

# Integral curves for contact and Engel structures

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We study integral curves in 3-dimensional contact space and Engel space. Comparing the proofs of the classification of topologically trivial Legendrian knots in the standard contact 3-space and the classification of integral circles in the standard Engel space, we discuss the differences and similarities of them.

*Keywords:* Engel structure, Horizontal curve, Legendrian knot.

## 1. Introduction

Submanifolds in manifolds with some geometric structures are often interesting objects for geometry, for example, Legendrian submanifolds in contact manifolds and Lagrangian submanifolds in symplectic manifolds. In this article, we study integral submanifolds in contact 3-manifolds and Engel manifolds. We discuss some relations between topologically trivial Legendrian knots in tight contact 3-manifolds, that is studied in [1], and embedded integral circles in the standard Engel space, that is studied in [2].

Contact structures and Engel structures are interesting objects for differential topology. A *contact structure* is a distribution of corank 1 on an odd-dimensional manifold which is maximally non-integrable. An *Engel structure* is a distribution of rank 2 on a 4-dimensional manifold which is maximally non-integrable (see Section 2 for precise definition). It is known that all contact structures on manifolds of the same dimension are locally equivalent (Darboux). Engel structures also enjoy the same important property as contact structures. All Engel structures are locally equivalent (Engel). Then, for contact and Engel structures, global study is important. These days, there are a lot of break through on 3-dimensional contact topology ( [3], [4]). However, as far as the author knows, global studies of Engel manifolds are not so many ( [5], [6], . . . , etc). Recently, sufficient condition for the existence of an Engel structure is obtained by Vogel [7]:

There exists an Engel structure on a 4-dimensional manifold if the manifold is parallelizable. Then, Engel manifolds must be going to be studied as an object for global differential topology.

We study the following objects in this article. An embedded circle in a 3-dimensional contact manifold which is everywhere tangent to the contact structure is called a *Legendrian knot*. An embedded circle in the standard Engel space which is everywhere tangent to the Engel structure is called a *horizontal loop*. Horizontal loops might be one of key objects to Engel topology like Legendrian knots to contact topology. Legendrian knots take important roles in constructions and classifications of contact manifolds. Then, horizontal loops should be good tools for constructions and classifications of Engel structures. Engel structures and contact structures on 3-dimensional manifolds are so closely related that mutual contributions between Engel topology and 3-dimensional contact topology are expected.

Now, we state the classification results after some basic notions needed to the statements (see Subsection 2.1, 2.3 for precise definitions).

First, we introduce some notions concerning 3-dimensional contact topology. Contact structures on 3-manifolds are divided into two conflicting classes, *tight* and *overtwisted*. For an overtwisted contact structure, there exists a so called overtwisted disk which is an embedded disk tangent to the contact structure at any boundary point. Two Legendrian knots  $\gamma_0, \gamma_1 : S^1 \rightarrow (M, \xi)$  in a contact 3-manifold  $(M, \xi)$  are said to be *Legendrian isotopic* if they are connected by a path of Legendrian knots. For Legendrian knots, there are two classical invariants called *Thurston-Bennequin's invariant* and *rotation number*.

The classification of topologically trivial Legendrian knots in a tight contact 3-manifold is obtained by Eliashberg and Fraser in [1]:

**Theorem 1.1.** *Let  $\gamma_0, \gamma_1 : S^1 \rightarrow (M, \xi)$  be topologically trivial Legendrian knots in a tight contact 3-manifold  $(M, \xi)$ . These Legendrian knots  $\gamma_0$  and  $\gamma_1$  are Legendrian isotopic if and only if they have the same Thurston-Bennequin's invariant and rotation number:  $\text{tb}(\gamma_0) = \text{tb}(\gamma_1)$ ,  $r(\gamma_0) = r(\gamma_1)$ .*

According to Theorem 1.1, topologically trivial Legendrian knots in a tight contact manifold are classified as follows. The possible pair  $(r(\gamma), \text{tb}(\gamma))$  of a rotation number and Thurston-Bennequin's invariant for a Legendrian knot  $\gamma$  is restricted. It is known that for a topologically trivial Legendrian knot, the pair  $(r(\gamma), \text{tb}(\gamma))$  takes its value in  $\{m, -|m| - 2k - 1 \mid k \geq 0, m : \text{integers}\}$  (see [1]). Further, for each value in the set, an example of

topologically trivial Legendrian knot is constructed. An image of a Legendrian submanifold by projection along Legendre fibers is called a front. A front of a Legendrian knot is an immersed circle with cusps of type  $(2, 3)$  generically. It is known that such a curve is lifted to a Legendrian knot.

**Remark 1.2.** For any pair  $(m, -|m| - 2k - 1)$  of integers with  $k \geq 0$ , there exists a topologically trivial Legendrian knot  $\gamma$  whose rotation number  $r(\gamma)$  is  $m$  and Thurston-Bennequin's invariant  $tb(\gamma)$  is  $-|m| - 2k - 1$ . In fact, we have the following catalog  $L_{s,t}$  of fronts of Legendrian knots (see Figure 1). By lifting an  $L_{s,t}$ -type front vertically, we obtain a Legendrian knot  $\bar{L}_{s,t}$  with Thurston-Bennequin's invariant  $tb(\bar{L}_{s,t}) = -(2t+s+1)$ , and rotation number  $r(\bar{L}_{s,t}) = \pm s$ . Thus, we conclude that topologically trivial

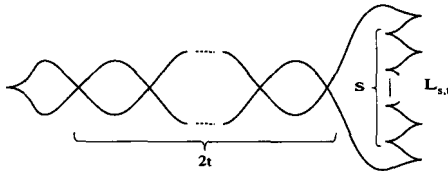


Fig. 1. Catalog of fronts

Legendrian knots in a tight contact 3-manifold are classified by Thurston-Bennequin's invariant and rotation number, and there corresponds a class to any pair  $(m, -|m| - 2k - 1)$  of integers with  $k \geq 0$ .

The standard Engel structure  $E$  on  $\mathbb{R}^4$  is defined as

$$E = \{dy - zdx = 0, dz - wdx = 0\},$$

where  $(x, y, z, w)$  are coordinates of  $\mathbb{R}^4$ . Two horizontal loops  $\gamma_0, \gamma_1: S^1 \rightarrow (\mathbb{R}^4, E)$  in the standard Engel space  $(\mathbb{R}^4, E)$  are said to be *horizontally homotopic* if they are connected by a path of horizontal loops. We classify horizontal loops in  $(\mathbb{R}^4, E)$  under this equivalence relation. An invariant under horizontal homotopy of horizontal loops is defined as follows. A *rotation number* of a horizontal loop  $\gamma: S^1 \rightarrow (\mathbb{R}^4, E)$  is defined as a degree of  $\gamma$  with respect to a trivialization of the Engel structure  $E$ . Let  $r(\gamma)$  denote it.

The following are proved in [2].

**Theorem 1.3.** Let  $\gamma_0, \gamma_1: S^1 \rightarrow (\mathbb{R}^4, E)$  be horizontal loops in the standard Engel space. These loops  $\gamma_0$  and  $\gamma_1$  are horizontally homotopic if and only if their rotation numbers coincide:  $r(\gamma_0) = r(\gamma_1)$ .

According to Theorem 1.3, we can classify horizontal loops in the following Remark 1.4. We claim that there actually is a horizontal loop with rotation number  $k$  for any integer  $k \in \mathbb{Z}$ . In order to describe horizontal loops, we use terms of fronts (see Subsection 2.2). The image of a horizontal loop in the standard Engel space by the projection  $\pi: (x, y, z, w) \mapsto (x, y)$  is called a *front* of the loop. Generically, it is a non-vertical immersed curve with cusps of type (2, 5). Conversely, we can lift such a plane curve to a horizontal loop in the standard Engel space.

**Remark 1.4.** For any integer  $n \in \mathbb{Z}$ , there is a horizontal loop  $\gamma$  whose rotation number is the given integer  $n$ :  $r(\gamma) = n$ . In fact, we have the following catalog  $F_k$  of fronts of horizontal loops (see Figure 2). By lifting an  $F_k$ -type front vertically, we obtain a horizontal loop  $\bar{F}_k$  with rotation number  $r(\bar{F}_k) = k$ . Thus, we conclude that horizontal loops in the standard

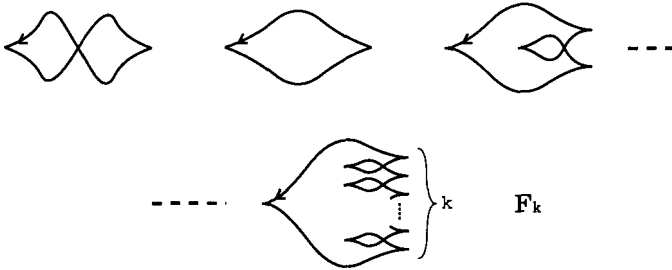


Fig. 2. Catalog of fronts

Engel space are classified by the rotation number, and that for any integer, there corresponds a class.

We can observe the results in this article from a view point of homotopy principle (h-principle for short). One of the most famous examples of h-principle is Whitney’s theorem on immersions of a circle into a plane ([8], [9]): regular homotopy classes of immersed circles are classified by their rotation numbers. The rotation number for horizontal curves is also the main tool for the classifications in this article, especially in Theorem 1.3. And, the rotation number for horizontal curves is defined by taking a certain projection to a plane. Then, by taking such a projection, we obtain a version of Whitney’s theorem with some conditions on some area (see Subsection 6.1). In other words, we find a new relation which satisfies h-principle. On the other hand, in order to show h-principle there is a strong

machinery, Gromov's method (see [10], [9]). A relation, homotopy among Legendrian immersions, similar to the relation in this article is studied by that method in [10]. The results in this article should also be proved by that method.

