Chapter 1

Correlation effects in one-dimensional systems

J.M.P. Carmelo

GCEP-Center of Physics, Universidade do Minho, Campus Gualtar, P-4710-057 Braga, Portugal

P.D. Sacramento

Departamento de Física and CFIF, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

D. Bozi

Instituto de Ciencia de Materiales, CSIC, Cantoblanco, E-28949 Madrid, Spain

L.M. Martelo

Departamento de Física, Faculdade de Engenharia, Universidade do Porto, P-4200-465 Porto, Portugal

We review developments concerning the effect of correlations on the electronic properties of one-dimensional systems, focusing our analysis on the one-dimensional Hubbard model. We consider methods used to describe the exotic properties of these systems, ranging from bosonization associated with the Tomonaga and Luttinger liquid behavior, to the Bethe ansatz solution, referring to all energy scales of solvable quantum problems and the pseudoparticle description. We use that description to study the model energy spectrum and the low-energy quantities. In the ensuing companion chapter we discuss the relation of the electronic operators to these quantum objects.

Contents

1.1. Effects of correlations

1.1.1. *Introduction*

This chapter is complementary to the ensuing companion chapter, Ref. 1, where some of the concepts and theoretical tools summarized here are applied to the study of low-dimensional correlated systems and a pseudofermion theory for the study of the finite-energy dynamical properties is reviewed.

The main point concerning the issues studied here and in the next chapter is that, when the electronic movements are restricted to low-dimensional geometries, the effects of the electron-electron interactions become non perturbative, and thus conventional Fermi liquid theory[2] does not apply. Instead, the low-energy physics of such interacting problems shows some basic similarities with that of the Tomonaga and Luttinger models.[3,4] The concept of a Tomonaga-Luttinger liquid[5] follows such similarities and refers to interacting low-dimensional models whose low-energy behavior belongs to the universality class of those models.

The one-dimensional Hubbard model[6] is one of the interacting electronic models whose low-energy physics belongs to such an universality class. Its importance is that it is the simplest lattice model which describes the effects of electronic correlations in low-dimensional complex materials. Indeed, the exotic non-Fermi-liquid behavior associated with the concept of a Tomonaga-Luttinger liquid is observed in some of such materials, see Refs. 7-11. One of the techniques used in the study of the low-energy physics of interacting models belonging to that universality class is bosonization, see Refs. 12-15. Some of these models have exact solutions which combined with their global symmetries provide the spectrum of all energy eigenstates. For instance, the global symmetry of the Hubbard model was recently shown to be $[SO(4) \times U(1)]/Z_2$.[16] Its exact energy spectrum and spectral properties can be calculated by combining symmetry with Bethe-ansatz techniques, see Refs. 17-31.

As further discussed in the ensuing chapter, Ref. 1, finite-energy spectral functions of the one-dimensional Hubbard model can be evaluated by expressing the generators of its energy eigenstates, associated with the model Bethe-ansatz solution and global symmetry, in terms of suitable rotated-electron operators.[28] Such operators are related to the electronic creation and annihilation operators by a unitary transformation.[28,32] Moreover, the combination of the rotated-electron basis with the information provided by the Bethe-ansatz solution and global symmetry reveals that the energy eigenstates correspond to simple occupancy configurations of exotic quantum objects which are closely related to the rotated electrons. The statistics of such objects can be defined in terms of a generalized Pauli principle.[33]

In addition to reviewing the several concepts and theoretical tools involved in the description of one-dimensional correlated electronic problems, below we also consider the specific case of the one-dimensional Hubbard model. Both here and in the following chapter we summarize how the low-energy and finite-energy physics of the model is described in terms of the above exotic objects. This provides important information about the non-perturbative microscopic processes which control the unconventional properties observed in many low-dimensional complex materials, which are described in Refs. 7-11.

1.1.2. *Fermi liquid theory*

In many three-dimensional systems the effect of interactions between fermionic particles is taken into account using the so-called Fermi liquid theory. In this theory[2] the excitation spectrum is fermionic but i) the quasiparticle parameters are renormalized with respect to the free system (like the effective mass), ii) the thermodynamic quantities have a behavior similar to the free electron case but also with renormalized parameters, iii) the lifetime of the quasiparticles is finite, except at the Fermi surface where it diverges like $\tau \sim (\epsilon - \epsilon_F)^{-2}$, (therefore the quasiparticles are well defined quantities for energies close to the Fermi surface) and iv) new collective modes emerge in the system. The lifetime of the quasiparticles is a consequence of the interactions and the energy of the excitations is expressed as,

$$\Delta E = \sum_{\vec{k}} \left(\epsilon_0(\vec{k}) - \mu \right) \Delta N(\vec{k}) + \frac{1}{2} \sum_{\vec{k},\vec{k}'} \Delta N(\vec{k}) f(\vec{k},\vec{k}') \Delta N(\vec{k}') + \cdots , \quad (1.1)$$

where $\Delta N(\vec{k})$ is the deviation of the quasiparticle distribution with respect to the equilibrium distribution and $f(\vec{k},\vec{k}')$ results from the residual interactions between the quasiparticles. The interactive term is of the same order of magnitude as the free term since the deviation of the energies, with respect to the chemical potential, is also small in the regime where the quasiparticles have a long lifetime. The fermionic nature of the quasiparticles implies that the excitation spectrum remains similar to the free electron case. The spectrum is a continuum of low energy excitations due to the excitations of particles and holes of arbitrarily small energies and momenta around the Fermi surface, in addition to the plasma mode at finite frequency.

