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Smooth manifolds and their applications in homotopy theory[1]

L. S. Pontrjagin

Introduction

The main goal of the present work is the homotopy classification of maps from the $(n+k)$-dimensional sphere Σ^{k+n} to the n-dimensional sphere S^n; here we solve this problem only for $k = 1$, 2. The method described below was published earlier in notes [1, 2]. It allowed V. A. Rokhlin [3] to solve the problem also for $k = 3$. One has not yet obtained the results for $k > 3$ in this way. The main obstruction comes from studying smooth (differentiable) manifolds of dimensions k and $k+1$. After [1–3], a series of works of French mathematicians [4] appeared, the authors succeeded much more in the classification of a sphere to a sphere of smaller dimensions. The methods of the French school principally differ from those applied here.

Smooth manifolds are the main, and, perhaps, the only subject of this research, thus we completely devote Chapter I to them; in that chapter we investigate them more widely, which is necessary for future applications. Besides main definitions, chapter I contains a simpler (resp. Whitney [5]) proof of embeddability of a smooth n-dimensional manifold into $(2n+1)$-dimensional Euclidean space; we also state and investigate the question concerning singular points of smooth mappings from an n-dimensional manifold to the Euclidean space of dimension less than $2n+1$.

In Chapter II we describe the way of applying smooth manifolds for solutions of homotopy problems. First of all, we show that for the homotopy

[1] Л. С. Понтрягин, Гладкие многообразия и их применения в теории гомотопий, Москва, Наука, 1976. Translated by V.O.Manturov

classification of mappings from one manifold to another one may restrict only to the case of smooth mappings and smooth deformations. Later on, we describe our method of applying smooth manifolds to the homotopy classification of mappings from the sphere Σ^{n+k} to the sphere S^n, which goes as follows.

A smooth closed manifold k-dimensional manifold M^k lying in $(n+k)$-dimensional Euclidean space E^{n+k} is called *framed* if for any point $x \in M^k$ a system $U(x) = \{u_1(x), \ldots, u_n(x)\}$ of n linearly independent vectors orthogonal to M^k and smoothly depending on x is given; notation: (M^k, U). Compactifying the space E^{n+k} by the infinite point q', we get the sphere Σ^{n+k}. Let $e_1, \ldots, e_n$ be a system of linearly independent vectors tangent to the sphere $S^n \subset E^{n+1}$ in its north-pole p. It turns out that there exists a smooth mapping f from Σ^{n+k} to the sphere S^n such that $f^{-1}(p) = M^k$, whence the mapping f_x obtained by linearisation of f at $x \in M^k$ maps the vectors $u_1(x), \ldots, u_n(x)$, to $e_1, \ldots, e_n$, respectively. The homotopy type of the mapping f enjoying these properties is uniquely defined by the framed manifold (M^k, U). For each homotopy type of the mapping of Σ^{n+k} to the sphere S^n there exists such a framed manifold that the corresponding mapping belongs to the prescribed homotopy type. Two framed manifolds (M_0^k, U_0) and (M_1^k, U_1) define one and the same homotopy type of mapping from the sphere Σ^{n+k} to the sphere S^n, when they are *homologous* in the following sense. Let $E^{n+k} \times E^1$ be the direct product of the Euclidean space E^{n+k} by the line E^1 of variable t. We think of the framed manifold (M_0^k, U_0) lying in the space $E^{n+k} \times 0$, and the framed manifold (M_1^k, U_1) lying in $E^{n+k} \times 1$. The framed manifolds (M_0^k, U_0) and (M_1^k, U_1) are thought to be homologous if in the strip $0 \leqslant t \leqslant 1$ there exists a smooth framed manifold (M^{k+1}, U) with boundary consisting of M_0^k and M_1^k, whose framing U coincides with the framings U_0 and U_1 on the boundary components.

The described construction allows one to reduce the homotopy classification question for mappings $\Sigma^{n+k} \to \Sigma^n$ to the homology classification of framed k-dimensional manifolds. The role of k-dimensional and $(k+1)$-dimensional manifolds is clear here. The homology classification of zero-dimensional framed manifolds is trivial; thus one easily classifies mappings from Σ^n to the sphere S^n. The homology classification of one- and two-dimensional manifolds is also not very difficult, and it leads to the homotopy classification of mappings from Σ^{n+k} to S^n for $k = 1, 2$. We describe this question in chapter IV of the present work. The homology classification of three-dimensional framed manifolds meets significant difficulties. It was obtained by V. A. Rokhlin [3].

For realising the homology classification of smooth manifolds in the present work, we use homology invariants of these manifolds. With a framed submanifold (M^k, U) of the Euclidean space E^{n+k} we associate a homology invariant of it, being at the same type a homotopy invariant of the corresponding mapping of the sphere Σ^{n+k} to the sphere S^n. For $n = k+1$ there is the well-known Hopf invariant γ for mappings Σ^{2k+1} to S^{k+1}. The invariant γ can be easily interpreted as a homology invariant of the framed manifold. In chapter III we give a definition of the invariant γ based on the smooth manifold theory, and also give its interpretation as a homologous invariant of framed manifolds. For $k = 1$, the Hopf invariant is a classifying one; this fact is proved (in a known way) in chapter IV. In chapter IV for $k = 1, 2$; $n \geqslant 2$ we construct an invariant δ. This invariant is a residue class modulo 2. From its existence, one deduces that the number of mapping classes $\Sigma^{n+k} \to S^n$ for $k = 1, 2$; $n \geqslant 2$ is at least two. The uniqueness of this invariant for all cases except $k = 1$, $n = 2$ is based on the uniqueness of γ for $k = 1$.

Chapter I

Smooth manifolds and their maps

§ 1. Smooth manifolds

Below, we first give the definition of smooth (differentiable) manifold of finite class and simplest relevant notions; besides, we consider some smooth manifolds playing an important role, more precisely: submanifolds of smooth manifolds, manifold of linear elements of a smooth manifold, the Cartesian product of two manifolds and the manifold of vector subspaces of a given dimension for a given vector space. Together with finite differentiable manifolds one can also define infinitely differentiable manifolds, for which the functions in questions are infinitely differentiable and also analytic manifolds where all functions in questions are analytic. In the present paper, infinitely differentiable and analytic manifolds play no role; thus they are out of question.

The notion of smooth manifold

A) Let E^k be a Euclidean space of dimension k provided with Cartesian coordinates $x^1, \ldots, x^k$. By a *half-space* of the space E^k we mean the set E_0^k, defined by the condition

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$$x^1 \leqslant 0. \tag{1}$$

By a *boundary* of the half-space E_0^k we mean the hyperplane E^{k-1} defined as

$$x^1 = 0. \tag{2}$$

A *domain* of the half-space E_0^k is an open subspace of it (which might not be open for the whole space E^k). The points of the half-space E_0^k, belonging to the boundary E^{k-1} are called its *boundary points.* A Hausdorff topological space M^k with a finite base is a *topological manifold* if each point a of it admits a neighbourhood U^k homeomorphic to a domain W^k of the half-space E_0^k or of a space E^k. Obviously, each domain of the space E^k is homeomorphic to some domain of the half-space E_0^k, but for coordinate systems, it is more convenient to consider both domain types. If a point a corresponds to a boundary point of the domain W^k, then it is called a *boundary point* for the manifold M^k as well as for its neighbourhood U^k. It is known that the notion of boundary point is invariant. A manifold having boundary point is said to be a manifold *with boundary*, otherwise it is called a manifold *without boundary.* A compact manifold without boundary is said to be closed. It is easy to check that the set of all boundary points of a manifold M^k is a $(k-1)$-dimensional manifold.

Definition 1. Let M^k be a topological manifold of dimension k and let U^k be some neighbourhood (being a subset) of this manifold homeomorphic to a domain W^k of the half-space E_0^k or of a space E^k. Defining a homeomorphism between U^k and W^k is equivalent to providing a coordinate system $X = \{x^1, \ldots, x^k\}$ for U^k corresponding to the coordinate system of the Euclidean space E^k. Herewith, two different coordinate systems X and Y in U^k are always connected by a one-to-one continuous transformation

$$y^j = y^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, k. \tag{3}$$

Fix a positive integer m and assume that functions (3) are not just continuous, but also m times continuously differentiable in the domain U^k and the Jacobian $\left|\dfrac{\partial y^j}{\partial x^i}\right|$ is non-zero. With that, we say that the coordinate systems X and Y belong to the same *smoothness class* of order m. Obviously, different classes do not intersect and each class is defined by any coordinate system belonging to it. If there is a preassigned class, then the neighbourhood U^k is called m times *continuously differentiable.* Thus, two m times continuously differentiable neighbourhoods U^k, V^k of the manifold M^k always induce two coordinate classes for its intersection; if these classes coincide, we say that the neighbourhoods U^k and V^k are *compatibly* differentiable. If all neighbourhoods of some bases for a manifold M^k are m times continuously differentiable and the classes are mutually compatible, then the manifold M^k is called m times *continuously differentiable* or *smooth* of class m; sometimes we refer just to smooth manifold without indicating m which is always assumed to be sufficiently large for our purposes. [Analogously, if the functions (3) are analytic, then the manifold is called *analytic*.]

As seen from the definition given above, setting the differentiable structure for a manifold is obtained by setting some bases for any neighbourhood. If two bases for a manifold define two smooth structures, they are thought to be equivalent iff the union of these bases satisfies the condition 1. Indeed, to define a smooth structure for a manifold, one should define it for any neighbourhood of some covering of the manifold. Obviously, such a covering defines the topology of the manifold as well. If we restrict ourselves to connected neighbourhoods, which is always possible, then in each neighbourhoods all coordinate systems are split into two classes, such that the transformation (3) inside one class has a positive Jacobian. Each of these two classes is called an *orientation* of the given neighbourhood. Obviously, a smooth manifold is orientable if and only if there exists a compatible orientation for all neighbourhoods. With each such choice, one associates an orientation of the manifold.

B) The boundary M^{k-1} of a smooth manifold M^k is itself a smooth manifold of the same class; this results from the following construction. Let U^k be a neighbourhood in M^k provided with a fixed coordinate system X such that the intersection $U^{k-1} = U^k \cap M^{k-1}$ is non-empty. The equation defining the subset U^{k-1} in U^k, obviously, looks like $x^1 = 0$; thus it is natural to take $x^2, \ldots, x^k$ as preassigned coordinates in U^{k-1}. Let V^k be another neighbourhood in M^k (possibly, coinciding with U^k) with a fixed coordinate system Y for which the intersection $V^{k-1} = V^k \cap M^{k-1}$ is non-empty. For the common part of neighbourhood U^k and V^k we have

$$y^j = y^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, k, \tag{4}$$

from which at $x^1 = 0$ we obtain

$$y^j = y^j(0, x^2, \ldots, x^k), \qquad j = 2, \ldots, k. \tag{5}$$

From differentiability of relations (4) one obtains the differentiability of relation (5). Furthermore, from the relation $y^1(0, x^2, \ldots, x^k) = 0$ we get (for $U^{k-1} \cap V^{k-1}$)

$$\frac{\partial(y^1, \ldots, y^k)}{\partial(x^1, \ldots, x^k)} = \frac{\partial y^1}{\partial x^1} \frac{\partial(y^2, \ldots, y^k)}{\partial(x^2, \ldots, x^k)}; \tag{6}$$

herewith, since the left-hand side is non-zero, we get $\dfrac{\partial(y^2, \ldots, y^k)}{\partial(x^2, \ldots, x^k)} \neq 0$. If the system X is orienting for the neighbourhood U^k, then we may take $x^2, \ldots, x^k$ to be the orienting system for U^{k-1}. Because $\dfrac{\partial y^1}{\partial x^1} > 0$ then from positivity of $\dfrac{\partial(y^1, \ldots, y^k)}{\partial(x^1, \ldots, x^k)}$ we obtain the positivity of $\dfrac{\partial(y^2, \ldots, y^k)}{\partial(x^2, \ldots, x^k)}$. Thus, the boundary of a smooth orientable manifold gets a natural orientation.

C) Let a be a point of a smooth manifold M^k. Each coordinate system defined in a neighbourhood U^k of the point a belonging to the preassigned class is called a *local coordinate system* at the point a. Obviously, each point a of the manifold M^k can be treated as a base point of some local coordinate system. By a *vector* (countervariant) on the manifold M^k at a we mean a function associating with each local corrdinate system at a a system of k real numbers called *vector components* with respect to this coordinate system, in such a way that the components $u^1, \ldots, u^k$ and $v^1, \ldots, v^k$ of the same vector seen from two coordinate systems $x^1, \ldots, x^k$ and $y^1, \ldots, y^k$ are connected by the relation

$$v^j = \sum_{i=1}^{k} \frac{\partial y^j(a)}{\partial x^j} u^i. \tag{7}$$

Obviously, the vector is uniquely defined by its components given in one local coordinate systems. Defining linear operations over vectors as linear operations over their components, we define the k-dimensional vector space structure R^k_a on the set of all vectors on the manifold M^k at the point a; this space is called *tangent* to the manifold M^k at the point a. With each local coordinate system at the point a one associates a basis in the tangent space, where all vectors have the same components as with respect to the coordinate system. If a point a belongs to the boundary M^{k-1} of the manifold M^k, then besides the tangent space R^k_a, one also defines the space R^{k-1}_a tangent to the manifold M^{k-1}. Take the parameters $x^2, \ldots, x^k$ to be local coordinates for M^{k-1} (see sect. «B») and associate with the vector from R^{k-1}_a having components $u^2, \ldots, u^k$ the vector from R^{k-1}_a having components $0, u^2, \ldots, u^k$; thus we obtain a natural embedding of the space R^{k-1}_a to R^k_a.