The result in this article can be generalized to higher dimensional cases. The notion of Engel structure is a special case of the notion of Goursat structure. The Goursat structure on a 3-manifold is a contact structure. However, the Darboux-type theorem, or local triviality, does not hold in higher dimensional cases. We extend Theorem 1.3 to the higher dimensional cases (see Subsection 6.2).

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## 2. Preliminaries

We introduce some basic miscellaneous notions in this section. First, we define some basic notions on contact structures. And then, we introduce some notions on Engel structures as extensions of those on contact structures.

### 2.1. Basic definitions

#### 2.1.1. Contact structures

A *contact structure* on a 3-manifold is a completely non-integrable tangent plane field. The complete non-integrability implies that this tangent plane field  $C$  is defined, at least locally, by a 1-form  $\alpha$  which satisfies the inequality  $\alpha \wedge d\alpha \neq 0$  everywhere. In terms of Lie brackets, the complete non-integrability is described as follows. The plane field  $C$  on a 3-manifold  $M$  can be regarded as a locally free sheaf of vector fields. Let  $[C, C]$  denote the sheaf of vector fields generated by all Lie brackets  $[X, Y]$  of vector fields  $X, Y$  which are cross-sections of  $C \subset TM$  as a bundle. Let  $\text{Ch}(C)$  be the *Cauchy characteristic subdistribution* of  $C$ , that is,  $\text{Ch}(C) := \{X \in C \mid [X, C] \subset C\}$ . A hyperplane field  $C$  is a *contact structure* if it satisfies  $C + [C, C] = TM$ ,  $\text{Ch}(C) = \{0\}$ .

The *standard contact structure* on  $\mathbb{R}^3$  is a hyperplane field  $C_0$  on  $\mathbb{R}^3$  defined by a 1-form  $\alpha_0 := dy - zdx$  as its kernel, where  $x, y, z$  are coordinates of  $\mathbb{R}^3$ . It is known that any contact structure on a 3-dimensional manifold is locally equivalent to this structure (Darboux).

A contact structure on a 3-manifold traces on an embedded surface a 1-dimensional foliation with singularities. Let  $F$  be an embedded surface

in a contact 3-manifold  $(M, C)$ . The contact structure  $C$  traces on  $F$  a 1-dimensional singular foliation. Such a foliation is called the *characteristic foliation* on  $F$  with respect to  $C$ , and  $F_C$  denotes it. When  $C = \{\alpha = 0\}$  and  $F$  are oriented,  $F_C$  is oriented by the vector field  $X$  on  $F$  which satisfies  $X \lrcorner \text{vol}_F = i^* \alpha$ , where  $\text{vol}_F$  is a volume form of  $F$ , and  $i: F \hookrightarrow M$  is the inclusion mapping. Generically, the characteristic foliation  $F_C$  has a finite number of singular points where  $F$  is tangent to  $C$ . A singular point is called *positive* or *negative* depending on whether the orientations of  $F$  and  $C$  coincide at the point or not. Generically, the index of the vector field which defines the characteristic foliation locally at a singular point is  $\pm 1$ . We call a singular point *elliptic* if its index is  $+1$ , and *hyperbolic* if it is  $-1$ . Because of the non-integrability of contact structures, the characteristic foliation  $F_C$  has topologically the focus type singularity at elliptic points, and  $F_C$  has the saddle type singularity at hyperbolic points. These singularities are called simple singularities.

Contact structures on 3-manifolds are classified into two conflicting classes. A contact structure  $C$  on  $M$  is said to be *overtwisted* if there exists an embedded disk  $\mathcal{D} \subset (M, C)$  the characteristic foliation  $\mathcal{D}_C$  on which has no limit cycle. A contact structure which is not overtwisted is called *tight*.

A *Legendrian knot* is an embedded circle in a 3-dimensional contact manifold which is everywhere tangent to the contact structure. Two Legendrian knots  $\gamma_0, \gamma_1: S^1 \rightarrow (M, C)$  are said to be *Legendrian isotopic* if there exists a smooth mapping  $H: S^1 \times I \rightarrow (M, C)$  which satisfies that, setting  $H_t(s) := H(s, t)$ ,  $H_0 = \gamma_0$  and  $H_1 = \gamma_1$ , and that  $H_t: S^1 \rightarrow (M, C)$  is a Legendrian knot for any  $t \in I = [0, 1]$ .

There are two classical invariant under Legendrian isotopy for Legendrian knots, Thurston-Bennequin's invariant and the rotation number. Let  $\gamma$  be an oriented Legendrian knot homologous to zero in a contact 3-manifold  $(M, C)$ , and  $F \subset M$  an embedded surface with boundary  $\partial F = \gamma$  which represents a relative homology class  $\beta \in H_2(M, \gamma)$ . Pushing  $\gamma$  slightly along a vector field transverse to  $C$ , we obtain a new knot  $\gamma'$ . The intersection number of  $\gamma'$  and  $F$  is called *Thurston-Bennequin's invariant* of  $\gamma$  with respect to  $\beta$ . Let  $\text{tb}(\gamma, \beta)$  denote it. Let  $X$  be a vector field tangent to  $\gamma$  which agrees with the orientation of  $\gamma$ . The degree of  $X$  with respect to a trivialization of the bundle  $C|_F$  depends only on  $\beta \in H_2(M, \gamma)$ . It is called the *rotation number* of  $\gamma$  with respect to  $\beta$ .  $r(\gamma, \beta)$  denotes it. When the choice of  $\beta \in H_2(M, \gamma)$  is clear, then we omit  $\beta$ . In particular, we study the rotation number of a Legendrian knot in the standard contact  $\mathbb{R}^3$  in Subsection 2.3. Studying Legendrian knots is one of important issues in

contact topology (see [11], [12], [1], [13] for example).

### 2.1.2. Engel structures

An Engel structure is a maximally non-integrable distribution of rank two on a 4-dimensional manifold. Generally, it is defined as follows. Let  $M$  be a 4-dimensional manifold, and  $D$  a distribution, or a subbundle of the tangent bundle  $TM$ , of rank 2. We can regard  $D$  as a locally free sheaf of vector fields on  $M$ . Let  $[D, D]$  denote the sheaf of vector fields generated by all Lie brackets  $[X, Y]$  of vector fields  $X, Y$  which are cross-sections of  $D$ . Set  $D^2 := D + [D, D]$ , and  $D^3 := D^2 + [D^2, D^2]$ . Then, an *Engel structure* on  $M$  is defined as a distribution  $D \subset TM$  of rank 2 which satisfies the following conditions:

$$\text{rank } D_p^2 = 3, \quad \text{rank } D_p^3 = 4 \quad (1)$$

at any point  $p \in M$ .

A certain Engel manifold is constructed from a 3-dimensional contact manifold. Let  $E$  be a contact structure on a 3-dimensional manifold  $N$ . By taking fibrewise projectivization of the contact structure  $E$ , we obtain a new 4-dimensional manifold  $\mathbb{P}E = \cup_{x \in N} \mathbb{P}(E_x)$ . On the 4-dimensional manifold  $\mathbb{P}E$ , an Engel structure  $D(E)$  is defined as  $D(E)_q := (d\pi)^{-1}l$ , where  $\pi: \mathbb{P}E \rightarrow N$  is the canonical projection,  $q = (p, l) \in \mathbb{P}E$  is a point, and  $l \subset T_p N$  is a line (see [5]). Such a procedure is called a *Cartan prolongation* (see [14], [5], [6]).

An Engel structure has a characteristic direction. Let  $D$  be an Engel structure on a 4-dimensional manifold  $M$ . From this Engel structure  $D$ , a line field is defined as follows:  $L(D) := \{X \in D \mid [X, D^2] \subset D^2\}$ . The line field  $L(D)$  is called the *Engel line field*. It is known that a contact structure is induced from an even-contact structure  $D^2$  on an embedded 3-dimensional manifold  $N \subset M$  which is transverse to the Engel line field  $L(D)$ . The contact structure is obtained as  $D^2 \cap TN$ . Such a procedure is called a *deprolongation* (see [5], [14]).

In this article, we work in the standard Engel space  $(\mathbb{R}^4, E)$ , that is, an ordinary 4-dimensional space  $\mathbb{R}^4$  endowed with the standard Engel structure. The *standard Engel structure* on  $\mathbb{R}^4$  is defined as a kernel of the following pair  $\omega_1, \omega_2$  of 1-forms:

$$\omega_1 = dy - zdx, \quad \omega_2 = dz - wdx, \quad (2)$$

where  $(x, y, z, w) \in \mathbb{R}^4$  are coordinates. Let  $E$  denote the standard Engel

structure on  $\mathbb{R}^4$ :

$$E := \{\omega_1 = 0, \omega_2 = 0\} = \text{Span} \left\{ \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}, \frac{\partial}{\partial w} \right\}.$$

We call the 4-dimensional space  $(\mathbb{R}^4, E)$  endowed with the standard Engel structure the *standard Engel space*. It is clear that the standard Engel structure  $E$  actually satisfies the condition (1) of the definition. In this case, the Engel line fields is  $L(E) = \text{Span} \{\partial/\partial w\}$ . With respect to the standard Engel structure on  $\mathbb{R}^4$ , the induced contact structure on  $\mathbb{R}^3 \subset \mathbb{R}^4$ , the  $(x, y, z)$ -space, is the standard contact structure  $C = \{\omega_1 = dy - zdx = 0\}$ .

