxiliary potential which is defined by the difference between the pure Coulomb potential and a screened Coulomb potential. We can not treat the Lippmann-Schwinger equation for the Coulomb potential but we can calculate the K-matrix equation for the Auxiliary potential and impose the condition that the on-and half on-shell amplitudes will vanish. By using such a full off-shell amplitude, we can calculate the core amplitude with a screened Coulomb potential in which the kernel of the Lippmann-Schwinger equation for the screened Coulomb potential is modified with the off-shell auxiliary amplitude. The phase shift introduced by the solution will be the Coulomb phase shift.

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