Smooth mappings

D) Let M^k and N^l be two m-smooth manifolds and let φ be a continuous mapping of the first manifold to the second manifold. At the point $a \in M^k$, choose a local coordinate system X; at the point $b = \varphi(a) \in N^l$ choose a local coordinate system Y; then in the neighbourhood of the point a the mapping φ will look as

$$y^j = \varphi^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, l. \tag{8}$$

If the function φ is n times continuously differentiable, $n \leqslant m$, then it will be n times continuously differentiable for any other choice of local coordinates; thus, one may speak of the n-smoothness class of the mapping φ. Later on, while speaking of smooth mapping, we shall always assume that n is sufficiently large. If the rank of the matrix $\left\| \dfrac{\partial \varphi^j}{\partial x^i} \right\|$ at the point a equals k, then the mapping φ is called *regular* at a. It is easy to see that if the point a belongs to the boundary M^{k-1} of the manifold M^k, then from the regularity of the mapping φ at a follows its regularity at the point a of the manifold M^{k-1}. If the mapping φ is regular at each point $a \in M^k$, then it is called *regular*. It is easy to check that if the mapping φ is regular at a, then it is regular and homeomorphic in some neighbourhood of the point a. A regular homeomorphic mapping is called a *smooth embedding*. The mapping φ is called *proper* at the point $a \in M^k$, if the rank of the matrix $\left\| \dfrac{\partial \varphi^j}{\partial x^i} \right\|$, $j = 1, \ldots, l$; $i = 1, \ldots, k$, equals l. It is easy to see that the set of all nonproper points of the mapping φ is closed in M^k. A point $b \in N^i$ is called *proper* for the mapping φ if the mapping φ is proper at any point of the set $\varphi^{-1}(b) \subset M^k$. The point a is a *singular* point of the mapping f if it is non-regular and nonproper at the same time, i.e. if the rank of the matrix $\left\| \dfrac{\partial \varphi^j}{\partial x^i} \right\|$, $j = 1, \ldots, l$; $i = 1, \ldots, k$, is less than any of k and l.

E) Each smooth mapping φ of a smooth manifold M^k to a smooth manifold N^l induces at each point $a \in M^k$ a linear mapping φ_a of the vector space R^k_a tangent to the manifold M^k at a, to the vector space R^l_0 tangent to N^l at $b = \varphi(a)$. Namely, if the local coordinate systems at points a and b, are X and Y, respectively, then to the vector $u \in R^k_a$ with components $u^1, \ldots, u^k$ in the system X one associates the vector $v = \varphi_a(u) \in R^l_0$ with components

$$v^j = \sum_{i=1}^{k} \frac{\partial \varphi^j(a)}{\partial x^i} u^i, \qquad j = 1, \ldots, l, \tag{9}$$

in the coordinate system Y. It is not easy to see that this correspondence is well defined, i.e. for any choice of local coordinate it results in one and the same mapping φ_a. If the mapping φ is regular at a, then the mapping

φ_a is one-to-one and defines an embedding of the same R_a^k into R_b^l. If φ is proper at a, then $\varphi_a(R_a^k) = R_b^l$.

Definition 2. A smooth mapping φ of class n from an m-smooth manifold M^k onto the smooth m-manifold N^k, $m \geqslant n$, is called *smooth homeomorphism* if it is regular. Obviously, if the mapping φ is a smooth homeomorphism of class n then the inverse mapping φ^{-1} is also a smooth homeomorphism of class n. Two manifolds are called *smoothly isomorphic* if there exists a smooth homeomorphism from one manifold onto the other.

Certain ways of constructing smooth manifolds

F) Let P^r be a subset of a smooth manifold M^k of class m, defined in the neighbourhood of any point belonging to it by a system of k—r independent equation. This means that for each point $a \in P^r$ there exists a neighbourhood U^k in the manifold M^k with local system X that the intersection $P^r \cap U^k$ consists of all points with coordinates satisfying the equations

$$\psi^j(x^1, \ldots, x^k) = 0, \qquad j = 1, \ldots, k - r. \tag{10}$$

Herewith we assume that the function ψ^j is m times smoothly differentiable and the functional matrix $\left\| \frac{\partial \psi^j(a)}{\partial x^i} \right\|$, $j = 1, \ldots, k - r;\ i = 1, \ldots, k$, has rank $k-r$; if a is a boundary point of the manifold M^k then we assume that the reduced functional matrix $\left\| \frac{\partial \psi^j(a)}{\partial x^i} \right\|$, $j = 1, \ldots, k - r;\ i = 2, \ldots, k$ has rank $k - l$. With the conditions above, the set P^r turns out to have a natural smooth r-dimensional m-smooth manifold structure; this manifold is smoothly embedded into M^k. Such a manifold P^r is called a *submanifold* of the manifold M^k. Furthermore, it turns out that the boundaries P^{r-1} and M^{k-1} of the manifolds P^r and M^k enjoy the relation

$$P^{r-1} = P^r \cap M^{k-1}, \tag{11}$$

and if $a \in P^{r-1}$ and R_a^k, R_a^{k-1}, R_a^r, R_a^{r-1} are tangent spaces to the manifolds M^k, M^{k-1}, P^r, P^{r-1} at the point a, then

$$R_a^{r-1} = R_a^r \cap R_a^{k-1}. \tag{12}$$

Here the spaces R_a^{k-1}, R_a^r, R_a^{r-1} are considered as subspaces of R_a^k (see «C» and «E»).

To prove that P^r is an r-dimensional manifold and to define the differentiable structure on it, we change, if necessary, the enumeration of coordinate for the Jacobian $\left| \frac{\partial \psi^j(a)}{\partial x^i} \right|$, $j = 1, \ldots, k - r;\ i = r + 1, \ldots, k$ to be non-zero; in the case of boundary point we may not change the number of the coordinate x^1. Then the system (10) will be uniquely resolvable in variables $x^1, \ldots, x^k$:

$$x^i = f^i(x^1, \ldots, x^r), \qquad i = r+1, \ldots, k. \tag{13}$$

In the case of boundary point, the coordinate x^1 is not among the independent variables. The functions f^i are defined, m times continuously differentiable in some domain W^r of the half-space E_0^r in variables $x^1, \ldots, x^r$ and define a homeomorphic mapping of this domain onto some neighbourhood U^r of the point a in P^r. Thus we have proved that P^r is an r-dimensional manifold. The differentiability for the neighbourhood U^r is defined by coordinates $x^1, \ldots, x^r$.

The natural inclusion of the manifold P^r in the manifold M^k is given in U^r by relations

$$\begin{aligned} x^i &= x^i, & i &= 1, \ldots, r; \\ x^i &= f^i(x^1, \ldots, x^r), & i &= r+1, \ldots, k, \end{aligned} \tag{14}$$

where the parameters $x^1, \ldots, x^r$ on the right-hand sides are thought to be coordinates in U^r and the parameters $x^1, \ldots, x^r$ on the left-hand side be the coordinates in U^k. The relation (11) is evident. Now, let $a \in P^{r-1}$; let us prove the relation (12). To local coordinates X, there correspond a certain basis $e_1, \ldots, e_k$ in R_a^k; the basis of the space R_a^{k-1} consists of vectors $e_2, \ldots, e_k$; the basis of the space R_a^k consists of vectors $e_i + \sum_{j=r+1}^k \dfrac{\partial f^j}{\partial x^i} e_j$, $i = 1, \ldots, r$; finally, the basis of the space R_a^{r-1} consists of the same vectors except for the first one. Considering these bases, we easily get to the relation (12).

To prove the compatibility of the coordinate systems we constructed for P^r consider together with the point a, another point $b \in P^r$ with local coordinates Y and neighbourhoods V^k and V^r analogous to the neighbourhoods U^k and U^r. The relations analogous to (13), will look like

$$y^i = g^i(y^1, \ldots, y^r), \qquad i = r+1, \ldots, k. \tag{15}$$

Suppose that U^r and V^r have a non-empty intersection. Then U^k and V^k also have a non-empty intersection; let

$$y^i = y^i(x^1, \ldots, x^k), \qquad i = 1, \ldots, k; \tag{16}$$

$$x^i = x^i(y^1, \ldots, y^k), \qquad i = 1, \ldots, k, \tag{17}$$

be the coordinate changes from X and Y and back. Substituting $x^{r+1}, \ldots, x^k$ from (13) for (16), we get for the first r variables y

$$y^i = \overset{*}{y}{}^i(x^1, \ldots, x^r), \qquad i = 1, \ldots, r. \tag{18}$$

In the same way substituting $y^{r+1}, \ldots, y^k$ from (15) for (17) we get

$$x^i = \overset{*}{x}{}^i(y^1, \ldots, y^r), \qquad i = 1, \ldots, r. \tag{19}$$

The coordinate changes (18) and (19) are m times continuously differentiable; since they are inverse to each other, their Jacobians are both nonzero.

This completes the proof of Statement «F».

G) Let M^k be a smooth manifold of class $m \geqslant 2$ and let L^{2k} be the set of all tangent vectors to it (see «C»), i.e. pairs of type (a, u), where $a \in M^k$, $u \in R^k_a$. The set L^{2k} naturally turns out to be a $2k$-dimensional manifold of class $m-1$ according to the following construction. Let U^k be a certain neighbourhood in the manifold M^k with local coordinate system X. By U^{2k}, denote the set of all pairs $(x, u) \in L^{2k}$ satisfying the condition $x \in U^k$. Take the set U^{2k} to be the neighbourhood in L^{2k}; the fixed coordinate system in it is constructed as follows. Let $x^1, \ldots, x^k$ be the coordinates of the point x in the system X and let $u^1, \ldots, u^k$ be the components of the vector u in the local coordinate system X; then the coordinates of the pair (x, u) are defined to be the numbers

$$x^1, \ldots, x^k,\ u^1, \ldots, u^k. \tag{20}$$

If V^k is a neighbourhood in M^k (possibly coinciding with U^k) with a fixed system Y, for which $x \in V^k$ and the coordinates of the pair (x, u) in the neighbourhood V^{2k} defined by Y are

$$y^1, \ldots, y^k,\ v^1, \ldots, v^k, \tag{21}$$

then the coordinate change from (20) to (21) is, evidently, given by the relation

$$y^j = y^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, k; \tag{22}$$

$$v^j = \sum_{i=1}^{k} \frac{\partial y^j}{\partial x^i} u^i, \qquad j = 1, \ldots, k \tag{23}$$

[see (9)]. These relations are $m-1$ times differentiable and have the Jacobian equal to $\left| \frac{\partial y^j}{\partial x^i} \right|^2$; this Jacobian is, evidently, positive. Since the neighbourhoods of type U^{2k} cover L^{2k}, the described construction turns L^{2k} into a smooth manifold of class $m-1$.

H) Let R^k be a vector space of dimension k. By a ray u^* in R^k passing through the vector $u \neq 0$ we mean the set of all vectors tu where t is some positive real number. Fix some basis for R^k and denote by R_i^{k-1} the coordinate hyperplane $u^i = 0$. If the ray u^* does not lie in R_i^{k-1}, then there exists a unique vector u on it satisfying the condition $|u^i| = 1$; call this vector the *basic* vector with respect to the plane R_i^{k-1}. The set of all rays for which the basic vector with respect to R_i^{k-1} satisfies $u^i = +1$ or $u^i = -1$, denote by U_{i1}^{k-1} or, by U_{i2}^{k-1}, respectively. For coordinates of the ray $u^* \in U_{ip}^{k-1}$, $p = 1, 2$, we take the components $u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^k$ of the basic vector u of this ray with respect to R_i^{k-1}. Since the system of all sets U_{ip}^{k-1} covers the set S^{k-1} of all rays, the set S^{k-1} becomes a smooth manifold evidently homeomorphic to the $(r-1)$-sphere.

I) Let M^k be a smooth manifold of class m. *Linear element manifold* of it is the set L^{2k-1} of all pairs (x, u^*), where $x \in M^k$, and u^* is a ray in R_x^k; the natural differential structure is defined according to the following construction. Let U^k be a neighbourhood in M^k with a fixed system X. In the vector space R_x^k tangent to M^k at $x \in U^k$ we have a basis corresponding to the local system X; thus, in the set S_x^{k-1} of rays of the space R_x^k we have domains $U_{ip,x}^{k-1}$ (see «H») endowed with coordinate systems. By U_{ip}^{2k-1} denote the set of all pairs (x, u^*) satisfying the condition $x \in U^k$, $u^* \in U_{ip,x}^{k-1}$, where the coordinates of the pair (x, u^*) in U_{ip}^{2k-1} are taken to be the numbers

$$x^1, \ldots, x^k,\ u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^k, \tag{24}$$

where $x^1, \ldots, x^k$ are the coordinates of x in the system X, and $u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^k$ are the coordinates of the ray u^* in $U_{ip,x}^{k-1}$. It can be easily checked that the system of neighbourhoods U_{ip}^{2k-1} covers L^{2k-1} and that the introduced coordinate systems are compatible with each other; thus L^{2k-1} is a $(2k-1)$-dimensional smooth manifold of class $m-1$.