A *horizontal curve*  $\Gamma \subset M$  in a Engel manifold  $(M, D)$  is a curve which is tangent to the Engel structure  $D$  everywhere:  $T_p\Gamma \subset D_p$  at any  $p \in \Gamma$ . Horizontal loops in the standard Engel space  $(\mathbb{R}^4, E)$  are dealt with in this article. A *horizontal loop* in  $(\mathbb{R}^4, E)$  is an embedding  $\gamma: S^1 \rightarrow (\mathbb{R}^4, E)$  of an oriented circle  $S^1$  into  $(\mathbb{R}^4, E)$  which satisfies the condition  $\gamma_*(T_p S^1) \subset (E)_{\gamma(p)}$ . In other words,  $\gamma$  satisfies the conditions  $\gamma^*\omega_1 = 0$  and  $\gamma^*\omega_2 = 0$ , where  $\omega_1, \omega_2$  are 1-forms defined as equations (2). For horizontal loops, we introduce an equivalence relation. Two horizontal loops  $\gamma_0, \gamma_1: S^1 \rightarrow (\mathbb{R}^4, E)$  are said to be *horizontally homotopic* if there exists a smooth mapping  $H: S^1 \times I \rightarrow (\mathbb{R}^4, E)$  which satisfies that, setting  $H_t(s) := H(s, t)$ ,  $H_0 = \gamma_0$  and  $H_1 = \gamma_1$ , and that  $H_t: S^1 \rightarrow (\mathbb{R}^4, E)$  is a horizontal loop for any  $t \in I = [0, 1]$ .

We introduce an invariant under horizontal homotopy for horizontal loops, in a similar way to the rotation number for Legendrian knots, in Subsection 2.3.

## 2.2. Vertical projections and fronts

We introduce the vertical projection of the standard Engel space to the standard contact 3-space, and the vertical projection of the standard contact 3-space to a 2-dimensional plane. The image of a Legendrian curve by the second vertical projection is called a *front* of the Legendrian curve. The image of a horizontal curve in the standard Engel space by the first vertical projection is a Legendrian immersion with singularities. The image of a horizontal curve in the standard Engel space by the double projections is also called a *front* of the horizontal curve. Fronts of Legendrian knots and those of horizontal loops are important tools to prove Theorem 1.1 and Theorem 1.3 respectively. Further, we introduce the vertical lift of a certain plane curve to a Legendrian curve and, further, to a horizontal curve in the standard Engel space.

The vertical projections are defined as follows. Let  $(x, y, z, w)$  be coordinates of the standard Engel space  $(\mathbb{R}^4, E)$  as above. Set  $Q_3 := \{w = 0\} \cong \mathbb{R}^3$ , and  $Q_2 := \{w = 0, z = 0\} \cong \mathbb{R}^2$ . Let  $\pi_1: \mathbb{R}^4 \rightarrow Q_3$ ,  $(x, y, z, w) \mapsto (x, y, z)$ , and  $\pi_2: Q_3 \rightarrow Q_2$ ,  $(x, y, z) \mapsto (x, y)$  be the canonical projections. We call them the first and the second *vertical projections*.

We should remark that the first projection is along the direction of the Engel line field, and the second one is along the Legendrian fiber with respect to the induced standard contact structure  $C = \{\omega_1 = dy - zdx = 0\}$  on  $Q_3 \cong \mathbb{R}^3$  (see Subsection 2.1). Let  $\Gamma: S^1 \rightarrow (\mathbb{R}^3, C)$  be a Legendrian knot in the standard contact 3-space. Then,  $\pi_2 \circ \Gamma: S^1 \rightarrow Q_2 \cong \mathbb{R}^2$  is an immersion with singularities. At a point where  $\Gamma$  is tangent to the  $z$ -direction, the projection  $\pi_2 \circ \Gamma$  has a singular point. The image of  $\pi_2 \circ \Gamma: S^1 \rightarrow \mathbb{R}^2$  is called a *wave front*, or *front* for short, of  $\Gamma$ . Generically, the singular points are cusps of type  $(2, 3)$ , where  $\pi_2 \circ \Gamma$  is locally diffeomorphic to a curve  $t \mapsto (t^2, t^3)$  at  $t = 0$ . Fronts of Legendrian knots are important tools to study Legendrian knots. We take further projections from the upper side to study horizontal loops. Let  $\gamma: S^1 \rightarrow (\mathbb{R}^4, E)$  be a horizontal loop in the standard Engel space. Then,  $\pi_1 \circ \gamma: S^1 \rightarrow (\mathbb{R}^3, C)$  is a Legendrian immersion with singularities. At a point where  $\gamma$  is tangent to the  $w$ -direction, the projection  $\pi_1 \circ \gamma$  has a singular point. Such Legendrian curves with singularities are studied by Ishikawa [15] and Zhitomirskii [16]. We call the image of further projection  $\pi_2 \circ \pi_1 \circ \gamma: S^1 \rightarrow Q_2 \cong \mathbb{R}^2$  a *front* of  $\gamma$ . It is a wave front of a Legendrian knot  $\pi_1 \circ \gamma$ . The mapping  $\pi_2 \circ \pi_1 \circ \gamma$  is an immersion with singular points. The singular points are cusps of type  $(2, 5)$ , that is, points where  $\pi_2 \circ \pi_1 \circ \gamma$  is locally diffeomorphic to a curve  $t \mapsto (t^2, t^5)$  at  $t = 0$ .

Contrary to the vertical projection, we can reconstruct a horizontal curve in the standard Engel space and a Legendrian knot from certain plane curves. We call these procedures *vertical lifts*. They are based on the prolongation procedure explained in Subsection 2.1.

First, we take vertical lifts to Legendrian knots. Let  $f: S^1 \rightarrow \mathbb{R}^2 \cong Q_2$ ,  $f(s) = (f_1(s), f_2(s))$ , be an immersion with cusps of type  $(2, 3)$ . Assume that the plane curve  $f$  is *nonvertical*, in other words, it satisfies  $df_1(s)/ds \neq 0$  at regular points and  $d^2 f_1(s)/ds^2 \neq 0$  at cusp points. We remark that the vertical projection of a Legendrian knot is actually nonvertical. Then, we can lift the given plane curve  $f(s) = (f_1(s), f_2(s))$  to a Legendrian curve  $\bar{f}: S^1 \rightarrow \mathbb{R}^3 \cong Q_3$ ,  $\bar{f}(s) = (f_1(s), f_2(s), f_3(s))$ , in  $\mathbb{R}^3$  with the standard contact structure  $C = \{dy - zdx = 0\}$  whose vertical projection is the given  $f(t)$ . Its  $z$ -coordinate  $f_3(s)$  is obtained as a slope of the curve  $f(s)$ .

Precisely, it is written down as follows:

$$f_3(s) = \begin{cases} \frac{df_2}{ds}(s) / \frac{df_1}{ds}(s) & \text{at regular points,} \\ \frac{d^2 f_2}{ds^2}(s) / \frac{d^2 f_1}{ds^2}(s) & \text{at cusp points.} \end{cases} \quad (3)$$

It is clear, from the definition of the standard contact structure, that the obtained curve  $\bar{f}(s)$  is Legendrian. Even at a cusp point, the curve is lifted to a smooth curve (see Example 2.1). When the plane curve  $f: S^1 \rightarrow \mathbb{R}^2$  has no self-tangency, it is lifted to an embedding, that is, a Legendrian knot.

Next, we observe vertical lifts to horizontal loops in the standard Engel space. Let  $f: S^1 \rightarrow \mathbb{R}^2 \cong Q_2$ ,  $f(s) = (f_1(s), f_2(s))$ , be an immersion with cusps of type  $(2, 5)$ . Assume that the same condition as above. Then, by the procedure as above, we obtain a Legendrian immersion with singularities. As cusp points of  $f$  are of type  $(2, 5)$ , we still have singularities (see Example 2.1). Furthermore, we can lift the obtained Legendrian curve  $\bar{f}(s) = (f_1(s), f_2(s), f_3(s))$  in  $(\mathbb{R}^3 \cong Q_3, C)$  to a horizontal loop  $\tilde{f}(s) = (f_1(s), f_2(s), f_3(s), f_4(s))$  in the standard Engel space  $(\mathbb{R}^4, E)$  whose vertical projection is the given plane curve  $f(s)$ . Its  $w$ -coordinate  $f_4(s)$  is obtained similarly to the above. It is obtained as a slope of a plane curve induced from  $\bar{f}(s)$  by the projection  $\text{pr}$  to the  $(x, z)$ -plane. Note that the projected plane curve  $\text{pr} \circ \bar{f}: S^1 \rightarrow \mathbb{R}^2$  is a nonvertical immersion with cusps of type  $(2, 3)$ , namely locally equivalent to a curve  $t \mapsto (t^2, t^3)$ . It is because of the assumption that  $f(s)$  is an immersion with cusps of type  $(2, 5)$  (see Example 2.1). Then, the  $w$ -coordinate of  $\tilde{f}(s)$  is obtained as follows:

$$f_4(s) = \begin{cases} \frac{df_3}{dt}(s) / \frac{df_1}{dt}(s) & \text{at regular points,} \\ \frac{d^2 f_3}{dt^2}(s) / \frac{d^2 f_1}{dt^2}(s) & \text{at cusp points.} \end{cases} \quad (4)$$

It is clear that the resulting curve  $\tilde{f}(s)$  is horizontal with respect to the standard Engel structure  $E = \{dy - zdx = 0, dz - wdx = 0\}$ . We should remark that, contrary to the Legendrian case, self-tangencies of order one are allowed for the plane curve  $f$ . Such points are lifted to different points with different  $w$ -coordinates.

**Example 2.1.** We observe these lifts at a cusp point. Suppose that a nonvertical cusp of type  $(2, 5)$  on the  $(x, y)$ -plane  $\mathbb{R}^2 \cong Q_2$  is parameterized as  $g(t) = (t^2, t^5)$  at  $t = 0$ . Then, it is lifted to a space curve  $\bar{g}(t) := (t^2, t^5, 5t^3/2)$  in  $\mathbb{R}^3 \cong Q_3$  which is Legendrian for the standard contact

structure  $C = \{dy - zdx = 0\}$ . The projection  $\text{pr} \circ \bar{g}(t) = (t^2, 5t^3/2)$  to the  $(x, z)$ -plane has a cusp of type  $(2, 3)$  at  $t = 0$ , where  $\text{pr}: (x, y, z) \mapsto (x, z)$  is a canonical projection. Then,  $\bar{g}(t)$  is lifted to a horizontal smooth curve  $\tilde{g}(t) := (t^2, t^5, 5t^3/2, 15t/4)$  in the standard Engel space  $(\mathbb{R}^4, E)$ .

**Example 2.2.** As a global circle, an immersed circle on the  $(x, y)$ -plane with cusps of type  $(2, 5)$  is lifted as in Figure 3.

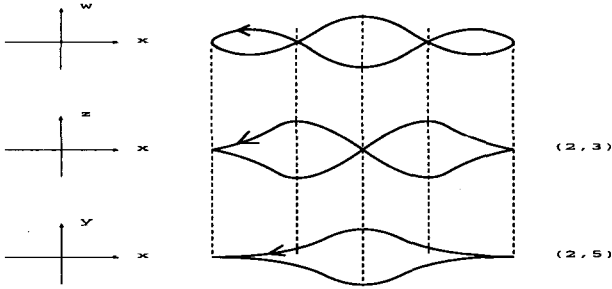


Fig. 3. projections and lifts.

We can calculate the Thurston-Bennequin invariant of a Legendrian knot, from its front. A vector field  $\partial/\partial y$  is transverse to the standard contact structure  $C = \{dy - zdx = 0\}$ . We obtain the front of Legendrian knot slightly perturbed from the given one along  $\partial/\partial y$  by perturbing the front of the given Legendrian knot along the  $y$ -direction. It is easy to check that the contribution to linking number is  $+1$  at a crossing and at a right-looking (or left-looking) cusp in the front. As a result, we conclude the Thurston-Bennequin invariant of the Legendrian knot obtained from the  $L_{s,t}$ -type front (see Figure 1 in Remark 1.2) is  $\text{tb}(L_{s,t}) = 2t + s + 1$ . Note that the sign is opposite from that in [12], [1]. It is because our standard contact structure is a negative one, that is,  $\omega_1 \wedge d\omega_1 = -dx \wedge dy \wedge dz$ .

### 2.3. Horizontal projections and rotation number

We introduce the horizontal projection of the standard Engel space to a 2-dimensional plane. We define the rotation number of a horizontal loop in the standard Engel space from the horizontal projection of the loop. Similarly to vertical lifts, we can lift a certain immersed circle in a plane to a Legendrian loop in the standard contact space, and to a horizontal loop in the standard Engel space. Further, we count the rotation number

of a horizontal loop obtained from a front of type  $F_k$  (see Figure 2) by the vertical lifts. Remark 1.4 is verified here.

The horizontal projection is defined as follows. Let  $(x, y, z, w)$  be coordinates of the standard Engel space  $(\mathbb{R}^4, E)$ . Set  $P_3 := \{y = 0\} \cong \mathbb{R}^3$ , and  $P_2 := \{y = 0, z = 0\} \cong \mathbb{R}^2$ . Let  $p_1: \mathbb{R}^4 \rightarrow P_3, (x, y, z, w) \mapsto (x, z, w)$ , and  $p_2: P_3 \rightarrow P_2, (x, z, w) \mapsto (x, w)$  be the canonical projections. We call them the first and the second *horizontal projections*.

We define an invariant, the rotation number, of a horizontal loop in the standard Engel space. It is similar to the rotation number for Legendrian knots (see [11]). Let  $\gamma: S^1 \rightarrow (\mathbb{R}^4, E)$  be a horizontal loop. The projected curve  $p_2 \circ p_1 \circ \gamma: S^1 \rightarrow P_2 \cong \mathbb{R}^2$  is an immersed plane curve because the standard Engel structure is transverse to  $y$  and  $z$ -axes. Then, we can calculate the degree of the immersed oriented curve  $p_2 \circ p_1 \circ \gamma: S^1 \rightarrow P_2 \cong \mathbb{R}^2$  with respect to  $x$  and  $w$ -axes, in other words, with respect to a trivialization of the Engel structure  $E$ . We call it the *rotation number* of  $\gamma$ . Let  $r(\gamma)$  denote it. This definition is independent of the choice of the trivialization by a similar reason to the case of Legendrian knots (see [11]). It is because  $H_1(\gamma)$  is null in  $H_1(\mathbb{R}^4)$ . Then, the rotation number is invariant under diffeomorphisms preserving Engel structure, and horizontal homotopies of horizontal loops.

**Example 2.3.** Let us consider a horizontal loop obtained in Example 2.2. The rotation number of the curve is 1. We can count by using the picture in Figure 3 on the  $(x, w)$ -plane.

We can count rotation numbers of horizontal loops obtained from fronts of type  $F_k$  (see Figure 2). According to Example 2.2, a cusp of a front corresponds to an immersed curve with one twist with the reversed orientation. Thus, in order to calculate the rotation number of a horizontal curve from a front of type  $F_k$ , we have only to count the degree of an immersed curve in Figure 4. As a result, we obtain that the rotation number of a horizontal

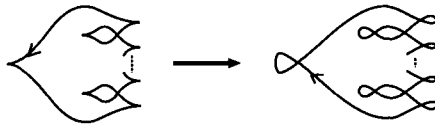


Fig. 4. How to count rotation numbers

loop obtained from a front of type  $F_k$  is  $k$ .

We can lift a certain immersed closed plane curve to an immersed horizontal loop in the standard Engel space. We call this procedure a *horizontal lift*. Let  $g: S^1 \rightarrow P_2 \cong \mathbb{R}^2$ ,  $g(s) = (g_1(s), g_4(s))$ , be an immersed closed curve in the  $(x, w)$ -plane. Suppose that the algebraic area bounded by the immersed curve is zero:

$$\int_{g(S^1)} g_4(s) dg_1(s) = 0. \quad (5)$$

The condition guarantees that the lifted curve is closed. We remark that horizontal projections of horizontal loops in  $(\mathbb{R}^4, E)$  and horizontal projections of Legendrian closed curves satisfy this condition. Then, the given immersed plane curve  $g(s) = (g_1(s), g_4(s))$  satisfying the condition (5) above is lifted to a Legendrian immersed circle  $\bar{g}(s) = (g_1(s), g_3(s), g_4(s))$  with singular points in the  $(x, z, w)$ -space  $P_3 \cong \mathbb{R}^3$  with the standard contact structure  $C' = \{\omega_2 = dz - wdx = 0\}$ . The horizontal projection of  $\bar{g}(s)$  is the original  $g(s)$ . The  $z$ -coordinate  $g_3(s)$  of  $\bar{g}(s)$  is obtained as follows:

$$g_3(s) := \int_0^s g_4(u) dg_1(u).$$

In order to lift the Legendrian curve  $\bar{g}(s)$  to an immersed horizontal loop in  $(\mathbb{R}^4, E)$ , we need an additional condition. To make it sure that the lifted curve is actually closed, we need a similar condition to equation (5). Suppose that the algebraic area bounded by the projection of  $\bar{g}(s)$  to the  $(x, z)$ -plane is zero:

$$\int_{\text{pr} \circ \bar{g}(S^1)} g_3(s) \cdot dg_1(s) = 0, \quad (6)$$

where  $\text{pr}: (x, z, w) \mapsto (x, z)$  is a canonical projection. Then, the Legendrian curve  $\bar{g}(s) = (g_1(s), g_3(s), g_4(s))$  in  $P_3 \cong \mathbb{R}^3$  can be lifted to an immersed horizontal loop  $\tilde{g}(s) = (g_1(s), g_2(s), g_3(s), g_4(s))$  in  $(\mathbb{R}^4, E)$  whose vertical projection is the given plane curve  $g(s)$ . The  $y$ -coordinate  $g_2(s)$  of  $\tilde{g}(s)$  is obtained as follows:

$$g_2(s) := \int_0^s g_3(u) \cdot dg_1(u).$$

**Example 2.4.** If an immersed curve as the top of Figure 3 is given, we can lift the curve from the top to the bottom of Figure 3.

We will consider this horizontal lift in Section 6.

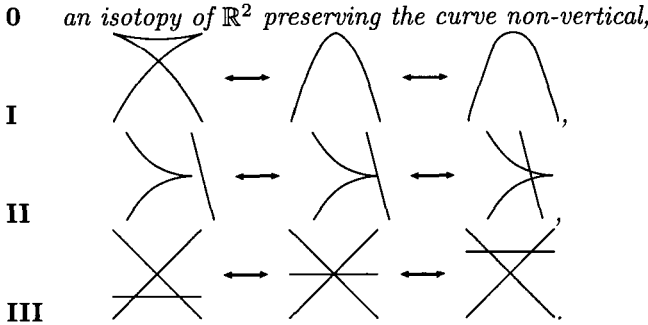
### 3. Reidemeister moves for contact and Engel structures

The Reidemeister moves for contact and Engel structures are introduced in this section. In other words, we determine finite number of moves of a front which can be lifted to Legendrian isotopies in the standard contact 3-space or horizontal homotopies in the standard Engel space.