J) Let M^k and N^l be two smooth manifolds of class m; suppose M^k has empty boundary. The direct product (Cartesian product) $P^{k+l} = M^k \times N^l$, i.e. the set of all pairs (x, y), where $x \in M^k$, $y \in N^l$, is naturally a smooth manifold of class m according to the following construction. Let U^k and V^l be arbitrary coordinate neighbourhoods in the manifolds M^k and N^l with coordinate systems X and Y. Consider the set $U^k \times V^l \subset M^k \times N^l$ as the coordinate neighbourhood in the manifold P^{k+1}: here the coordinates of the point $(x, y) \in U^k \times V^l$ are set to be the numbers $x^1, \ldots, x^k, y^1, \ldots, y^l$, where $x^1, \ldots, x^k$ are the coordinates of the point x

in the system X and $y^1, \ldots, y^l$ are the coordinates of y with respect to Y. It follows from a straightforward check that the coordinate neighbourhood system constructed above defines in P^{k+l} a smooth structure of class m. If M^k and N^l are orientable manifolds and the systems X and Y correspond to the orientations of these manifolds, we define the orientation of P^{k+l} by the system X, Y. Herewith, the Cartesian product acquires a natural orientation. If N^{l-1} is the boundary of the manifold N^l, then the boundary of the manifold $M^k \times M^l$ turns out to be $M^k \times M^{l-1}$.

K) Let E^{k+l} be a vector space of dimension $k+l$ and let $G(k,l)$ be the set of all k-dimensional vector subspaces of it. The set $G(k,l)$ is a smooth (even analytic) manifold with respect to the following construction. Let $E_0^k \in G(k,l)$ and let $e_1, \ldots, e_k, f_1, \ldots, f_l$ be a basis of the space E^{k+l} such that the vectors $e_1, \ldots, e_k$ lie in E_0^k. Denote the linear span of vectors $f_1, \ldots, f_l$ by E^l. Denote by U^{kl} the set of all vector subspaces $E^k \in G(k,l)$ the intersection of which with E^l consists of only the origin of coordinates. If $E^k \in U^{kl}$ then there exists a basis $e'_1, \ldots, e'_k$ of the vector space E^k defined by the relations

$$e'_i = e_i + \sum_{j=1}^{i} x_i^j f_j, \qquad i = 1, \ldots, k,$$

where $\left\| x_i^j \right\|$ is a real number matrix. Consider the elements x_i^j, $i = 1, \ldots, k$, $j = 1, \ldots, l$, of this matrix as coordinates of the element E^k in the coordinate neighbourhood U^{kl}. It can be checked straightforwardly that the set of coordinate neighbourhoods of type U^{kl} defines an analytic structure in $G(k,l)$; this $G(k,l)$ is an analytic manifold of dimension kl.

§ 2. Embedding of a manifold into Euclidean space

In the present subsection we show that any compact k-dimensional smooth manifold of class $m \geqslant 2$ can be regularly homeomorphically mapped into the Euclidean space R^{2k+1} of dimension $2k+1$ and can be regularly mapped into R^{2k}; here the smoothness class of these mappings equals m. These statements in a stronger form, i.e. for $m \geqslant 1$ and without compactness assumptions, were proved by Whitney [5]; the proof given below is somewhat easier.

In the proof, we shall rely on the following quite elementary Theorem 1.

Smooth mapping of a manifold to a manifold of larger dimension

Theorem 1. *Let M^k and N^l be two smooth manifolds of dimensions k and l, respectively, where $k < l$, and let φ be a smooth mapping of class 1 of the manifold M^k to the manifold N^l. It turns out that the set $\varphi(M^k)$ has the first category in N^1, i.e. it can be represented as a sum of countably many nowhere dense sets in N^l. In particular, if the manifold M^k is compact, then the set $\varphi(M^k)$ is compact as well; thus $N^l \setminus \varphi(M^k)$ is a domain everywhere dense in N^l.*

Proof. Suppose $a \in M^k$, let $b = \varphi(a)$, V_b^i be some coordinate neighbourhood of the point b in N^l and let U_a^k be such a coordinate neighbourhood of the point a in M^k that $\varphi(U_a^k) \subset V_b^l$. Choose neighbourhoods U_{a1}^k and U_{a2}^k of the point a in M^k such that $\overline{U}_{a1}^k \subset U_a^k$, $\overline{U}_{a2}^k \subset U_{a1}^k$ and such that the set $\overline{U}_{a1}^k$ is compact. The domains $\overline{U}_{a2}^k$, $a \in M^k$, cover the manifold M^k. From this cover, one can take a countable subcover; thus, in order to prove the theorem it suffices to show that for any arbitrary choice of the point a from M^k, the set $\varphi(\overline{U}_{a2}^k)$ is nowhere dense in V_b^l. Since the domain U_a^k is the homeomorphic image of a domain of the Euclidean subspace E_0^k, we shall assume that U_{a2}^k itself is a domain of the subspace E_0^k. In the same way, we assume that V_b^l is a domain of the Euclidean subspace E_0^l. Thus the mapping φ can be treated as a smooth mapping of class 1 from a domain U_a^k to the Euclidean space E^l; thus it suffices to show that the set $\varphi(\overline{U}_{a2}^k)$ is nowhere dense in E^l. Let us prove this.

The smoothness of φ and compactness of the mapping $\overline{U}_{a1}^k$ result in the existence of a positive constant c such that for any two arbitrary points x and x' from $\overline{U}_{a1}^k$, the inequality

$$\rho(\varphi(x), \varphi(x')) < c\rho(x, x') \tag{1}$$

holds. Chose some ε-cubature of the Euclidean subspace E_0^k, i.e. tile the subspace E_0^k into right-angled cubes with edge ε. Denote the set of all cubes intersecting $\overline{U}_{a2}^k$ by Ω. As the set $\overline{U}_{a2}^k$ is compact and hence is bounded by a rather large cube, the number of cubes in Ω does not exceed c_1/ε^k where c_1 is some positive constant independent of ε. Let δ be the distance between the sets $E_0^k \setminus U_{a1}^k$ and $\overline{U}_{a2}^k$. Suppose that the diagonal length $\varepsilon\sqrt{k}$ of each cube from Ω is less than δ. Then each cube K_i from Ω lies in the domain U_{a1}^k and, by virtue of (1), the set $\varphi(K_i)$ is contained in some cube L_i of the space E^l with edge length $c\sqrt{k} \cdot \varepsilon$; the volume of the latter cube equals $c^l k^{l/2} \cdot \varepsilon^{l-k}$. Thus the whole set $\varphi(\overline{U}_{a2}^k)$ is contained in the union of cubes L_i, whose number does not exceed c_1/ε^k; thus the total volume of the set $\varphi(\overline{U}^k_{a2})$ does not exceed the number $c_1 c^l k^{l/2} \cdot \varepsilon^{l-k}$. Since ε is chosen arbitrarily small, from above it follows that the set $\varphi(\overline{U}^k_{a2})$ does not contain any domain and, being compact, it should be nowhere dense in E^l.

Thus, Theorem 1 is proved.

The projection operation in the Euclidean space

Later on, the projection operation will play a key role. Let C^r be a vector space and let B^q be its vector subspace. Regarding the space C^r as an additive group and the space B^q as a subgroup of it, we obtain a tiling of the space C^r into conjugacy classes according to B^q; these conjugacy classes form a vector space A^p of dimension $p = r - q$. Associating with any element $x \in C^r$ the corresponding conjugacy class $\pi(x) \in A^p$, we get a linear mapping π of the space C^r onto the space A^p called the *projection* along the projecting subspace B^q. More intuitively, the space A^p can be realized as a linear subspace of dimension p of the space C^r intersecting the space B^q only in the origin; then the operation π is just the original projection. If the space C^r is Euclidean, then defining B^q to be the orthogonal complement to the given space $A^p \subset C^r$, we get an orthogonal projection π of the space C^r to the subspace A^p.

A) Let φ be a smooth mapping of a smooth manifold M^k to some vector space C^r regular at a point $a \in M^k$, and let π be the projection of the space C^r along the one-dimensional subspace B^1 to the space A^{r-1}. It turns out that the mapping $\pi\varphi$ from M^k to A^{r-1} is not regular at a (see § 1, «D») if and only if the line $\varphi(a) + B^1$ passing through $\varphi(a)$ parallel to B^1 is tangent to $\varphi(M^k)$ at the point $\varphi(a)$.

To prove this, choose some local coordinates $x^1, \ldots, x^k$ in the neighbourhood of a; endow C^r with rectilinear coordinates $y^1, \ldots, y^r$ such that the last axis coincides with B^1. In the chosen coordinate system, the mapping φ looks like: $y^j = \varphi(x^1, \ldots, x^k)$, $j = 1, \ldots, r$, where the rank of the matrix $\left\| \dfrac{\partial \varphi^j}{\partial x^i} \right\|$, $j = 1, \ldots, r$; $i = 1, \ldots, k$, at the point a, is, by regularity assumption, equal to k. With each vector u on M^k at a one associates the vector $v = \varphi_a(u) \in C^r$, which is tangent to $\varphi(M^k)$ at the point $\varphi(a)$ and has components $v^1, \ldots, v^r$ [i.e. defined by relations (9) § 1, $l = r$]. Now, if the mapping $\pi\varphi$ is not regular at the point a, then the rank of the matrix $\left\| \dfrac{\partial \varphi^j}{\partial x^i} \right\|$, $j = 1, \ldots, r-1$, $i = 1, \ldots, k$, is less than k; thus there exists a vector $u \neq 0$ such that for the vector $v = \varphi_a(u)$ we have $v^1 = \ldots = v^{r-1} = 0$, $v^r \neq 0$; the latter means that $v \in B^1$. If, on the contrary, there exists a vector $v = \varphi_a(u) \neq 0$ belonging to B^1 then the rank of the matrix $\left\|\dfrac{\partial\varphi^j}{\partial x^i}\right\|$, $j = 1, \ldots, r-1$; $i = 1, \ldots, k$ is less than k, i.e. the mapping $\pi\varphi$ is not regular in a.

B) Let φ be a smooth regular mapping of class 2 from a smooth manifold M^k to the vector space C^r of dimension $r > 2k$, and let $B^q \in \in G(q, r-q)$ be the subspace of dimension $q \leqslant r - 2k$ of projection for C^r onto the space A^p. Denote the projection by π. By Ω'_q we denote the set of all such projecting spaces B^q for which the mapping $\pi\varphi$ is not regular. It turns out that the set Ω'_q has first category in the manifold $G(q, r-q)$ of all projecting directions.

Let (x, u^*) be an arbitrary linear element of the manifold M^k (see § 1, «I») and let u be some non-zero vector of the ray u^*. To the vector u, according to (9) § 1, there corresponds the vector $v = \varphi_x(u) \neq 0$. The ray v^* of the space C^r defined by the vector v depends only on the linear element (x, u^*), and we set $v^* = \Phi(x, u^*)$. It can be easily checked that the mapping Φ from the manifold L^{2k-1} (see § 1, «I») to the manifold S^{r-1} (see § 1, «H») has smoothness class one, thus $\Phi(L^{2k-1})$ is of first category in S^{r-1} (since $r - 1 > 2k - 1$, see Theorem 1). Thus, by virtue of «A», we get «B» for $q = 1$.

Applying this construction consequently, we get the proof of the statement «B» for any arbitrary $q \leqslant r - 2k$.

C) Let φ be a smooth of class one one-to-one mapping from the smooth manifold M^k to the vector space C^r and let $B^q \in G(q, r-q)$ be the projecting subspace of the dimension $q \leqslant r - 2k - 1$. Denote the projection by π. By Ω''_q, denote the set of all projecting subspaces B^q such that the mapping $\pi\varphi$ is not one-to-one. It turns out that Ω''_q has first category in the manifold $G(q, r-q)$.

Let x and y be two arbitrary different points of the manifolds M^k. By $\Phi'(x, y)$ denote the ray consisting of all vectors of the type $t(\varphi(y) - \varphi(x))$, where t is a positive number. Thus we get a mapping Φ' from the manifold M^{2k} of all ordered pairs (x, y), $x \neq y$, to the manifold S^{r-1} of all rays of the space C^r. In the manifold M^{2k} one naturally introduces differentiability, and it can be easily checked that the mapping Φ' is smooth of class 1. Thus, $\Phi'(M^{2k})$ turns out to be of first category in S^{r-1} (see Theorem 1), from which follows «C» for $q = 1$. Applying this construction consequently we get the proof of «C» for arbitrary $q \leqslant r - 2k - 1$.

From «B» and «C» one straightforwardly gets

D) Let φ be a smooth one-to-one regular mapping of class 2 of a smooth manifold M^k to the vector space C^r and let $B^q \in G(q, r-q)$ be the projecting space of dimension $q \leqslant r - 2k - 1$. Denote the projection mapping by π, and denote by Ω_q the set of all projecting spaces B^q such that the mapping $\pi\varphi$ is not one-to-one and regular. Since $\Omega_q = \Omega'_q \cup \Omega''_q$ Ω_q has first category in the manifold $G(q, r-q)$.

The embedding theorem

E) Let $\varphi_1, \ldots, \varphi_n$ be smooth (of class m) mappings of the smooth manifold M^k to vector spaces $C_1, \ldots, C_n$, respectively. Denote by C the direct sum of the spaces $C_1, \ldots, C_n$ consisting of all systems $[u_1, \ldots, u_n]$, with $u_i \in C_i$. Define the direct sum φ of mappings $\varphi_1, \ldots, \varphi_n$ by $\varphi(x) = [\varphi_1(x), \ldots, \varphi_n(x)]$, $x \in M^k$. It is easy to see that φ is an m-smooth mapping of the manifold M^k to C. It can be easily checked that if at least one mapping $\varphi_1, \ldots, \varphi_n$ is regular in $a \in M^k$ then so is φ. Furthermore, it can be easily checked that if two points a and b from M^k are mapped to different points by one of the mappings $\varphi_1, \ldots, \varphi_n$ then they have different images under φ.