#### 3.1. Reidemeister moves for contact structures

The Reidemeister moves for Legendrian knots are given in the following theorem. It is proved in [17].

**Theorem 3.1 (Legendrian Reidemeister moves, [17]).** *Let  $\Gamma_0, \Gamma_1: S^1 \rightarrow (\mathbb{R}^3, C)$  be Legendrian knots in the standard contact 3-space. These knots are Legendrian isotopic if and only if the front  $\pi_2 \circ \Gamma_0(S^1)$  is moved to the other front  $\pi_2 \circ \Gamma_1(S^1)$  via a finite sequence of the following moves:*



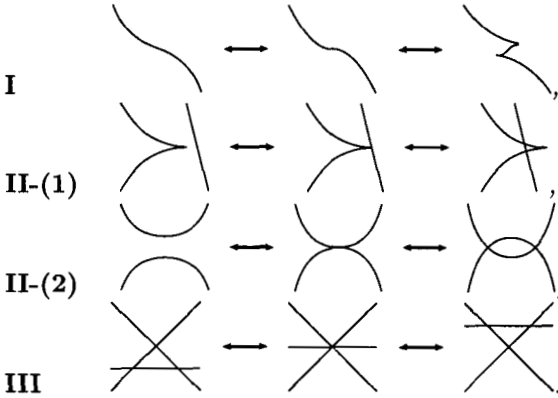
Note that cusps in the list above are of type (2, 3).

#### 3.2. Reidemeister moves for Engel structures

The Reidemeister moves for horizontal loops in the standard Engel space is given in the following theorem. It is proved in [2].

**Theorem 3.2 (Engel Reidemeister moves).** *Let  $\gamma_0, \gamma_1: S^1 \rightarrow (\mathbb{R}^4, E)$  be horizontal loops in the standard Engel space. These loops are horizontally homotopic if and only if the front  $\pi_2 \circ \pi_1 \circ \gamma_0(S^1)$  is moved to the other front  $\pi_2 \circ \pi_1 \circ \gamma_1(S^1)$  via a finite sequence of the following moves:*

0 an isotopy of  $\mathbb{R}^2$  preserving the curve non-vertical,



Note that cusps in the list above are of type  $(2, 5)$ . Moves I and II-(2) never occur in the Legendrian Reidemeister moves, and that move I in Theorem 3.1 never occurs in the Engel case.

We should remark that move II-(1)' below (see Figure 5) can be obtained by a combination of the moves II-(1) and II-(2) in Theorem 3.2 (see Figure 6).



Fig. 5. II-(1)'



Fig. 6. combination of II-(1) and II-(2)

Then, the move II-(1)' is not essential in the list of moves in Theorem 3.2.

#### 4. Classification of trivial Legendrian knots

In this section, a rough sketch of the proof of the classification of topologically trivial Legendrian knots, namely Theorem 1.1, is given. The arguments in this section are mainly due to Eliashberg and Fraser [1].

**4.1. Characteristic foliations on a spanning disk**

We observe characteristic foliations on a disk which spans the given topologically trivial Legendrian knot. Let  $\Gamma: S^1 \rightarrow (M, C)$ , or its image  $L := \Gamma(S^1)$ , be a topologically trivial Legendrian knot in a tight contact 3-manifold  $(M, C)$ . Since  $L$  is topologically trivial there exists a disk  $D \subset (M, C)$  which spans  $L$ . We observe characteristic foliation  $D_C$  by perturbing the disk  $D$ . The goal of this subsection is to arrange the characteristic foliation to a certain form.

The form we are going to obtain is defined as follows. The characteristic foliation  $D_C$  is said to be in *elliptic form* if it satisfies the following conditions (see Figure 7 for example):

- the signs of singular points on the boundary  $L = \partial D$  is alternating,
- all singular points on the interior  $\text{int}D$  are elliptic,
- each singular point on the boundary  $L = \partial D$  is connected to singular points in the interior  $\text{int}D$  by leaves of  $D_C$ , except connections with two singular points on  $L = \partial D$  next to it,
- each elliptic point in the interior  $\text{int}D$  is connected to at least two hyperbolic points on the boundary  $L = \partial D$ .

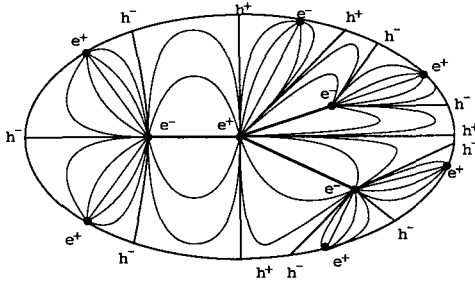


Fig. 7. elliptic form

It is proved in [1] that any topologically trivial Legendrian knot  $L$  in a tight contact manifold  $(M, C)$  has a spanning disk  $D \subset (M, C)$  characteristic foliation on which is in elliptic form. A rough sketch of the proof is introduced in this subsection.

First, we should introduce an important tool to deal with characteristic foliations. The following Elimination Lemma is proved by Giroux and improved by Fuchs (see [18], [19]).

**Lemma 4.1 (Elimination Lemma).** *Let  $F$  be an embedded surface in a contact 3-manifold  $(M, C)$ , and  $p, q \in F$  elliptic and hyperbolic points of the characteristic foliation  $F_C$  with the same sign. If there is a trajectory  $\gamma$  of  $F_C$  joining  $p$  and  $q$ , then there exists a  $C^0$ -small isotopy  $h_t: F \rightarrow (M, C)$ ,  $t \in [0, 1]$ , which has the following properties:*

- $h_t$  is the identity on  $\gamma$  and outside a neighborhood  $U$  of  $\gamma$ ,
- $h_0 = \text{id}$ ,
- the characteristic foliation  $\tilde{F}_C$  on  $\tilde{F} := h_1(F)$  has no singular point in  $\tilde{F} \cap U$ .

(see Figure 8)

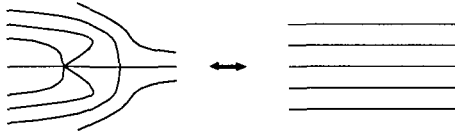


Fig. 8. Elimination of singularities

Now, a rough story of the proof of the existence of a spanning disk with a characteristic foliation in elliptic form is introduced as follows.

First, take a spanning disk  $D$  of the given Legendrian knot so that the characteristic foliation  $D_C$  on  $D$  has a certain form near the boundary  $\partial D = L$ . The characteristic foliation  $D_C$  is said to be in *normal absorbing form* along  $L = \partial D$  if signs of singular points on  $L$  is alternating and the positive ones are hyperbolic and the negative ones are elliptic. We can put the disk  $D$  so that the characteristic foliation  $D_C$  is in normal absorbing form along  $L$  by twisting  $D$  around  $L = \partial D$  keeping  $L$  fixed.

Then, we apply the Elimination Lemma 4.1. Since the characteristic foliation  $D_C$  on the spanning disk  $D$  is in normal absorbing form along  $L \subset \partial D$ , there exists a transverse knot  $L'$  on  $D$  near  $L$  (see Figure 9). Then, we can apply arguments in [12] for transverse knots. According to [12], we can eliminate all negative elliptic and positive hyperbolic singular points inside by the Elimination Lemma 4.1. Such a reduced foliation on  $D$  is said to be *reduced with normal absorbing form on the boundary*. We should remark that, according to [12], there appears a Legendrian tree on  $D$  whose vertices are positive elliptic points and whose edges are separatrices of negative hyperbolic points. Tightness of the contact structure is important for the argument in this paragraph.

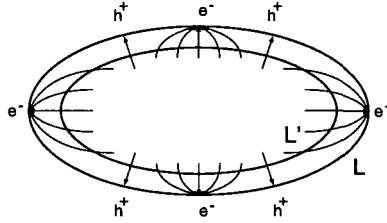


Fig. 9. Normal absorbing form

It is proved in [1] that the spanning disk  $D$  of  $L$  whose characteristic foliation  $D_C$  is in reduced form with normal absorbing boundary is perturbed to have a characteristic foliation in elliptic form. An important trick is the converse of the Elimination Lemma occurring at a boundary singular point. It is carefully observed in [20] (elliptic-hyperbolic conversion). By this trick, we can exchange an elliptic (hyperbolic) point on the boundary with a hyperbolic (elliptic) point with the same sign and another elliptic (hyperbolic) point with the same sign (see Figure 10). Then, we can eliminate hyperbolic

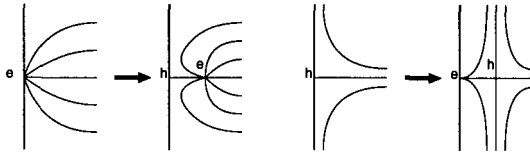


Fig. 10. elliptic-hyperbolic conversion

points inside and some other extra things by the Elimination Lemma 4.1. Thus, we obtain a characteristic foliation in elliptic form. We should remark that there appears a Legendrian tree on the perturbed  $D$  with a characteristic foliation  $D_C$  in elliptic form. In this case, the vertices are positive and negative elliptic points and the edges are leaves of  $D_C$  connecting elliptic points directly with no singular point on them. We observe this tree in the next subsection.

#### 4.2. Deformation of corresponding fronts

In this subsection, we deal with fronts which correspond to the obtained characteristic foliation on the spanning disk of the Legendrian knot in elliptic form. Characteristic foliation in elliptic form is not enough to discuss the classification. So, we use the arguments on fronts of Legendrian knots.

The goal of this subsection is to reduce the front corresponding to the characteristic foliation in elliptic form to one of the fronts in the catalog (see Figure 1).

A tree with twigs is induced from a characteristic foliation in elliptic form. Let  $D$  be a spanning disk of a Legendrian knot  $L$  characteristic foliation on which is in elliptic form. Then, we can take a Legendrian tree  $T$  whose vertices are interior elliptic points of  $D_C$  and whose edges are leaves of  $D_C$  connecting interior elliptic points directly. Note that there are at least two separatrices of hyperbolic points on the boundary meeting at each interior elliptic point. We consider them *twigs* of the tree. Let us embed twigged trees into the  $(x, y)$ -plane so that the slope of edges are sufficiently horizontal and two twigs at each vertex are vertical. And, give each vertex the same sign as the corresponding elliptic point (see Figure 11 and Figure 7).

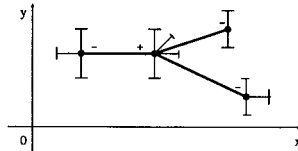


Fig. 11. twigged tree

A front is induced from a twigged tree embedded in  $\mathbb{R}^2$  as follows. First, we should be careful about the twisting of the spanning disk  $D$ . Then, it is useful to change the twigged tree a little. Reverse the relative positions of edges and twigs to the right of each positive vertex and to the left of each negative vertex (see Figure 11 and Figure 15 for example). Now, we apply the following procedure to obtain a front from a twigged tree.

- (1) To each end vertex, we give the following (Figure 12) part of front according to the sign of the point. At a positive or negative vertex, fronts go upward or downward respectively:
- (2) To each non-end vertex, with exactly two twigs, we give a crossing of fronts. At a positive or negative vertex, fronts go upward or downward respectively, like operation (1) above (see Figure 13):
- (3) To each extra twigs, we give a zig-zag of fronts. The zig-zag goes downward for a twig at a positive vertex and goes upward for a twig at a negative vertex (see Figure 14):

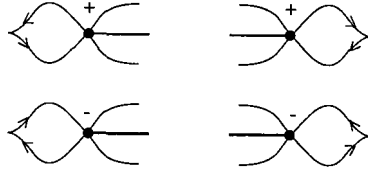


Fig. 12. fronts corresponding to the ends

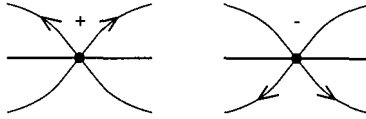


Fig. 13. fronts corresponding to non-end vertices

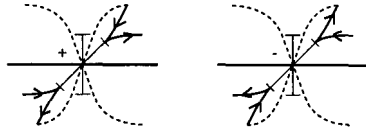


Fig. 14. fronts corresponding to twigs

(4) Connecting them suitably, we obtain a front for the twigged tree (see Figure 15, 7, and 11):

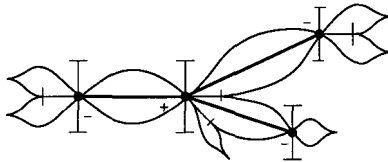


Fig. 15. example of corresponding front

Thus, we obtain a front from a twigged tree. Note that the correspondence is one to one. Also, we should remark that we can construct a Legendrian knot by the Legendrian lift, which has a spanning disk  $D$  with the same twigged tree as the given one on it.

Now, we handle fronts instead of characteristic foliations. We can sim-

ply the obtained front to one of the front in the list given in Remark 1.2 (see Figure 1). Basically, we make the given tree linear by the Legendrian Reidemeister moves (Theorem 3.1). The procedure is discussed in [12] (see Figure 16, for example).

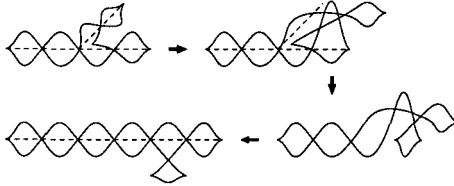


Fig. 16. linearization of a tree

The problem is how to deal with zig-zags which appear in the third procedure of constructing fronts, or which correspond to twigs. It is proved in [1] that we can move a zig-zag to anywhere we want. Then, applying the argument above, we obtain a front corresponding to linear tree with zig-zags on one end. A cancelation of different type of zig-zags is discussed in [1]. Then, as a result, we obtain a front in the catalogue in Figure 1.

The proof of Theorem 1.1 is completed in [1] by proving the following lemma:

**Lemma 4.2.** *Let  $L, L'$  be Legendrian knots bounding disks  $D, D'$  respectively, which have the diffeomorphic characteristic foliation in elliptic form. Then,  $L$  and  $L'$  are Legendrian isotopic.*

The rough story of the proof is the following. By taking slightly different spanning disks (so called exceptional spanning surfaces) from what we discussed, the Legendrian knots  $L$  and  $L'$  shrink arbitrary close to the diffeomorphic skeletons which consist of the trees on the spanning disks and elliptic points on the boundary of the disks. It is proved those skeletons with germs of surface coincide by an ambient contact isotopy.

## 5. Classification of horizontal loops in the standard Engel space

Theorem 1.3 is treated in this section. It is clear that rotation numbers of two horizontally homotopic loops are the same. We show that any horizontal loop is horizontally homotopic to one of horizontal loops of type  $\bar{F}_k$ ,  $k \in \mathbb{Z}$ .

The proof in [2] is organized as follows. First, we reduce the problem to a problem of Legendrian knots with singularities. Then, we can apply arguments of Eliashberg and Fraser [1] for Legendrian knots. By applying their arguments, the given loop is homotoped horizontally to a horizontal loop with a front of a certain type (Subsection 5.1). Next, such a front is moved to that of  $F_k$ -type by Engel Reidemeister moves (Subsection 5.2).

### 5.1. Legendrian knots with singularities

Horizontal loops in the standard Engel space  $(\mathbb{R}^4, E)$  can be identified with Legendrian knots with certain singularities in the standard contact space  $(\mathbb{R}^3, C)$ . Let  $\gamma: S^1 \rightarrow (\mathbb{R}^4, E)$  be a horizontal loop. Then,  $\pi_1 \circ \gamma: S^1 \rightarrow (\mathbb{R}^3, C)$  is a Legendrian knot with some singularities. In other words, a space curve  $\pi_1 \circ \gamma$  is obtained as a single vertical lift of the front  $\pi_2 \circ \pi_1 \circ \gamma$  (see Subsection 2.2). The Legendrian curve  $\pi_1 \circ \gamma: S^1 \rightarrow (\mathbb{R}^3, C)$  is not an immersion at points corresponding to cusps of the front  $\pi_2 \circ \pi_1 \circ \gamma$ , and has self-intersection at points corresponding to self-tangency of the front. If we care about these points, we can apply some arguments on Legendrian knots. That is, homotopy among such Legendrian knots can be lifted to horizontal homotopy.

First of all, we remark that a self-intersection of a Legendrian knot corresponding to a horizontal loop is canceled by horizontal homotopy of the horizontal loop. As we mentioned above, a self-intersection of the corresponding Legendrian knot is a self-tangency of the front of the horizontal loop. In terms of fronts, a self-tangency of a front appears as that of regular curves, a regular curve and a cusp, or cusps. The last one is canceled easily. The former two are canceled by Engel-Reidemeister moves II-(2) and II-(1)' (see Subsection 3.2).

An important fact is that any horizontal loop is horizontally homotopic to a horizontal loop corresponding to a topologically trivial Legendrian knot with singularities. It is known, in knot theory, that applying  $X$ -moves, we can make a knot topologically trivial (see [21]). An  $X$ -move is an operation on knot diagram, projection of knots to planes, defined as in Figure 17. We obtain the  $X$ -move for Legendrian knots, which is obtained by Engel Reidemeister moves I, II-(1), and II-(2) (see Figure 18). This implies that any horizontal loop is horizontally homotopic to a horizontal loop whose corresponding Legendrian knot is topologically trivial. Then, considering topologically trivial Legendrian knots with singularities is sufficient to show Theorem 1.3.

We can apply arguments due to Eliashberg and Fraser in [1] for topo-



Fig. 17. X-move

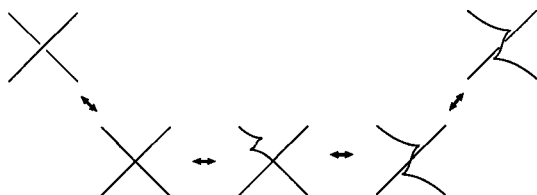


Fig. 18. X-move for Legendrian knots

logically trivial Legendrian knots. Roughly speaking, the argument due to Eliashberg and Fraser [1] consists of two parts. Manipulations of characteristic foliations on a disk spanning the given topologically trivial Legendrian knot, and deformations of fronts. As for the manipulations of characteristic foliations, we can apply almost the same arguments to topologically trivial Legendrian knots induced from horizontal loops. On the other hand, arguments on deformations of fronts for horizontal loops are different from those for Legendrian knots. Reidemeister moves for fronts of horizontal curves are different from those for fronts of Legendrian curves (see Theorem 3.2, Theorem 3.1).