Theorem 2. *Let M^k be a smooth compact manifold of class $m \geqslant 2$. There exists a smooth embedding of class of the manifold M^k into a finite-dimensional Euclidean space.*

Proof. Denote by $\varkappa(t)$ some real function in the real variable t, which is infinitely differentiable and satisfies the following properties:

$$\varkappa(t) = 1 \text{ for } |t| \leqslant 1/2; \qquad \varkappa(t) = 0 \text{ for } |t| \geqslant 1;$$

for $-1 \leqslant t \leqslant -1/2$ the function $\varkappa(t)$ monotonously increases; for $1/2 \leqslant t \leqslant 1$, the function $\varkappa(t)$ monotonously decreases. Such a function can be easily constructed.

Set

$$\varkappa^i(t^1, t^2, \ldots, t^k) = t^i \cdot \varkappa(t^1) \cdot \varkappa(t^2) \ldots \varkappa(t^k),$$

for $i = 1, \ldots, k$ and

$$\varkappa^{k+1}(t^1, t^2, \ldots, t^k) = \varkappa(t^1) \cdot \varkappa(t^2) \ldots \varkappa(t^k).$$

Let R^k be the Euclidean space with Cartesian coordinates $t^1, \ldots, t^k$ and let R^{k+1} be the Euclidean space with Cartesian coordinates $y^1, \ldots, y^{k+1}$. Denote by Q the cube in the space R^k defined by the inequalities $|t^i| < 2$, denote by Q' the cube of the same space defined by the inequalities $|t^i| < 1$ and by Q'' the cube defined as $|t^i| < 1/2$. By Q_0 we denote the half-cube cut out from the cube Q by the inequality $t^1 \leqslant 0$. Now, define the mapping from R^k to the space R^{k+1} by the relations

$$y^j = \varkappa^j(t^1, t^2, \ldots, t^k), \qquad j = 1, \ldots, k+1. \tag{2}$$

It can be easily checked that this mapping is infinitely differentiable, maps the set $R^k \setminus Q'$ to the coordinate origin of the space R^{k+1}, its restriction to the cube Q' is a continuous and one-to-one mapping and its restriction to the cube Q'' is regular.

Now, let a be an arbitrary point of M^k and U_a^k be some coordinate neighbourhood of it endowed with a coordinate system X having origin at a; finally, let ε be a small positive number such that under the mapping

$$t^i = \frac{x^i}{\varepsilon}, \qquad i = 1, \ldots, k, \tag{3}$$

of the neighbourhood U_a^k to the space R^k the image of this neighbourhood covers the whole cube Q, whence a is an interior point of M^k or the whole half-cube Q_0, whence a is a boundary point of M^k. Denote the pre-images of the cubes Q' and Q'' under this mapping by Q'_a and Q''_a, respectively.

Define the mapping φ_a of the manifold M^k to the Euclidean space R^{k+1} by

$$y^j = \varkappa^j\left(\frac{x^1}{\varepsilon}, \frac{x^2}{\varepsilon}, \ldots, \frac{x^k}{\varepsilon}\right)$$

for the point $x \in U_a^k$ with coordinates $x^1, \ldots, x^k$ and by $y^j = 0$ for the point $x \in \in M^k \setminus U_a^k$. It can be easily checked that φ_a is an m-smooth mapping of M^k to R^{k+1}, which is homeomorphic on Q'_a and regular on Q''_a.

Selecting among neighbourhoods Q''_a a finite cover $Q''_{a_1}, \ldots, Q''_{a_n}$ of the manifold M^k and taking the direct sum of mappings corresponding to these cubes, $\varphi_{a_1}, \ldots, \varphi_{a_n}$ (see «E»), we get the desired mapping φ of the manifold M^k to a finite-dimensional Euclidean space.

From the statements proved above the theorem formulated earlier, follows straightforwardly. Indeed, the manifold M^k can be regularly and homeomorphically embedded into a vector space C of rather high dimension (see Theorem 2). Furthermore, in the space C^r there exists such a projecting direction B^{r-2k-1}, such that the obtained projection of the manifold M^k to the space A^{2k+1} is regular and homeomorphic (see «D»). In the same way, in the space C^r there exists a projecting direction B^{r-2k} such that the projection of the manifold M^k to the space A^{2k} is regular (see «B»). Below we prove a stronger Theorem 3 showing that for any smooth mapping of a manifold M^k to a Euclidean space C^{2k+1} there exists an arbitrarily close regular and homeomorphic mapping of the same manifold, and for any smooth mapping of M^k to the Euclidean space C^{2k} there exists an arbitrarily close regular mapping. For the precise formulation of Theorem 3, one needs to introduce the notion of m-neighbourhood for mappings, taking into account all derivatives up to order m, inclusively.

First note that if f is a smooth mapping of the domain W^k of the

Euclidean half-space E_0^k to a vector space C^r then the partial derivatives of the vector function $f(x) = f(x^1, \ldots, x^k)$ are vectors of the space C^r.

F) Let M^k be an m-smooth compact manifold and E^l be a vector space, P be the set of all m-smooth mappings of the manifold M^k to the space E^l. Introduce the topology for P by setting a metric depending on an arbitrary choice of some constructed elements. Let $U_s, V_s, s = 1, \ldots, n$, be a finite set of coordinate domains of the manifold M^k such that the domains $U_s,\ s = 1, \ldots, n$, cover M^k and the inclusions $\overline{U}_s \subset V_s,\ s = 1, \ldots, n$ hold, wherever in each domain V_s a preassigned coordinate system X_s is chosen. Furthermore, let Y be a Cartesian coordinate system of the space E^l. Define the distance $\rho(f, g)$ between two mappings f and g from P (depending on the choice of U_s, V_s, coordinate systems $X_s,\ s = 1, \ldots, n$, and the coordinate system Y). To do this, let us write the mappings f and g of the domain V_s in coordinate form by setting

$$y^j = f_s^j(x) = f_s^j(x^1, \ldots, x^k), \tag{4}$$

$$y^j = g_s^j(x) = g_s^j(x^1, \ldots, x^k). \tag{5}$$

Let $i_1, \ldots, i_k$ be a set of non-negative integers with sum not exceeding m. Set

$$\omega_s^j(x; i_1, \ldots, i_k) = \left| \frac{\partial^{i_1 + \ldots + i_k}(f_s^j(x) - g_s^j(x))}{(\partial x^1)^{i_1} \ldots (\partial x^k)^{i_k}} \right| .$$

Denote the maximum of the function $\omega_s^j(x; i_1, \ldots, i_k)$ in the variable x at $x \in \overline{U}_s$ by $\omega_s^j(i_1, \ldots, i_k)$, and define the distance $\rho(f, g)$ between f and g to be the supremum of all numbers $\omega_s^j(i_1, \ldots, i_k)$, where $i_1, \ldots, i_k$, s, j run over all admissible values. It can be easily checked that the topology of the space P does not depend on the arbitrary choice of the system of $U_s,\ V_s,\ s = 1, \ldots, n$, and coordinate systems $X_s,\ s = 1, \ldots, n,\ Y$. The topological space P is called the *class m mapping space* of the manifold M^k to the space E^l. The statement that for the map f there is an arbitrary close map enjoying some property A means that in any neighbourhood of the point f in the space P there exists a map enjoying the property A.

Theorem 3. *Let M^k be a class $m \geqslant 2$ smooth k-dimensional compact manifold, let A^p be a vector space of dimension p and let P be the class m mapping space from the manifold M^k to the space A^p. The set of all regular mappings from the set P denoted by Π'; denote the set of all regular and homeomorphic maps belonging to P by Π. It turns out that the sets Π' and Π are domains in the space P. Furthermore, if $p \geqslant 2k$ then the domain Π' is everywhere dense in P and if $p \geqslant 2k + 1$ then the domain Π is everywhere dense in P.*

Proof. First, show that the sets Π' and Π are everywhere dense in the space P for the values of p indicated in the theorem. Let $f \in P$ and let e be a class m regular and homeomorphic mapping of the manifold M^k to a vector space B^q of sufficiently large dimension (see Theorem 2). Denote the direct sum of the vector spaces A^p and B^q by C^r; here we assume the spaces A^p and B^q to be linear subspaces of the space C^r. The mapping h, being a direct sum of the mappings f and e (see «E») is regular and homeomorphic, and its projection to A^p along B^q coincides with the mapping f. By virtue of statements «B» and «D», in any neighbourhood of the projecting direction B^q there exists a projecting direction B_1^q such that the projection g of the mapping h is regular if $p \geqslant 2k$; it is regular and homeomorphic if $p \geqslant 2k+1$. Thus, for a given map f there exists an arbitrarily close map g enjoying the desired properties.

Let us show that Π' is a domain. Let $f \in \Pi'$. Since the mapping f is regular at $x \in U_s$ the rank of the matrix $\left\| \dfrac{\partial f_s^j}{\partial x^i} \right\|$ at this point equals k (see § 1, «F»). Consequently, the rank of a matrix close to the matrix $\left\| \dfrac{\partial f_s^j}{\partial x^i} \right\|$ also equals k. Thus, there exists such a small positive number ε' such that for $\rho(f,g) < \varepsilon'$ the mapping g is regular at the point x. Since the first derivatives of the functions $f_s^j(x)$ are continuous and the sets $\overline{U}_s$ are compact and one can choose a finite number of them to cover M^k, there exists a small positive number ε such that for $\rho(f,g) < \varepsilon$, the mapping g is regular at each point $x \in M^k$.

To prove that Π is a domain, first note the following:

a) In the set Q of all linear mappings of the Euclidean vector space E^k to the Euclidean vector space A^p, let us introduce the metrics according to some coordinate systems X and Y in these spaces. Let φ and ψ be elements from Q written in coordinates as

$$y^j = \sum_{i=1}^{k} \varphi_i^j x^i, \qquad j = 1, \ldots, p;$$

$$y^j = \sum_{i=1}^{k} \psi_i^j x^i, \qquad j = 1, \ldots, p.$$

Define the distance $\rho(\varphi, \psi)$ as the maximum of $|\varphi_j^i - \psi_j^i|$. It turns out that for any compact set F of non-degenerate mappings there exists a small positive δ such that for $\rho(F, \psi) < \delta$ we have

$$|\psi(x)| > \delta \cdot |x|,$$

where x is an arbitrary vector from E^k.

Taking into account the continuity, one easily proves this statement by reductio ad absurdum.

Let $f \in \Pi$. It turns out that there exist small numbers δ and ε such that for $\rho(f, g) < \varepsilon$ (see «F») the equality

$$\rho(g(a), g(x)) \geqslant \delta\rho(f(a), f(x)) \tag{6}$$

holds; here a and x are two arbitrary points from M^k.

Indeed, when $\rho(f(a), f(x)) < \alpha$, where α is a positive constant the mappings f and g in the neighbourhood of a are very exactly approximated by linear ones, herewith this can be done uniformly with respect to $a \in M^k$. In this case the inequality (6) easily follows from statement «A». In the case when $\rho(f(a), f(x)) \geqslant \alpha$, the inequality (6) follows from the bijectivity of f for ε being reasonably small. From inequality (6) and the bijectivity of f one gets the bijectivity for any map g reasonable close to f.

Thus, Theorem 3 is proved.

§ 3. Nonproper points of smooth maps

First, recall the definition of nonproper point for a map (see § 1, «D»). Let φ be a smooth mapping from a manifold M^k to a manifold N^l. A point a of the manifold M^k is called nonproper for the mapping φ if the functional matrix of the mapping φ at the point a has rank strictly less than l. A point b of the manifold N^l is called nonproper with respect to φ if the whole pre-image $\varphi^{-1}(b)$ of this point contains at least one nonproper point $a \in M^k$ of φ. Thus, one should distinguish between nonproper points of φ in M^k and nonproper points of φ in N^l. If F is the set of all nonproper points of φ in the manifold M^k, then $\varphi(F)$ is the set of all nonproper points of the mapping φ in the manifold N^l. Theorem 4 below due to Dubovitsky [6] states that the set $\varphi(F)$ has first category in the manifold N^l, i.e. it can be represented as a countable union of compact sets nowhere dense in N^l. It follows from this that the set $N^l \setminus \varphi(F)$ of all proper points of the mapping φ in the manifold N^l has second category N^l, i.e. «widely spread» and, in any case, everywhere dense. Informally speaking this can be formulated by saying that the points of the manifold N^l are, in general, proper. Theorem 4 has some important applications in smooth manifold theory; there are many corollaries saying that in general position some "good" property obtains. To prove any result of such type one should properly define the manifolds M^k and N^l together with a mapping φ. This choice can be described by Statement «A» given below: rather general and thus, not very formal.

General position argument

A) Let Q be a smooth manifold and let P be a set of operations over Q that constitutes a smooth manifold as well. While performing an operation $p \in P$ over Q some point $q \in Q$ can be singular in a certain sense, which should be clearly described. The pair (p, q), $p \in P$, $q \in Q$ is *marked* if the point q is singular with respect to the operation p. It is assumed that the set of all marked pairs (p, q) constitutes a smooth submanifolds M^k of the manifold $P \times Q$ (see § 1, «J», «E»). With each point $(p, q) \in M^k$, one associates the point $\varphi(p, q) = p$. Thus one gets a mapping φ from the manifold M^k to the manifold $N^l = P$. If the point $p_0 \in P$ is a proper point of the mapping φ in the manifold $P = N^l$, then any point $q \in Q$ singular with respect to p_0, is in some sense *typical*, and the set Q_0 of all points q of the manifold Q which are singular with respect to the operation p_0 consists of typical singular points.

There are many applications of the construction «A»; some of them are to be demonstrated in § 4. A very simple application of the construction «A» having illustrative character is given below as Statement «B».

B) Let A^r and B^s be two smooth submanifolds of the vector space E^n. One says that at a point $a \in A^r \cap B^s$ the manifolds A^r and B^s are in *general position* if tangent planes to the manifolds A^r and B^s have intersection of dimension $r + s - n$. One says that the manifolds A^r and B^s are in general position if they are in general position at any common point. It can be shown straightforwardly that if the manifolds A^r and B^s are in general position then their intersection $A^r \cap B^s$ is a submanifold of dimension $r+s-n$ in the space E^n. Let $p \in E^n$. Denote by A^r_p the manifold consisting of all points of type $p + x$, where $x \in A^r$. Thus the manifold A^r_p is obtained from the manifold A^r by shifting along the vector p. It turns out that the set of all vectors $p \in E^n$, for which the manifolds A^r_p and B^s are in general position, is the set of second category in E^n; thus there exist an arbitrarily small shift p for which the manifolds A^r_p and B^s are in general position.

To prove Statement «B», let us use construction «A» by setting $Q = A^r \times B^s$, $P = E^n$ and assuming the point $q = (a, b) \in A^r \times B^s$ to be singular with respect to the operation $p \in E^n$ if $p + a = b$. The set M^k of all marked pairs (p, q) where $p \in E^n$, $q = (a, b) \in A^r \times B^s$ is thus defined by $p = b - a$, i.e. the pair (p, q) is uniquely defined by the point $q = (a, b)$; thus there is a natural smooth homeomorphism of the manifolds M^k and $A^r \times B^s$ that allows us to identify these manifolds. The mapping φ of the manifold $M^k = A^r \times B^s$ to the manifold $P = E^n$ is defined according to the formula $\varphi(a,b) = b - a$. Simple calculations show that a point $q = (a,b) \in M^k$ is a proper point of the map φ if and only if the manifolds A^r_{b-a} and B^s are in general position at their intersection point b. Thus, the point $p_0 \in E^n$ is a proper point of the mapping φ if and only if the manifolds $A^r_{p_0}$ and B^s are in general position. From that and from Theorem 4 proved below, one gets the claim of «B».

The Dubovitsky Theorem

In the formulation of the Dubovitsky theorem, the smoothness class m of the map $\varphi : M^k \to N^l$ is defined as $m = k - l + 1$ and not as (1), as given below. In this sense Theorem 4 is weaker than Dubovitsky's theorem. Since the exact estimate for the smoothness class m is not important, below we give a weaker estimate (1), which allows us to simplify the proof.

Theorem 4. *Let M^k and N^l be two smooth manifolds of positive dimensions k and l and let φ be an*

$$m = m(k,l) = 2 + \frac{(k-l)(k-l+1)}{2} \tag{1}$$

class smooth mapping from M^k to N^l. It turns out that the set of all nonproper points of φ in the manifold N^l is of first category in N^l. In particular, if the manifold M^k is compact then the complement to this set is an everywhere dense domain in the manifold N^l.

Proof. First consider the case when the manifold M^k has no boundary. Let $a \in M^k$, $b = \varphi(a)$, and let V^l_b be some coordinate neighbourhood of the point b in the manifolds N^l; let U^k_a be a coordinate neighbourhood of the point a in the manifold M^k such that $\varphi(U^k_a) \subset V^l_b$. Let us choose neighbourhoods U^k_{a1} and U^k_{a2} of the point a in M^k that $\overline{U}^k_{a1} \subset U^k_a$, $\overline{U}^k_{a2} \subset U^k_{a1}$ and such that the set $\overline{U}^k_{a1}$ is compact. The domains U^k_{a2}, $a \in M^k$, cover the manifold M^k. Among them, one can select a finite cover, thus, to prove the theorem, it suffices to prove it for mappings φ from $U^k_{a2} \subset M^k$ to the manifold V^l_b. Since the domain U^k_a is a homeomorphic image of a domain in the Euclidean space E^k, we may just assume that U^k_a is a domain in the space E^k. Analogously, we assume that V^l_b is a domain in the Euclidean space E^l. From this point of view, φ is an m-smooth mapping of the domain U^k_a to the Euclidean space E^l, and it suffices to show that the set of nonproper points has first category in E^l. Let us do it.

Fix the point a and remove the index a from the notation. The mapping φ of the domain U^k of E^k to E^l has the following form in Cartesian coordinates:

$$y^j = \varphi^j(x) = \varphi^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, l. \tag{2}$$

Here the functions φ^j are m times continuously differentiable. By F_0 we denote the set of all points $x \in U_2^k$ where the functional matrix $\left\|\dfrac{\partial\varphi^j}{\partial x^i}\right\|$, $i = 1, \ldots, k$, $j = 1, \ldots, l$ has rank less than l. For $k < l$, Theorem 4 becomes Theorem 1 which has already been proved. Thus we will assume that $k \geqslant l$. Set $s = k - l + 1$. The function φ^l will play a special role. From (1) it follows that $m > s$; thus the function φ^l is $s + 1$ times continuously differentiable. Let r be a positive integer less than or equal to s. Denote by F_r the set of all points from F_0, where all the partial derivatives of orders $1, 2, \ldots, r$ of the function φ^l equal zero. Then we evidently have

$$F_0 \supset F_1 \supset \ldots \supset F_s.$$

We will show that the images of all sets $F_0 \setminus F_1, \ldots, F_{s-1} \setminus F_s$ under φ have first category in E^l. This will prove that the set $\varphi(F_0)$ of nonproper points of the mapping φ is of first category in E^l as well.

First, let us consider the set F_s. The Taylor decomposition for φ^l at the point $p \in F_s$ does not contain terms of degrees $1, 2, \ldots, s$. From this and from compactness of the set $\overline{U}_1$, it follows that there exists a constant c such that for $p \in F_s, x \in \overline{U}_1$ we have

$$|\varphi^l(x) - \varphi^l(p)| < c \cdot (\rho(p, x))^{s+1}. \tag{3}$$

For the remaining functions φ^j, $j = 1, \ldots, l - 1$, the equalities

$$|\varphi^j(x) - \varphi^j(p)| < c\rho(p, x) \tag{4}$$

hold; they result from the continuity of the first derivatives and the compactness of the set $\overline{U}_1$. The constant c in inequalities (3) and (4) is common for all functions φ^j, $j = 1, 2, \ldots, l$. Choose a certain ε-cubature for E^k, i.e. tile the space E^k into proper cubes with edge length ε, and denote by Ω the set of all closed cubes of this cubature intersecting the set F_s. Since the set $\overline{F}_s$ is compact, the number of cubes from Ω does not exceed $\dfrac{c_1}{\varepsilon^k}$, where c_1 is a positive constant independent of ε. Let δ be the distance between the sets $E^k \setminus U_1^k$ and $\overline{U}_2$. Assume that $\varepsilon < \delta/\sqrt{k}$; then each cube K_q from Ω is contained in U_1^k. From that and from the fact that K_q contains the point $p \in F_s$, and from inequalities (3), (4) it follows that the set $\varphi(K_q)$ is contained in some orthogonal parallelepiped L_q of the space E^l having one edge length equal to $2c\sqrt{k} \cdot \varepsilon^{s+1}$ and the remaining $l - 1$ edges equal to $2c\sqrt{k} \cdot \varepsilon$. The volume of this parallelepiped L_q equals $2^l c^l k^{l/2} \cdot \varepsilon^{l+s}$. The compact set $\varphi(\overline{F}_s)$ is contained in the sum of closed parallelepipeds of type L_q; the number of them does not exceed c_1/ε^k. It follows that the volume of the set $\varphi(\overline{F}_s)$ does not exceed $c_1 \cdot \varepsilon^{l+s-k} = c_2 \cdot \varepsilon$ (c_2 does not depend on

ε). Thus, since ε is chosen arbitrarily small, the compact set $\varphi(\overline{F}_s)$ does not contain any domain of the space E^l, thus is nowhere dense in E^l.

If $k = 1$, then, since $k \geqslant l \geqslant 1$ we have $l = 1$, $s = 1$. In this case $F_s = F_0$, and we arrive at the statement of the theorem for $k = 1$. This gives us the induction hypothesis on k. We suppose that the theorem is true for the case when the source manifold has dimension less than k. Let us prove the theorem for dimension k.

Let us prove that for $0 \leqslant r < s$, the set $\varphi(F_r \setminus F_{r+1})$ is of first category in the space E^l. This is precisely the part of the proof to be done by induction. Let $p \in F_r \setminus F_{r+1}$. Since p does not belong to the set F_{r+1}, there exists a partial derivative of order $r+1$ of the function φ^l taking a non-zero value at p. Denote the value of this derivative at $x \in U^k$ by $\omega_1(x)$. Since $\omega_1(x)$ is a derivative of order $r+1$ then $\omega_1(x) = \partial\omega(x)/\partial x^i$, where $\omega(x)$ is the derivative of order r for $r > 0$ or the function $\varphi^l(x)$ itself for $r = 0$. For definiteness, assume $i = k$. Set

$$z^i = x^i,\ i = 1, \dots, k-1; \quad z^k = \omega(x) = \omega(x^1, \dots, x^k). \tag{5}$$

It follows from $\partial\omega(p)/\partial x^k \neq 0$ that the functional determinant of (5) is non-zero at p; thus, this transformation introduces in some neighbourhood W_p^k of p new coordinates $z^1, \dots, z^k$. We shall assume that W_p^k does not intersect F_{r+1} and choose a neighbourhood W_{p1}^k of the point p such that its closure $\overline{W}_{p1}^k$ is compact and is contained in W_p^k. By varying the point p, we can cover the set $F_r \setminus F_{r+1}$ by a countable system of neighbourhoods of type W_{p1}^k. Thus, to prove that the set $\varphi(F_r \setminus F_{r+1})$ has first category, it is sufficient to show that $\varphi(F_r \cap \overline{W}_{p1}^k)$ is nowhere dense in E^l. Let us prove this fact.

Let us fix the point p and omit the index p in the notation. Substituting in (2) the expressions $x^1, \dots, x^k$ in terms of $z^1, \dots, z^k$, we get the expression for φ in coordinates $z^1, \dots, z^k$ for the domain W^k. Suppose this expression looks like

$$y^j = \varphi^j(x) = \psi^j(z^1, \dots, z^k). \tag{6}$$

Here $z^1, \dots, z^k$ are the new coordinates of the point x. Consider the domain W^k with coordinates $z^1, \dots, z^k$ as a smooth manifold. It follows from (5) that the mapping φ from W^k to the space E^l given by (6) has smoothness type $m(k,l) - r$. For $r = 0$ the smoothness class of the map φ equals $m(k,l) = m(k-1,l-1)$ [see (1)]. Choosing for $r > 0$ the worst estimate for the smoothness class, that is, $r = s-1 = k-l$, we see that for $r > 0$ the smoothness class of the considered map φ equals $m(k,l) - (k-l) = m(k-1,l)$ [see (1)]. The set $H \subset W^k$

of all nonproper points of the mapping φ in the manifold W^k is defined by $H = W^k \cap F_0$. This follows from the non-degeneracy of (5) at W^k. Denote by W_t^{k-1} the submanifold of the manifold W^k defined by the equation $z^k = t$. Note that the smoothness class of the mapping from W_t^{k-1} to E^l equals $m(k-1, l-1)$ for $r = 0$ and it equals $m(k-1, l)$ for $r > 0$. Let us consider the cases $r = 0$ and $r > 0$ separately.

Assume $r = 0$. Then $\omega(x) = \varphi^l(x) = z^k$. Thus, the expression (6) for the mapping φ turns into

$$y^j = \psi^j(z^1, \ldots, z^k), \quad j = 1, \ldots, l-1; \quad y^l = z^k. \tag{7}$$

Denote by E_t^{l-1} the linear subspace of the space E^l defined by the equation $y^l = t$. It follows from the relations (7) that $\varphi(W_t^{k-1}) \subset E_t^{l-1}$. Denote by $H_t \subset W_t^{k-1}$ the set of all nonproper points of the mapping φ from the manifold W_t^{k-1} to the space E_t^{l-1}. It follows from the relations (7) that $H_t = H \cap W_t^{k-1}$. If the set $\varphi(F_0 \cap \overline{W}_1^k)$ contained a domain, then there would exist a value t such that the intersection $\varphi(F_0 \cap \overline{W}_1^k) \cap E_t^{l-1}$ would contain a domain in E_t^{l-1}. However, this is impossible because

$$\varphi(F_0 \cap \overline{W}_1^k) \cap E_t^{l-1} \subset \varphi(H) \cap E_t^{l-1} = \varphi(H \cap W_t^{k-1}) = \varphi(H_t),$$

and the set $\varphi(H_t)$ has first category in E_t^{l-1} according to the induction assumption. Thus, the set $\varphi(F_0 \cap \overline{W}_1^k)$ is nowhere dense in E^l; the case $r = 0$ is discussed completely.

Now assume $r > 0$. Then $\omega(x)$ is a derivative of order r of the function φ^l; thus $\omega(x) = 0$ for $x \in F_r$. Since for the neighbourhood W^k we have $\omega(x) = z^k$ then

$$F_r \cap W^k \subset W_0^{k-1}. \tag{8}$$

Let $H' \subset W_0^{k-1}$ be the set of all nonproper points of the mapping $\varphi : W_0^{k-1} \to E^l$. It is easy to see that $H \cap W_0^{k-1} \subset H'$ [see (6)] and, since $F_r \cap W_1^k \subset H$ then it follows from (8) that $F_r \cap W_1^k \subset H'$. By virtue of the induction hypothesis, the set $\varphi(H')$ has first category in E^l. Since $F_r \cap W_1^k \subset H'$ the set $\varphi(F_r \cap W_1^k)$ is nowhere dense in E^l. Thus we have completed the proof for the case $r > 0$.

Thus, Theorem 4 is proved when M^k has no boundary.