We manipulate characteristic foliations on a disk  $D \subset \mathbb{R}^3$  spanning the given topologically trivial Legendrian knot  $\pi_1 \circ \gamma: S^1 \rightarrow (\mathbb{R}^3, C)$  with singularities which is induced from a horizontal loop  $\gamma: S^1 \rightarrow (\mathbb{R}^4, E)$ . Such a curve has singular points where projections to the  $(x, y)$ -plane are  $(2, 5)$ -cusps and projections to  $(x, z)$ -plane are  $(2, 3)$ -cusps generically. We take a spanning disk  $D$  so that the characteristic foliation  $D_C$  has elliptic singular points (see Figure 19) at singular points of  $\pi_1 \circ \gamma$ . We can take such

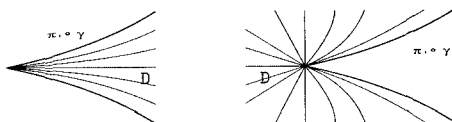


Fig. 19. elliptic corner

a disk  $D$  as follows. First, we take surfaces around singular points of the

curve  $\pi_1 \circ \gamma$  so that the curve is leaves of the characteristic foliation on the surfaces, and that the singular points are elliptic corners. We extend such small surfaces to an annulus along the curve  $\pi_1 \circ \gamma$  so that it can be extended to a disk. And then, we extend the annulus to a disk. Thus, we obtain a disk  $D$  spanning  $D$  with elliptic corners as above. An elliptic corner can be changed to a smooth elliptic point by a  $C^0$ -small perturbation of  $D$  keeping  $D_C$  homeomorphic, according to [19]. Applying such perturbations to all singular points, we obtain a topologically trivial smooth Legendrian knot  $\Gamma: S^1 \rightarrow (\mathbb{R}^3, \mathcal{C})$  with a spanning disk  $\tilde{D}$ , characteristic foliation  $\tilde{D}_C$  on which is homeomorphic to  $D_C$ . Then, we apply the argument in [1] to  $\Gamma$  which is  $C^0$ -close to  $\pi_1 \circ \gamma$ . For the Legendrian knot  $\Gamma$ , there corresponds a front. Note that the given Legendrian knot  $\pi_1 \circ \gamma$  with singularities is induced from the horizontal loop  $\gamma$ . Then, the corresponding front must have the same number of cusps as the number of singular points of  $\pi_1 \circ \gamma$ . The number of cusps of the front corresponding to the smooth curve  $\Gamma$  might be less than the number of singular points of  $\pi_1 \circ \gamma$  because of the Legendrian Reidemeister moves I. In such a case, we create pairs of singular points of  $D_C$  on  $\Gamma = \partial D$  corresponding to the Legendre Reidemeister moves I above by the converse operation of the Elimination Lemma. Then, the characteristic foliation on  $D$  corresponds to a front with the same number of cusps as the number of singular points of  $\pi_1 \circ \gamma$ . Perturbing the disk  $D$  we arrange the characteristic foliation  $D_C$  so that the points on  $\Gamma = \partial D$  corresponding to singular points of  $\pi_1 \circ \gamma$  correspond to cusps of the obtained front. If such a point is a boundary elliptic point of  $D_C$ , then we can return  $\Gamma$  to  $\pi_1 \circ \gamma$  around the point. If such a point is a boundary hyperbolic point, we make it elliptic by applying the elliptic-hyperbolic conversion (see Figure 20). Then, we can return  $\Gamma$  to  $\pi_1 \circ \gamma$ . We should remark that a convex (resp.

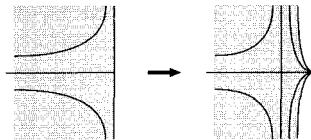


Fig. 20. cusp for boundary hyperbolic point

concave) elliptic corner for the initial spanning disk  $D$  might be changed to a concave (resp. convex) elliptic corner for the resulting disk  $\hat{D}$ . In this case, we have just retaken the spanning disk and never changed the given Legendrian knot  $\pi_1 \circ \gamma$ . Consequently, we obtain a disk  $\hat{D}$  with singularity

spanning  $\pi_1 \circ \gamma$  the characteristic foliation on which is in elliptic-like form. In other words, the characteristic foliation  $\widehat{D}_C$  is in elliptic form almost everywhere with extra pairs of elliptic and hyperbolic points of the same sign (see Figure 8) on the boundary which correspond to pairs of cusps in the Legendrian Reidemeister move I, and with extra hyperbolic elliptic conversions (see Figure 20). Then, the front we obtained is the front induced from a characteristic foliation in elliptic form with some pair of cusps in the figure of the Legendrian Reidemeister move I whose cusps are of type (2, 5).

We continue the arguments in [1] on deformations of fronts. They conclude that any topologically trivial Legendrian knot in  $(\mathbb{R}^3, \mathcal{C})$  is isotopic to a Legendrian knot whose front is of type  $L_{s,t}$  (Remark 1.2, see Figure 1). In their arguments, they apply Legendrian Reidemeister moves (Theorem 3.1). By one of those moves, a pair of cusps are canceled. In our case, such cancellations are impossible by the Engel Reidemeister moves (Theorem 3.2). As a result, we obtain a front of type  $L_{s,t}$  with some pair of cusps as in move I of the Legendrian Reidemeister move (Theorem 3.1). We should remark that such a pair of cusps does not affect any other Engel Reidemeister moves. In fact, a regular curve can move through the pair of cusps freely (See Figure 21).

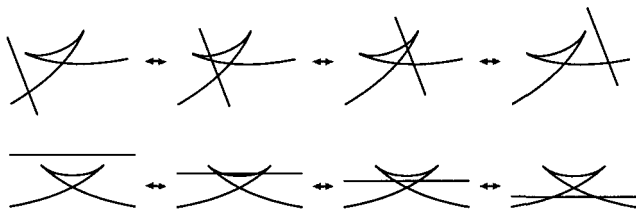


Fig. 21. cusps do not affect

### 5.2. Deformations of fronts

In this subsection, we deform a front of type  $L_{s,t}$  with certain pairs of cusps to a front of type  $F_k$  by Engel Reidemeister moves. Our goal is a front of type  $F_k$ .

Before describing the required moves, we observe important properties of the pairs of cusps in Legendrian Reidemeister move I. As we checked above, a regular curve can move through the pair of cusps freely. Further, a pair of cusps can move through a cusp. It is realized by canceling and

creating a zig-zag by Engel Reidemeister move I (see Figure 22). Last of

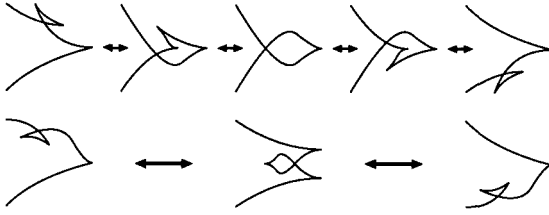


Fig. 22. cusps move through a cusp

all, two consecutive pairs of cusps are canceled if they are on different side of a curve each other (see Figure 23).

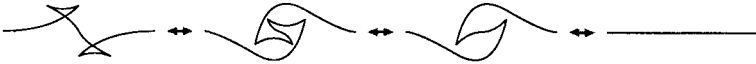


Fig. 23. cancellation of cusps

Now, we describe the move from a front of type  $L_{s,t}$  with certain pair of cusps to a front of type  $F_k$ .

First, we reduce a front of type  $L_{s,t}$  to that of type  $L_{0,0}$ . As is mentioned above, those pairs of cusps do not affect to Engel Reidemeister moves. Therefore, we can apply Engel Reidemeister moves directly to a front of type  $L_{s,t}$  ignoring those pairs of cusps. We can cancel zig-zags on the right hand side of an  $L_{s,t}$ -type front by the Engel Reidemeister move I, and make two consecutive twists one by move II-(2). Then, a front of type  $L_{s,t}$  is reduced to that of type  $L_{0,0}$ , that is, a flying saucer or lips.

Now, we deform such a front to a front of type  $F_k$ . We can cancel one pair of cusps inside of  $L_{0,0}$ -type front and one pair of cusps outside of  $L_{0,0}$ -type front together by the move in Figure 23. Thus, we obtain a front of type  $L_{0,0}$  which has pairs of cusps only inside or outside. In the first case, we obtain a front of type  $F_k$  by pulling one cusp of each pair to the outside as in Figure 22 by the Engel Reidemeister moves II-(1), (2). In the second case, we consider in two steps. If there is only one pair of cusps outside of an  $L_{0,0}$ -type front, we obtain an  $F_0$ -type front by the move in Figure 22. If there are  $k + 1 > 1$  pairs of cusps outside, we obtain an  $F_0$ -type front with  $k$  pairs of cusps. Applying the move in Figure 22 to the other side, we

obtain a  $-L_{0,0}$ -type front with  $k$  pairs of cusps inside and one pair of cusps outside, where  $-L_{0,0}$  is a type of fronts which is the same as  $L_{0,0}$  but with the reversed orientation. Then, canceling a pair of pairs of cusps, we obtain a front of type  $-L_{0,0}$  with  $k - 1$  pairs of cusps inside. We should remark that the orientation is reversed by the operation above (see Figure 24). In

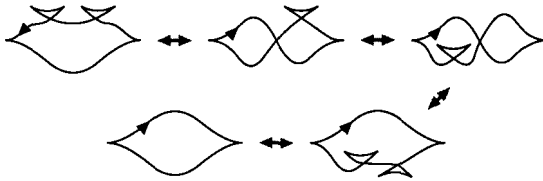


Fig. 24. cusps outside

the same way as in the first case, we obtain a front of type  $F_k$ .

Last of all, in order to finish the proof of Theorem 1.3, we need an argument similar to Lemma 4.2. The important operation is to make a Legendrian knot with singular points induced from a horizontal loop shrink to the skeleton on the spanning disk in Subsection 5.1. Since we obtained a disk with a characteristic foliation in elliptic-like form, we can deform the spanning disk so that it is an so called exceptional spanning surface almost every where. The differences are pairs of elliptic and hyperbolic points and additional hyperbolic points (see Subsection 5.1). Note that we obtain such a surface by small deformation of an exceptional spanning surface. We can apply the same argument to the elliptic corners as the argument for boundary smooth elliptic points. For each Legendrian knots without additional pairs of hyperbolic and elliptic points and hyperbolic points shrinking to the skeleton, we can make additional points. Then, the given Legendrian knot shrinks sufficiently close to the skeleton.