Finally, suppose the manifold M^k has a non-empty boundary M^{k-1}. Suppose $F' \subset M^{k-1}$ is the set of all nonproper points of the mapping $\varphi : M^{k-1} \to N^l$ and $F \subset M^k$ is the set of all nonproper points of the mapping $\varphi : M^k \to N^l$. It is easy to see that

$$F \cap M^{k-1} \subset F'.$$

Thus,

$$F \subset (F \setminus M^{k-1}) \cup F'.$$

The set $F \setminus M^{k-1}$ consists of all nonproper points of the mapping φ in the manifold $M^k \setminus M^{k-1}$ with boundary deleted. Analogously, the set F' consists of all nonproper points of the mapping φ on M^{k-1} without boundary. Thus, both sets $\varphi(F \setminus M^{k-1})$ and $\varphi(F')$ have first category in N^l. The set $\varphi(F)$ is contained in their union, thus it has the first category in N^l.

Therefore, Theorem 4 is proved.

§ 4. Non-degenerate singular points of smooth mappings

Let f be a smooth mapping from a manifold M^k to a manifold N^l. Let $a \in M^k$ and $b = f(a) \in N^l$ be interior (non-boundary) points of the manifolds M^k and N^l. In the neighbourhoods of a and b, let us introduce local coordinates $x^1, \ldots, x^k$ and $y^1, \ldots, y^l$ taking these points to be coordinate origins. Let

$$y^j = f^j(x) = f^j(x^1, \ldots, x^k)$$

be the coordinate expression for f in the chosen coordinate systems.

Suppose a is a regular point of f, i.e. that the rank of the matrix $\left\| \dfrac{\partial f^j(a)}{\partial x^i} \right\|$, $j = 1, \ldots, l$, $i = 1, \ldots, k$, equals k; to be more precise, we shall assume that the determinant $\left| \dfrac{\partial f^j(a)}{\partial x^i} \right|$, $i, j = 1, \ldots, k$ is non-zero. With this assumption the relations

$$\xi^i = f^i(x^1, \ldots, x^k), \qquad i = 1, \ldots, k,$$

may serve to define in the neighbourhood of a the new coordinates $\xi^1, \ldots, \xi^k$ of the point x. Let

$$\begin{aligned} y^j &= \xi^j, && j = 1, \ldots, k; \\ y^j &= \varphi^j(\xi^1, \ldots, \xi^k), && j = k+1, \ldots, l, \end{aligned}$$

be the expression of the mapping f in these new coordinates. Let us introduce in the neighbourhood of the point b the new coordinates $\eta^1, \ldots, \eta^l$, by setting

$$\begin{aligned} \eta^j &= y^j, && j = 1, \ldots, k; \\ \eta^j &= y^j - \varphi^j(y^1, \ldots, y^k), && j = k+1, \ldots, l. \end{aligned}$$

In coordinates $\xi^1, \ldots, \xi^k, \eta^1, \ldots, \eta^l$ the mapping f looks like

$$\eta^j = \xi^j,\ j = 1, \ldots, k; \quad \eta^j = 0,\ j = k+1, \ldots, l. \tag{1}$$

Now, let us assume that the point a is proper, i.e. the rank of the matrix $\left\| \dfrac{\partial f^j(a)}{\partial x^j} \right\|$, $j = 1, \ldots, l$, $i = 1, \ldots, k$ equals l, and assume for definiteness that the determinant $\left| \dfrac{\partial f^j(a)}{\partial x^i} \right|$, $i, j = 1, \ldots, l$ is non-zero. Then the relations

$$\xi^i = f^i(x^1, \ldots, x^k),\ i = 1, \ldots, l; \quad \xi^i = x^i,\ i = l+1, \ldots, k,$$

may serve for introducing in a neighbourhood of a the new coordinates $\xi^1, \ldots, \xi^k$ of the point x. Furthermore, assuming

$$\eta^j = y^j, \quad j = 1, \ldots, l,$$

we see that in coordinates $\xi^1, \ldots, \xi^k$, $\eta^1, \ldots, \eta^l$ the mapping f can be written as

$$\eta^j = \xi^j, \quad j = 1, \ldots, l. \tag{2}$$

Thus, if the manifold M^k is closed and $b \in N^l$ is a proper point of the mapping f, then $f^{-1}(b)$ is a smooth $(k-l)$-dimensional submanifold of the manifold M^k with local coordinates $\xi^{l+1}, \ldots, \xi^k$ in the neighbourhood of a. In the case when the manifolds M^k and N^l are oriented and their orientations are given by the coordinate systems $\xi^{l+1}, \ldots, \xi^k, \xi^1, \ldots, \xi^l$ and $\eta^1, \ldots, \eta^l$, then the manifold $f^{-1}(b)$ gets a natural orientation given by the coordinate system $\xi^{l+1}, \ldots, \xi^k$.

We see that both in the case of a regular point a and in the case of proper point a the mapping is written quite simply in the properly chosen coordinate systems [see (1), (2)].

It was shown in § 2 that in any neighbourhood of any arbitrary smooth mapping from M^k to the vector space A^{2k} there exists a regular mapping, and all mappings sufficiently close to a regular one, are regular as well (see Theorem 3). In this sense, singular points (see § 1, «D») of mappings $M^k \to A^{2k}$ are unbalanced, that is, they are removable by a small perturbation. For mappings from M^k to the vector space A^{2k-1} we have another situation: singular points that occur there are, generally, balanced: they cannot be removed by a small perturbation. This problem was solved by Whitney. Here we give a simpler proof of his theorem (see Theorem 6). We will not use this theorem in the sequel. The question about typical singular points is solved here also for mappings of a manifold M^k to the one-dimensional space A^1, i.e. to the line (see Theorem 5; it will have applications in the homotopy theory of mappings, see § 3, Chapter 4). Thus, the question about typical singular points of a mapping is solved for mappings from manifolds of dimension k to space of dimension $2k-1$ or 1. For other dimension, it remains a quite actual open problem.

Generally, a regular mapping from M^k to the vector space A^{2k} is not homeomorphic: it has self-intersections, which might be non-removable by small perturbation of the initial mapping. The question whether a self-intersection is typical is also solved here (see «A» and «B»); these statements will be used in the sequel.

For proving Theorems 5 and 6, and also Statement «B» we significantly use the construction «A» (see page 21) and Theorem 4.

Typical self-intersection points of mappings $M^k \to E^{2k}$

A) Let f be a regular smooth mapping of class $m \geqslant 1$ from a closed manifold M^k to the vector space A^{2k} and let a and b be two different points from M^k having the same image $f(a) = f(b) \in A^{2k}$. Furthermore, let U and V be neighbourhoods of points a and b in M^k such that the mapping f is homeomorphic for any of these neighbourhoods, and T_a^k and T_b^k are tangent planes at points $f(a)$ and $f(b)$ to the manifolds $f(U)$ and $f(V)$, respectively. Say that for a self-intersection pair (a, b) the mapping f is *typical* if the tangent planes T_a^k and T_b^k are in general position, i.e. they intersect precisely at one point $f(a) = f(b)$. Obviously, in this case for sufficiently small neighbourhoods U and V, the manifolds $f(U)$ and $f(V)$ have a unique common point $f(a) = f(b)$ as well (implicit function theorem), and small perturbations of the mapping preserve typical self-intersections. If f is typical for any self-intersection pair and, furthermore, no three pairwise different points have the same image, we say that f is *typical.* It follows from closeness of the manifold M^k that, for a mapping f typical for any self-intersection pair, there exists only a finite number of self-intersection pairs.

B) Let f be a closed homeomorphic mapping of a closed manifold M^k to a vector space C^{2k+1}. The set P^{2k} of all pairs (x, y), where $x \in M^k$, $y \in M^k$, $x \neq y$, naturally forms a smooth manifold of dimension $2k$. With each point $(x, y) = P^{2k}$, associate a point $\sigma(x, y) = (f(y) - f(x))^* \in S^{2k}$, i.e. the ray of the vector $f(y) - f(x)$ (see § 1, «H»). Let e be an arbitrary non-zero vector from the space C^{2k+1} and let π_e be the projection along the one-dimensional space e^{**} containing e. It turns out that the regular mapping $\pi_e f$ is typical for any self-intersection pair (see «A») if and only if the mapping σ from the manifold P^{2k} to the manifold S^{2k} is proper in the point $e^* \in S^{2k}$. From this, by virtue of Theorem 4, it follows that for any given one-dimensional projection direction there exists an arbitrarily closed projection direction e^{**} that the mapping $\pi_e f$ is typical for each self-intersection pair. Furthermore, it turns out that for any one-dimensional projection direction there is an arbitrarily close direction e_0^{**} that the mapping $\pi_{e_0} f$ is typical.

Let us prove Statement «B». Let $e_1, \ldots, e_{2k+1}$ be a basis of a vector space C^{2k+1}. Denote by W the set of all vectors $u = \sum_{n=1}^{2k+1} u^n e_n$ of the space C^{2k+1} for which $u^{2k+1} > 0$, and denote by W^* the set of all rays u^* for $u \in W$. For coordinates of the ray $u^* \in W^*$, we take the numbers $u^{*n} = u^n/u^{2k+1}$, $n = 1, \ldots, 2k$. Herewith we introduce local coordinates for the domain W^* of the manifold S^{2k} (see § 1,«H»). Now, let a and b be two different points of the manifold M^k. Choose a basis $e_1, \ldots, e_{2k+1}$ in such a way that $e_{2k+1} = e = f(b) - f(a)$. In neighbourhoods of points a and b of the manifold M^k, let us choose local coordinates $x^1, \ldots, x^k$ and $y^1, \ldots, y^k$; let

$$u^n = f_a^n(x^1, \ldots, x^k) = f_a^n(x), \qquad n = 1, \ldots, 2k+1; \tag{3}$$

$$u^n = f_b^n(y^1, \ldots, y^k) = f_b^n(y), \qquad n = 1, \ldots, 2k+1, \tag{4}$$

be a coordinate expression of the mapping f in the neighbourhoods of a and b, respectively. While projecting along the vector $e = f(b) - f(a)$, the points b and a merge: $\pi_e f(a) = \pi_e f(b)$; thus the condition that $\pi_e f$ is typical for the self-intersection pair (a, b), evidently, means that the determinant

$$\begin{vmatrix} \dfrac{\partial f_a^1(a)}{\partial x^1} & \cdots & \dfrac{\partial f_a^{2k}(a)}{\partial x^1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial f_a^1(a)}{\partial x^k} & \cdots & \dfrac{\partial f_a^{2k}(a)}{\partial x^k} \\ \dfrac{\partial f_b^1(b)}{\partial y^1} & \cdots & \dfrac{\partial f_b^{2k}(b)}{\partial y^1} \\ \dfrac{\partial f_b^1(b)}{\partial y^k} & \cdots & \dfrac{\partial f_b^{2k}(b)}{\partial y^k} \end{vmatrix} \tag{5}$$

is non-zero. For a neighbourhood of the point (a, b) of the manifold P^{2k} we may use the coordinate system consisting of numbers $x^1, \ldots, x^k$, $y^1, \ldots, y^k$; thus the mapping σ has the following coordinate form:

$$u^{*n} = \frac{f_b^n(y) - f_a^n(x)}{f_b^{2k+1}(y) - f_a^{2k+1}(x)}, \qquad n = 1, \ldots, 2k. \tag{6}$$

In these coordinates, the functional determinant of the mapping σ at the point (a, b), evidently, coincides with the determinant (5) up to sign. Thus we have proved that regular mapping $\pi_e f$ is typical for each self-intersection pair if and only if the mapping σ is proper at the point e^*.

Now, choose the ray e^* in such a way that the vector e is not parallel to any vector tangent to the manifold $f(M^k)$ and that the mapping σ is proper at the point $e^* \in S^{2k}$. By virtue of Theorems 1 and 4, the set of rays enjoying the above properties, is everywhere dense in the manifold S^{2k}. Suppose there exist three pairwise distinct points a, b, c of the manifold M^k such that $\pi_e f(a) = \pi_e f(b) = \pi_e f(c)$. In the neighbourhood of c in M^k, let us introduce the local coordinates $z^1, \ldots, z^k$, and let

$$u^n = f_c^n(z^1, \ldots, z^k) = f_c^n(z), \qquad n = 1, \ldots, 2k+1, \tag{7}$$

be the coordinate expression of the mapping f in the neighbourhood of c, analogous to the expressions (3) and (4). Now, if x, y, z are three points of the manifold M^k close to a, b, c, respectively, such that the points $f(x)$, $f(y)$, $f(z)$ lie on the same line then we have

$$\frac{f_a^n(x) - f_c^n(z)}{f_a^{2k+1}(x) - f_c^{2k+1}(z)} = \frac{f_b^n(y) - f_c^n(z)}{f_a^{2k+1}(y) - f_c^{2k+1}(z)}, \qquad n = 1, \ldots, 2k. \tag{8}$$

Here we have $2k$ equation. We may assume that these equations implicitly define the functions $x^1, \ldots, x^k$, $y^1, \ldots, y^k$ in independent variables $z^1, \ldots, z^k$. For the initial value $z = c$ we have the solution $x = a$, $y = b$. For these initial values of the functions and independent variables, the functional determinant of the system (8) is non-zero, since so is the determinant (5). Thus, the system (8) satisfies the condition of the implicit function theorem. It follows now that the set of triples x, y, z closed to the triple a, b, c and satisfying the condition that $f(x)$, $f(y)$, $f(z)$ lie on the same line, forms a k-dimensional manifold. Thus, by virtue of Theorem 1, we see that for the point e^* of the manifold S^{2k} there is an arbitrarily close point e_0^* satisfying the conditions of the Statement «B».