Thus, Theorem 1.3 has been proved.

## 6. Observations

We observe some related issues and some generalization.

### 6.1. Variants of Whitney's theorem

We obtain a variant of the Whitey theorem concerning regular homotopy classes of plane curves as a corollary of Theorem 1.3.

We, first, review the original Whitney theorem (see [8]). Let  $f_0, f_1: S^1 \rightarrow \mathbb{R}^2$  be immersions. These two immersions are said to be *regularly homotopic* if there exists a mapping  $H: S^1 \times I \rightarrow \mathbb{R}^2$  which satisfies the following conditions:

- $H(s, 0) = f_0(s), H(s, 1) = f_1(s)$  for any  $s \in S^1$ ,
- setting  $f_t(s) := H(s, t), f_s: S^1 \rightarrow \mathbb{R}^2$  is an immersion for any  $t \in I$ .

A *rotation number* is defined as the degree of an immersion of an oriented circle to a plane, which is an invariant for regular homotopies of immersed circles. Let  $r(f)$  denote the rotation number of an immersion  $f: S^1 \rightarrow \mathbb{R}^2$ . The following theorem is known as Whitney's theorem.

**Theorem 6.1 (Whitney).** *Let  $f_0, f_1: S^1 \rightarrow \mathbb{R}^2$  be immersions. They are regularly homotopic if and only if they have the same rotation number:  $r(f_0) = r(f_1)$ .*

A version of the Whitney theorem is obtained as a corollary of Theorem 1.3. In Theorem 1.3, we dealt with horizontal homotopies of embeddings of circles horizontal to the Engel structure. Let us recall that we obtain immersions of  $S^1$  to  $\mathbb{R}^2$  by taking projections of horizontal loops (see Subsection 2.3). Then, by taking horizontal projections, we obtain a regular homotopy from a horizontal homotopy. We should remark that such an immersion  $f := p_2 \circ p_1 \circ \gamma: S^1 \rightarrow \mathbb{R}^2$ , obtained from some horizontal loop  $\gamma$ , are characterized in terms of volumes as follows (see Subsection 2.3). An immersed closed curve  $g: S^1 \rightarrow \mathbb{R}^2, g(s) = (g_1(s), g_4(s))$ , is a horizontal projection of some Legendrian immersed knot in the standard contact 3-space if and only if it satisfies the following condition (see equations (5)):

$$(1) \int_0^{2\pi} g_4(s) dg_1(s) = 0.$$

Furthermore,  $g$  is an immersed loop in the standard Engel space if and only if it satisfies the following condition additionally (see equation (6)):

$$(2) \int_0^{2\pi} \left( \int_0^s g_4(u) dg_1(u) \right) dg_1(s) = 0.$$

Condition (1) implies that the algebraic area bounded by the curve  $g$  is zero. Condition (2) implies that the algebraic area bounded by the plane curve  $\bar{g}(s) := (g_1(s), g_3(s))$ , where  $g_3(s) := \int_0^s g_4(u) dg_1(u)$ , is zero. As a result, we obtain the following as a corollary of Theorem 1.3. We should remark

that the statements of Theorem 1.3 is valid even for immersed horizontal loops.

**Theorem 6.2.** *Let  $f_0, f_1: S^1 \rightarrow \mathbb{R}^2$ ,  $f_i(s) = (f_i^1(s), f_i^4(s))$ ,  $i = 0, 1$ , be immersions satisfying conditions (1) and (2). They are homotopic among immersed circles satisfying the conditions (1) and (2) above if and only if they have the same rotation number:  $r(f_0) = r(f_1)$ .*

### 6.2. General Goursat structure case

We can generalize the arguments of this article from the arguments on contact and Engel structures to those on general Goursat structures.

First, we review the definition of a Goursat structure. Let  $M$  be a manifold, and  $D \subset TM$  a distribution of rank 2. We defined distributions  $D^2, D^3$  in Subsection 2.1. Furthermore, we define distributions  $D^i$ ,  $i = 4, 5, 6, \dots$  inductively as  $D^i := D^{i-1} + [D^{i-1}, D^{i-1}]$ . The distribution  $D$  is called a *Goursat structure* on  $M$  if it satisfies the following conditions: (1) there exists a positive integer  $s \in \mathbb{Z}$  for which  $D^s = TM$ , (2)  $\text{rank } D_p^{i+1} = \text{rank } D_p^i + 1$ ,  $i = 1, 2, \dots, s-1$ , at any point  $p \in M$ . A Goursat structure on 3-dimensional manifold is a contact structure. And an Engel structure is considered as a Goursat structure on a 4-dimensional manifold. The standard Goursat structure  $D_0$  on  $\mathbb{R}^n$  is given as

$$D_0 = \{dx_2 - x_3dx_1 = 0, dx_3 - x_4dx_1 = 0, \dots, dx_{n-1} - x_n dx_1 = 0\}$$

$$= \text{Span} \left\{ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + \dots + x_n \frac{\partial}{\partial x_{n-1}} \right\}.$$

An important property of Goursat structure is that there are singular points where Goursat structures are not locally equivalent to the standard one if the dimension of the manifold is greater than 4. Classification problem of the germs of Goursat structures is still open (see [22]).

Similarly to the case for an Engel structure, we can define a horizontal loop and some related notions. An invariant, the rotation number, under horizontal homotopies is also defined in a similar way to the Engel case. Horizontal projections  $p_1: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ ,  $p_2: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-2}, \dots, p_{n-2}: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ , are defined as canonical projections  $p_i: (x_1, x_{i+1}, x_{i+2}, \dots, x_n) \mapsto (x_1, x_{i+2}, \dots, x_n)$ ,  $i = 1, 2, \dots, n-2$ . The *rotation number* for a horizontal loop  $\gamma: S^1 \rightarrow (\mathbb{R}^n, D_0)$  is defined as a degree of an immersion  $p_{n-2} \circ p_{n-3} \circ \dots \circ p_1 \circ \gamma: S^1 \rightarrow \mathbb{R}^2$ . Let  $r(\gamma)$  denote it. Vertical projections  $\pi_1: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ ,  $\pi_2: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-2}, \dots, \pi_{n-2}: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ , are defined as canonical projections  $\pi_i: (x_1, x_2, \dots, x_{n-i}, x_{n-i+1}) \mapsto (x_1, x_2, \dots, x_{n-i})$ ,

$i = 1, 2, \dots, n - 2$ . The image  $\pi_{n-2} \circ \pi_{n-3} \circ \dots \circ \pi_1 \circ \gamma: S^1 \rightarrow \mathbb{R}^2$  by these vertical projections of a horizontal loop  $\gamma: S^1 \rightarrow (\mathbb{R}^n, D_0)$  is called a *front* of  $\gamma$ . Generically, it is an immersion with cusps of type  $(2, 2n - 3)$ .

We can generalize some results in this article for Engel structures to the case of Goursat structures. We remark that  $\pi_{n-4} \circ \dots \circ \pi_1 \circ \gamma: S^1 \rightarrow (\mathbb{R}^4, E)$  is a horizontal loop for the standard Engel structure with singularities, and that  $\pi_{n-3} \circ \pi_{n-4} \circ \dots \circ \pi_1 \circ \gamma: S^1 \rightarrow (\mathbb{R}^3, C)$  is a Legendrian knot for the standard contact structure with singularities. Then, we can apply the same arguments concerning fronts as the Engel case since we can lift a front to a horizontal curve in  $(\mathbb{R}^n, D_0)$  by taking further Cartan prolongations (see Subsection 2.1). As a corollary of Theorem 1.3, we obtain the following:

**Corollary 6.3.** *Let  $\gamma_0, \gamma_1: S^1 \rightarrow (\mathbb{R}^n, D_0)$  be horizontal loops in the standard  $n$ -dimensional Goursat space,  $n \geq 4$ . These loops  $\gamma_0$  and  $\gamma_1$  are horizontally homotopic if and only if their rotation numbers coincide:  $r(\gamma_0) = r(\gamma_1)$*

We can also apply similar arguments concerning horizontal projections. The conditions under which an immersed circle  $g(s) = (g_1(s), g_n(s))$ ,  $s \in S^1 \cong [0, 2\pi]/\sim$ , into  $\mathbb{R}^2$  can be lifted to an immersed horizontal loop in  $(\mathbb{R}^n, D_0)$  is described as follows. We define functions  $g_i(s)$ ,  $i = 2, 3, \dots, n-1$ , inductively as follows:

$$g_{n-1}(s) := \int_0^s g_n(u) dg_1(u),$$

$$g_i(s) := \int_0^s g_{i+1}(u) dg_1(u), \quad i = n-2, n-3, \dots, 2.$$

Then the required condition is

$$g_i(2\pi) = 0, \quad i = 1, 2, \dots, n-1, n. \quad (7)$$

Then, as a corollary of Theorem 1.3, we obtain the following:

**Corollary 6.4.** *Let  $f_0, f_1: S^1 \rightarrow \mathbb{R}^2$ ,  $f_i(s) = (f_i^1(s), f_i^n(s))$ ,  $i = 0, 1$ , be immersions satisfying condition (7),  $n \geq 4$ . They are homotopic among immersed circles satisfying the conditions (7) above if and only if they have the same rotation number:  $r(f_0) = r(f_1)$ .*

We remark the case when  $n = 4$  is the case related to horizontal immersed circles in the Engel space.

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