Typical critical points of a real-valued function on a manifold

C) Let f be a class m smooth mapping ($m \geqslant 2$) from a manifold M^k to the one-dimensional Euclidean space E^1, or, what is the same, to the line. By choosing a coordinate system on the line E^1, we write down the mapping f as $y^1 = f^1(x)$, $x \in M^k$, where f^1 is a real-valued function of class m, defined on M^k. In a neighbourhood of a certain point $a \in M^k$, let us introduce local coordinates $x^1, \ldots, x^k$ with the origin at a, and let

$$y^1 = f^1(x) = f^1(x^1, \ldots, x^k)$$

be the expression for f in these coordinates. The point a is called a *critical point* of the function f^1, and the number $f^1(a)$ is called the *critical value* of the function f^1 at the point a if all derivatives of the first order of the function f^1 are zeros at a or, which is the same, if a is a singular point of the function f (see § 1, «D»). Taking the Taylor decomposition for the function f^1 at the critical point a, we get

$$f^1(x) = f^1(a) + \sum_{i,j} a_{ij} x^i x^j + \dots . \tag{9}$$

If the determinant $|a_{i,j}| \neq 0$, then the critical point a is called *non-degenerate*. It can be checked straightforwardly that for a critical point a of the function f, any arbitrary coordinate change the matrix $\|a_{i,j}\|$ is transformed as coefficients of quadratic form. From this it follows, in particular, that the non-degeneration of the singular point is its invariant property, i.e. it does not depend on the choice of the coordinate system.

D) Let h be an m-smooth mapping ($m \geqslant 2$) of a manifold M^k to the Euclidean vector space C^{q+1}. Let u be a non-zero vector from C^{q+1} and let u^{**} be the one-dimensional subspace containing the vector u. Denote by π_u the orthogonal projection of the space C^{q+1} to the line u^{**}. The set N^q of all pairs (x, u^*), where $x \in M^k$, and u^* is a ray orthogonal to the manifold $h(M^k)$ at the point $h(x)$ can be naturally seen as an $(m-1)$-smooth manifold of dimension q. With each point $(x, u^*) \in N^q$, associate a point $\nu(x, u^*) = u^* \in S^q$ (see § 1, «H»). The mapping ν is a smooth mapping of class $m-1$ from N^q to S^q. It turns out that the point $a \in M^k$ is a singular point of the mapping $\pi_u h$ from M^k to u^{**} if and only if the ray u^* is orthogonal to the manifold $h(M^k)$ at the point $h(a)$. Furthermore, if the ray u^* is orthogonal to the manifold $h(M^k)$ at the point $h(a)$ then the singular point a of the mapping $\pi_u h$ is non-degenerate if and only if (a, u^*) is a proper point of the mapping ν.

Let us prove Statement «D». Denote the scalar product of vectors u and v from C^{q+1}, as usual, by (u, v). Let $u \in C^{q+1}$ and $(u, u) = 1$. Indeed, the real-valued function $(u, h(x))$ in variable $x \in M^k$ defined on M^k, corresponds to the mapping $\pi_u h$ of the manifold M^k to the axis u^{**}. In the local coordinates $x^1, \dots, x^k$ defined in a neighbourhood of a, one has

$$\frac{\partial}{\partial x^i}(u, h(a)) = \left(u, \frac{\partial h(a)}{\partial x^i}\right), \qquad i = 1, \dots, k. \tag{10}$$

The fact that the left-hand sides of all relations (10) are all zeros means that a is a singular point of the mapping $\pi_u h$; the fact that all right-hand sides are zeros means that the vector u is orthogonal to the manifold $h(M^k)$ at the point $h(a)$. Thus, we have proved that the point a is a singular point for the mapping $\pi_u h$ if and only if the ray u^* is orthogonal to $h(M^k)$ at the point $h(a)$.

To establish a criterion whether a singular point a of the mapping $\pi_{u_0}h$ is degenerate, let us choose in the space C^{q+1} such an orthonormal basis $e_1, \ldots, e_{q+1}$ that the vectors $e_1, \ldots, e_k$ are tangent to the manifold $h(M^k)$ at the point $h(a)$, and the vector e_{q+1} coincides with u_0. In the corresponding coordinates $y^1, \ldots, y^{q+1}$ of the space C^{q+1} the map h in the neighbourhood of a looks like

$$y^j = h^j(x) = h^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, q+1. \tag{11}$$

Since the vectors $e_1, \ldots, e_k$ are tangent to the manifold $h(M^k)$ at the point $h(a)$, it follows directly that

$$\left| \frac{\partial h^j(a)}{\partial x^i} \neq 0 \right|, \qquad i = 1, \ldots, k,$$

From that we see that the relations

$$\xi^i = h^i(x^1, \ldots, x^k), \qquad i = 1, \ldots, k$$

may serve for introducing new coordinates $\xi^1, \ldots, \xi^k$ of the point x in the neighbourhood of the point a in M^k. In these coordinates, the mapping h looks like

$$h(x) = \sum_{i=1}^{k} \xi^i e_i + \sum_{j=1}^{q+1-k} \varphi^j(x) \cdot e_{k+j}. \tag{12}$$

Because the vectors $e_1, \ldots, e_k$ are tangent to $h(M^k)$ at the point $h(a)$, it follows that

$$\frac{\partial \varphi^j(a)}{\partial \xi^i} = 0, \quad i = 1, \ldots, k;\ j = 1, \ldots, q+1-k. \tag{13}$$

Let (x, u^*) be a point of the manifold N^q close to the point $(a, u^*) = (a, e^*_{q+1})$. On the ray u^*, let us choose a vector u satisfying the condition

$$(u, e_{q+1}) = 1.$$

Denote the remaining q components of the vector u in the basis $e_1, \ldots, e_{q+1}$ by $u^1, \ldots, u^q$: $u^i = (u, e_i)$, $i = 1, \ldots, q$. The orthogonality condition for the vector u and $h(M^k)$ at the point $h(x)$ now looks like

$$0 = \left(u, \frac{\partial h(x)}{\partial \xi^i} \right) = u^i + \sum_{j=1}^{q-k} u^{k+j} \frac{\partial \varphi^j(x)}{\partial \xi^i} + \frac{\partial \varphi^{q+1}(x)}{\partial \xi^i},\ i = 1, \ldots, k. \tag{14}$$

This relation shows that for coordinates of the element (x, u^*) of the manifold N^q we can choose the coordinates $\xi^1, \ldots, \xi^k$ of the point x and the components $u^{k+1}, \ldots, u^q$ of the vector u. For coordinates of the ray u^* in the manifold S^q, let us take the first q components of the vector u and denote these components by $v^1, \ldots, v^k$ in order not to mix them up with the coordinates $u^{k+1}, \ldots, u^q$ of the element (x, u^*) in the manifold N^q. Since $v^i = u^i$, $i = 1, \ldots, q$, then in the chosen coordinate system the mapping $\nu : N^q \to S^q$ looks like [see (14)]

$$v^i = -\sum_{j=1}^{q-k} u^{k+j} \frac{\partial \varphi^j(x)}{\partial \xi^i} - \frac{\partial \varphi^{q+1}(x)}{\partial \xi^i},\ i = 1, \ldots, k,$$

$$v^{k+j} = u^{k+j}, \qquad j = 1, \ldots, q-k.$$

The direct calculation [see (13)] shows that the Jacobian of the mapping ν at the point (a, e^*_{q+1}) is equal to $(-1)^k \left| \frac{\partial^2 \varphi^{q+1}(a)}{\partial \xi^i \partial \xi^\alpha} \right|$, $\quad i, \alpha = 1, \ldots, k$. Thus, the point (a, u_0) is a proper point of the mapping ν if and only if the following equation holds:

$$\left| \frac{\partial^2 \varphi^{q+1}(a)}{\partial \xi^i \partial \xi^\alpha} \right| \neq 0. \tag{15}$$

Since the mapping $\pi_{u_0} h$ from M^k to the axis u_0^{**} is associated with the function $\varphi^{q+1}(x)$, the condition (14) coincides with the non-degeneracy condition for the singular point a of the mapping $\pi_{u_0} h$. This completes the proof of «D».

Theorem 5. *Let M^k be a smooth compact manifold of class $m \geqslant 3$ with boundary M^{k-1} consisting of two closed manifolds M_0^{k-1} and M_1^{k-1}, each of which possibly empty. Let f^1 be a real-valued function of class m defined on M^k. Suppose that the function f^1 takes the same value c_i, $i = 0, 1$ in all points of the manifold M_i^{k-1} and $c_0 < c_1$, and that for any non-boundary point $x \in M^k$ the inequality $c_0 < f(x) < c_1$ holds. Moreover, suppose that no critical point of f^1 lies on the boundary M^{k-1}. It turns out that for the function f^1 there exists an arbitrarily m-class close (see § 2, «F») function g^1 coinciding with f^1 in some neighbourhood of the boundary such that all critical points of the function g^1 are not degenerate and critical values in different critical points are pairwise distinct.*

Proof. With the function f^1, let us associate the mapping f from the manifold M^k to the one-dimensional vector space A^1. Let e be a homeomorphic regular class m mapping from M^k to the Euclidean space B^q

(see Theorem 2). Denote the direct sum of vector spaces A^1 and B^q by C^{q+1}; let us consider the spaces A^1 and B^q as orthogonal subspaces of the space C^{q+1}. Denote the direct sum of mappings f and e (see § 2, «E») by h. The mapping h is a regular homeomorphic class m mapping from the manifold M^k to the Euclidean space C^{q+1} such that the orthogonal projection π of h to the line A^1 coincides with f: $f = \pi h$. First of all, let us show that in any neighbourhood of the line A^1 there exists a line in the orthogonal projection to which generates a function having only non-degenerate critical values. The desired function in the formulation of the theorem is to be obtained from this by some modifications.

Let N^q be a manifold of all normal elements (x, u^*) of the manifold $h(M^k)$, as defined in «D», and let ν be the mapping from the manifold N^q to the manifold S^q also defined in «D». Let us show that if $u^* \in S^q$ is a proper point of ν then all singular points of $\pi_u h$ are non-degenerate. Indeed, if a is a singular point of the mapping $\pi_u h$, then the ray u^* is orthogonal to $h(M^k)$ at the point $h(a)$; thus $(a, u^*) \in N^q$. Since the mapping ν is proper at (a, u^*) of the manifold N^q, then the singular point a is non-degenerate (see «D»). Let ε be a given positive number and let u be such a unit vector of the spaces C^{q+1} that the function $h^1 = (u, h(x))$ is class m ε-close to f^1 and that $u^* \in S^q$ is a proper point of the mapping ν so that all critical points of the function h^1 are non-degenerate. By Theorem 4, such a vector u does exist.

Let δ be such a small positive number that for $f^1(x) < c_0 + 3\delta$ and for $f^1(x) > c_1 - 3\delta$ the point x is not a critical point of the function f^1. The existence of such δ follows from the conditions of the theorem, since neither the boundary M^{k-1} nor its small neighbourhood contains critical points of f^1. Furthermore, suppose $\chi(t)$ is a real-valued class m function in variable t equal to zero at $t \leqslant c_0 + \delta$ and $t \geqslant c_1 - \delta$ and equal to one at $c_1 - 2\delta \geqslant t \geqslant c_0 + 2\delta$. Set

$$h^2(x) = f^1(x) + \chi(f^1(x))(h^1(x) - f^1(x)). \tag{16}$$

It is easy to see that if ε that we have taken for constructing the function $h^1(x)$, is chosen to be reasonably small then all critical points of the function $h^2(x)$ defined by (16) coincide with the critical points of the function $h^1(x)$ and thus they are non-degenerate. Since for $t \leqslant c_0 + \delta$ and for $t \geqslant c_1 - \delta$ the function $\chi(t)$ equals zero it follows that for some neighbourhood of the boundary M^{k-1} the functions $h^2(x)$ and $f^1(x)$ coincide.

Typical singularities of mappings $M^k \to E^{2k-1}$

E) Let f be an m-class smooth ($m \geqslant 2$) mapping from the manifold M^k to the vector space A^{2k-1}. Let a be a singular point of the mapping f and let $x^1, \ldots, x^k$ be a local coordinate system in its neighbourhood such that

$$\frac{\partial f(a)}{\partial x^1} = 0. \tag{17}$$

Such a coordinate system in a singular point neighbourhood always exists. If the system

$$\frac{\partial^2 f(a)}{\partial x^1 \partial x^i}, \ \frac{\partial f(a)}{\partial x^j}, \quad i = 1, \ldots, k; \ j = 2, \ldots, k, \tag{18}$$

of $2k-1$ vectors of the space A^{2k-1} is linearly independent, then the singular point a is called *non-degenerate*. Later on, we shall show that the non-degeneracy of a singular point is invariant, i.e. this notion is independent of the coordinate system: if some coordinate system $\xi^1, \ldots, \xi^k$ defined in a neighbourhood of a satisfies the condition

$$\frac{\partial f(a)}{\partial \xi^1} = 0, \tag{19}$$

then the vector systems (18) and

$$\frac{\partial^2 f(a)}{\partial \xi^1 \partial \xi^i}, \ \ \frac{\partial f(a)}{\partial \xi^j}, \ \ i = 1, \ldots, k; \ j = 2, \ldots, k, \tag{20}$$

are either both linearly dependent or both linearly independent. It turns out that in a sufficiently small neighbourhood of a non-degenerate singular point there are no other singular points.

Let us prove that the non-degeneracy is invariant. Assume that the relations (17) and (19) hold and that the vector system (20) is linearly independent. Let us show that the system (18) is also linearly independent. We have

$$\frac{\partial f(a)}{\partial x^1} = \sum_{\alpha} \frac{\partial f(a)}{\partial \xi^\alpha} \frac{\partial \xi^\alpha(a)}{\partial x^1},$$

from which, according to the assumption above, we deduce

$$\frac{\partial \xi^\alpha(a)}{\partial x^1} = 0, \ \alpha = 2, \ldots, k. \tag{21}$$

Since the Jacobian $\left| \dfrac{\partial \xi^\alpha(a)}{\partial x^i} \right|$, $\alpha, i = 1, \ldots, k$, is non-zero, it follows from (21) that

$$\frac{\partial \xi^1(a)}{\partial x^1} \neq 0, \quad \left| \frac{\partial \xi^\alpha(a)}{\partial x^i} \right| \neq 0, \quad \alpha, i = 2, \ldots, k. \tag{22}$$

From the relations (21) we get

$$\frac{\partial f(a)}{\partial x^j} = \sum_{\alpha=2}^{k} \frac{\partial f(a)}{\partial \xi^\alpha} \frac{\partial \xi^\alpha(a)}{\partial x^j}, \quad j = 2, \dots, k. \tag{23}$$

Furthermore, taking into account (21) and (19), we get

$$\frac{\partial^2 f(a)}{\partial x^1 \partial x^i} = \sum_{\beta=1}^{k} \frac{\partial^2 f(a)}{\partial \xi^1 \partial \xi^\beta} \frac{\partial \xi^1(a)}{\partial x^1} \frac{\partial \xi^\beta(a)}{\partial x^i} + \sum_{\alpha=2}^{k} \frac{\partial f(a)}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha(a)}{\partial x^1 \partial x^i}, \quad i = 1, \dots, k. \tag{24}$$

From the relations (23), (24), (22) and linear independence of the system (20) one gets the linear independence of the system (18).

Now, let us show that the singular point a is isolated. To do that, view a as the origin for the coordinate system $x^1, \dots, x^k$ and consider the Taylor decomposition for vectors $\dfrac{\partial f(x)}{\partial x^i}$, $i = 1, \dots, k$, in the neighbourhood of a in coordinates $x^1, \dots, x^k$:

$$\frac{\partial f(x)}{\partial x^1} = \sum_{\alpha=1}^{k} \frac{\partial^2 f(a)}{\partial x^1 \partial x^\alpha} x^\alpha + \varepsilon_1, \tag{25}$$

$$\frac{\partial f(x)}{\partial x^i} = \frac{\partial f(a)}{\partial x^i} + \varepsilon_i, \quad i = 2, \dots, k, \tag{26}$$

where ε_1 is second-order small with respect to $\varrho = \sqrt{(x^1)^2 + \ldots + (x^k)^2}$, and $\varepsilon_2, \dots, \varepsilon_k$ are first-order small with respect to ϱ. Since the vectors of the system (19) are linearly independent, it follows from (25) and (26) that the vectors $\dfrac{\partial f(x)}{\partial x^1}, \dots, \dfrac{\partial f(x)}{\partial x^k}$ are linearly independent for all points $x \neq a$ sufficiently close to a.

F) Let h be a regular class m mapping ($m \geqslant 2$) from the manifold M^k to the vector space C^{2k}. Denote the manifold of all rays u^* of the manifold C^{2k} by S^{2k-1} (see § 1, «H») and denote by L^{2k-1} the manifold of all linear elements of the manifold $h(M^k)$, i.e. the manifold of all pairs (x, u^*), where $x \in M^k$, and u^* is the ray tangent to $h(M^k)$ at $h(x)$. Define the mapping τ from the manifold L^{2k-1} to the manifold S^{2k-1} by setting $\tau(x, u^*) = u^*$. Denote the projection of the space C^{2k} along the line u^{**} containing u by π_u. As noticed before (see § 2, «A»), the point $a \in M^k$ is a singular point of $\pi_u h$ if and only if the ray u^* is tangent to $h(M^k)$ at the point $h(x)$, i.e. if $(a, u^*) \in L^{2k-1}$. It turns out that the singular point a of the mapping $\pi_u h$ is non-degenerate if and only if the mapping τ is proper at the point $(a, u^*) \in L^{2k-1}$.

Let us prove the last statement. Let a be a singular point of $f = \pi_{u_0} h$. Choose a basis $e_1, \ldots, e_{2k}$ of the vector space C^{2k} in such a way that the vectors $e_1, \ldots, e_k$ are tangent to $h(M^k)$ at the point $h(a)$ and so that the vector e_1 coincides with u_0. Let $y^j = h^j(x) = h^j(x^1, \ldots, x^k)$ be the expression of the mappings h in the coordinates $y^1, \ldots, y^{2k}$ with respect to the basis $e_1, \ldots, e_{2k}$. Note that the absolute value of the Jacobian $\left|\dfrac{\partial h^j(a)}{\partial x^i}\right|$, $i, j = 1, \ldots, k$, differs from zero, thus the relations

$$\xi^i = h^i(x^1, \ldots, x^k), \ m = 1, \ldots, k,$$

can be used to introduce new coordinates $\xi^1, \ldots, \xi^k$ of x in the neighbourhood of a. In the new coordinates, the vector $h(x)$ will look like

$$h(x) = \sum_{i=1}^{k} \xi^i e_i + \sum_{i=1}^{k} \varphi^j(x) e_{k+j}, \tag{27}$$

where the functions $\varphi^j(x)$ satisfy the condition

$$\frac{\partial \varphi^j(x)}{\partial \xi^i} = 0, \quad i, j = 1, \ldots, k. \tag{28}$$

Let (x, u^*) be an element of the manifold L^{2k-1} close to the element (a, u_0). The vector u is tangent to $h(M^k)$ at the point $h(x)$; thus it can be written as

$$u = \sum_{i=1}^{k} u^i \frac{\partial h(x)}{\partial \xi^i} = \sum_{i=1}^{k} u^i e_i + \sum_{i,j=1}^{k} u^i \frac{\partial \varphi^j(x)}{\partial \xi^i} e_{j+k}. \tag{29}$$

On the ray u^*, let us choose a vector u such that $u^1 = 1$; then the expression (29) looks like

$$u = e_1 + \sum_{i=2}^{k} u^i e_i + \sum_{j=1}^{k} \frac{\partial \varphi^j(x)}{\partial \xi^1} e_{k+j} + \sum_{j=1}^{k} \sum_{i=2}^{k} u^i \frac{\partial \varphi^j(x)}{\partial \xi^i} e_{j+k}. \tag{30}$$

For coordinates of the elements (x, u^*) in L^{2k-1} we may take the numbers $u^2, \ldots, u^k, \xi_1, \ldots, \xi^k$. Since the first component of the vector u in the space C^{2k} equals one [see (30)], the coordinates of the row u^* in the manifold S^{2k-1} can be set to be the remaining components $v^2, \ldots, v^{2k}$ of the vector u in the space C^{2k}. In the chosen coordinates, the mapping τ is written (according to (30)) as

$$v^i = u^i, \quad i = 2, \dots, k;$$
$$v^{k+j} = \frac{\partial \varphi^j(x)}{\partial \xi^1} + \sum_{i=1}^{k} u^i \frac{\partial \varphi^j(x)}{\partial \xi^i}, \quad j = 1, \dots, k. \tag{31}$$

A simple calculation shows that the Jacobian of the mapping τ at the point (a, u_0) is equal to

$$\left| \frac{\partial^2 \varphi^j(a)}{\partial \xi^1 \partial \xi^i} \right|, \quad i, j = 1, \dots, k. \tag{32}$$

Consider now the mapping $\pi_{u_0} h$. Let us assume that it is a projection to some vector space A^{2k-1} with basis $e_2, \dots, e_{2k}$ along some line e_1^{**}. Then we have [see (27)]

$$f(x) = \pi_{u_0} h(x) = \sum_{i=2}^{k} \xi^i e_i + \sum_{\alpha=1}^{k} \varphi^\alpha(x) e_{k+\alpha}. \tag{33}$$

Thus we deduce

$$\frac{\partial^2 f(a)}{\partial \xi^1 \partial \xi^i} = \sum_{\alpha=1}^{k} \frac{\partial^2 \varphi^\alpha(x)}{\partial \xi^1 \partial \xi^i} \cdot e_{k+\alpha}, \quad i = 1, \dots, k,$$
$$\frac{\partial f(a)}{\partial \xi^j} = e_j, \quad j = 2, \dots, k.$$

Thus, in this case the vectors of the system (19) are linearly independent if and only if the Jacobian (32) is non-zero.

Statement «F» is proved.

Theorem 6. *Let f be an m-class smooth $(m \geqslant 3)$ mapping from a compact manifold M^k of dimension k to the vector space A^{2k-1} of dimension $2k-1$. It turns out that for the mapping f there is an arbitrarily m-close mapping g with all singular points non-degenerate and not lying on the boundary M^{k-1} of the manifold M^k.*

PROOF. Let us treat the vector space A^{2k-1} as a subspace of the vector space C^{2k} of dimension $2k$. Let B^1 be some one-dimensional subspace of the space C^{2k} not lying in A^{2k-1}. Denote the projection of the space C^{2k} to the space A^{2k-1} along B^1 by π. Fix a positive number ε; let h be a regular mapping of M^k to the vector space C^{2k} such that the mapping πh is ε-close to f (see Theorem 3). Let L^{2k-1} be the manifold of linear elements of the manifold $h(M^k)$ (see «F»); let L^{2k-2} be the submanifold of L^{2k-1} consisting of all elements of the type (x, u^*) where $x \in M^{k-1}$, and let τ be the mapping from L^{2k-1} to the sphere S^{2k-1} constructed in «F». It follows from «F» that if $u^* \in S^{2k-1}$ is not a singular point of the mapping τ and does not belong to the set $\tau(L^{2k-2})$ then all singular points of the mapping $\pi_u h$ are non-degenerate and do not belong to the boundary of the manifold M^k. By virtue of Theorems 4 and 1, there exists a vector u such that u^* satisfies the conditions described above and the mapping $\pi_u h$ is ε-close to πh. Thus, there is a 2ε-close to f mapping $g = \pi_u h$ satisfying the conditions of the theorem.

Theorem 6 is proved.

Canonical form of typical critical points and typical singular points

In Statements «C» and «E», several singular points of mappings from manifolds M^k to vector spaces A^1 and A^{2k-1}, were found to be non-degenerate. In Theorems 5 and 6, it was shown that all degenerate singular points of the considered mappings are not balanced, i.e. removable by small perturbations. However, we did not prove that those singular points called non-degenerate are balanced, i.e. they are preserved by small perturbations. The proof of this fact is not difficult, but we shall omit it. Also, we have not described the structure of the mapping in the neighbourhood of a non-degenerate singular point. It is not easy in the general situation; below we present the results without proving them.

G) Let a be a non-degenerate critical point of a real-valued function $f^1(x)$ defined on a manifold M^k. As noticed in Statement «A», the Taylor decomposition of the function $f^1(x)$ in the neighbourhood of the point a looks like (9). It turns out that (see [7]) by a coordinate change in the neighbourhood of a this Taylor decomposition can be transformed to that of the type

$$f^1(x) = f^1(a) + (x^1)^2 + \ldots + (x^s)^2 - (x^{s+1})^2 - \ldots - (x^k)^2, \qquad (34)$$

where the number s of positive squares is an invariant of the point a, i.e. does not depend on the coordinate choice in the neighbourhood of this point, and is not changed by a small perturbation. Thus, the function defined on a k-dimensional manifold has $k+1$ possible types of critical points ($s = 0, \ldots, k$). Since the mapping f of the manifold M^k does not define the function $f^1(x)$ directly, then the points of different type for the function may happen to be of different type for a mapping. Indeed, changing the sign of the function $f^1(x)$ interchanges the roles of s and $k-s$; thus the corresponding critical points belong to the same type of mapping critical points. It is worth mentioning that, in the general situation, one cannot get from the expression (9) to the expression (34) by a linear coordinate change, as it might seem. An evident linear transformation is just the first step of the transformation of (9) to (34). Under linear transformation the third-order and higher-order terms are preserved, whence they are absent in (34).

H) Let a be a non-degenerate critical point of $f : M^k \to A^{2k-1}$ (see «E»). It turns out (see [8]) that in the neighbourhoods of the points a and $f(a)$ one can change the coordinate systems (generally, the coordinate change is not linear) such that the mapping f in the neighbourhood of a has the following coordinate form:

$$\begin{aligned} &y^1 = (x^1)^2, \quad y^2 = x^1 x^2, \dots, \quad y^k = x^1 x^k, \\ &y^{k+1} = x^2, \quad y^{k+2} = x^3, \dots, \quad y^{2k-1} = x^k. \end{aligned} \tag{35}$$

Here the points a and $f(a)$ are taken to be the coordinate origins.

Statement «H» is quite a difficult theorem.

By using the expression (35), one can visualize the geometry of the mapping f in the neighbourhood of a, especially in the case when $k = 2$.

Chapter II

Framed manifolds

§ 1. Smooth approximations of continuous mappings and deformations

In the present section, we shall show that while studying the homotopy types of mappings from one manifold to another it is sufficient to consider only smooth mappings and smooth homotopies. This results from the following facts. Let M^k and N^l be two m-smooth closed manifolds. It turns out that in any homotopy class of mappings from N^l to M^k there exists an $(m-1)$-smooth mapping, and if two $(m-1)$-smooth mappings from the manifold N^l to the manifold M^k are homotopic, then there exists an $(m-3)$-smooth homotopy between these mappings. Thus, while studying mappings of smoothness class m, one has to consider the homotopies of class $m-3$. This loss of smoothness class can be avoided by using several tricks, but since the results of this section are to be used only for studying maps from sphere to sphere and the sphere is an analytic manifold, we need not worry about the loss of smoothness class; thus there is no sense in giving a more difficult proofs of a more precise statements.