1.1.3. *One-dimensional systems*

The one-dimensional case is special due to the fact that the Fermi surface has only two points. An important consequence is that any instability of momentum $2k_F$ couples the two states at the Fermi surface and may lead to a gap in the spectrum (Peierls instability). The resulting spectrum is qualitatively different from the spectrum of the Fermi gas (which has no gap) showing that the interactions have an important role in the one-dimensional case. In general, in systems of higher dimension, any instability that couples two points of the Fermi surface has a null measure and therefore is not relevant to the behavior of the system (with the exception of nesting). The Peierls instability in one dimension suggests that the excitations of the system may have a different nature and can be described in terms of collective bosonic excitations.

Two models of one-dimensional conductors are normally considered in the literature. The first is a continuum model where one considers electrons with weak interactions and where the electrons occupy states that are extended. The other model is suitable in the opposite limit, where the electrons have wave functions which are more localized, typically with a strong atomic character. The model is immersed in a lattice and describes situations of narrow bands where the interactions (or at least the correlations) between the electrons are typically strong. The continuum model was originally considered by Tomonaga[3] and Luttinger[4] and the lattice model was introduced by Hubbard.[6] Actually, the Tomonaga-Luttinger model also constitutes a good starting point in situations where the interactions between the electrons are not weak, as long as one is interested in the low-energy and small-momentum properties, where the lattice details are not important.

1.1.4. *Tomonaga and Luttinger models*

The dispersion relation of the free electron gas is such that only the electrons close to the Fermi surface are important. It is therefore usual to linearize the dispersion relation in the form $\epsilon_r(k) = v_F(rk - k_F)$ introducing two branches ($r = \pm 1$) around the two Fermi points $\pm v_F$. It is then necessary to introduce a cut-off k_0. Such a procedure leads to the Tomonaga model.[3] It is also possible (and convenient) to consider a dispersion relation without cut-off, extending the bands to $\pm\infty$ (note that considering only the regime of small energies these additional states are not expected to affect the behavior). This choice corresponds to the Luttinger model.[4] This model describes therefore two types of fermions corresponding to the two branches $r = \pm 1$, whose densities interact via two interactions: one term corresponds to interactions between electrons in the same branch (g_2) and the other between electrons in different branches (g_4). The Luttinger model is defined by $H = H_0 + H_2 + H_4$ where,

$$\begin{aligned}
H_0 &= \sum_{r,k,s} v_F(rk - k_F) : c^\dagger_{rks} c_{rks} : , \\
H_2 &= \frac{1}{L} \sum_{p,s,s'} \left(g_{2,||}(p)\delta_{s,s'} + g_{2,\perp}(p)\delta_{s,-s'} \right) \rho_{+,s}(p)\rho_{-,s'}(-p) \, , \\
H_4 &= \frac{1}{2L} \sum_{r,p,s,s'} \left(g_{4,||}(p)\delta_{s,s'} + g_{4,\perp}(p)\delta_{s,-s'} \right) : \rho_{r,s}(p)\rho_{r,s'}(-p) : \, .
\end{aligned} \tag{1.2}$$

The first term is the usual kinetic operator and in the interacting terms the spin, s, dependence on the interactions is considered in the forms $||$ or $\perp$. The operators have to be normal ordered to eliminate the infinite number of states introduced in the model, by subtracting the average value in the ground state.

Besides these terms that only include low-momentum scattering, the Luttinger model may be extended considering additional terms that take into account the possibility of finite momentum excitations, where the two branches are coupled, allowing backscattering or considering Umklapp processes. The model without these additional terms is exactly solvable.[5] The Hamiltonian conserves the total charge, the spin, and also these quantities separately in each branch. Therefore the charge and spin currents are also conserved.

1.1.5. *Bosonization*

One way to solve the Luttinger model results from the property that the commutator of the density operators is given by,

$$[\rho_{r,s}(p), \rho_{r',s'}(-p')] = -\delta_{r,r'}\delta_{s,s'}\delta_{p,p'} \frac{rpL}{2\pi} \, , \tag{1.3}$$

where L is the system length. Therefore the density operators satisfy bosonic commutation relations. Defining charge and spin density operators in each branch $\rho_r(p) = \left(\rho_{r,\uparrow}(p) + \rho_{r,\downarrow}(p)\right)/\sqrt{2}$, $\sigma_r(p) = \left(\rho_{r,\uparrow}(p) - \rho_{r,\downarrow}(p)\right)/\sqrt{2}$, and using Kronig's identity, where the kinetic term of the Hamiltonian can be written in a bilinear form in the density operators, one obtains that the Luttinger model can be diagonalized via a Bogoliubov-Valatin transformation leading to a separation of the charge and spin degrees of freedom $\tilde{H} = \tilde{H}_\rho + \tilde{H}_\sigma + \tilde{H}_c$,

$$\tilde{H}_\nu = \frac{2\pi v_\nu}{L} \sum_{p\neq 0} : (\tilde{\nu}_+(p)\tilde{\nu}_+(-p) + \tilde{\nu}_-(p)\tilde{\nu}_-(p)) : ,$$
$$\tilde{H}_c = \frac{\pi}{2L} \sum_{\nu=\rho,\sigma} \left(\frac{v_\nu}{K_\nu}(N_{+,\nu} + N_{-,\nu})^2 + v_\nu K_\nu (N_{+,\nu} - N_{-,\nu})^2 \right) , \quad (1.4)$$

($\nu = \rho, \sigma$) where the charge and spin velocities are given by,

$$v_\nu = \sqrt{\left(v_F + \frac{1}{2\pi}(g_{4||} \pm g_{4\perp})\right)^2 - \left(\frac{1}{2\pi}(g_{2||} \pm g_{2\perp})\right)^2} , \quad (1.5)$$

($+, -$ corresponds to ρ, σ respectively), $N_{r,\nu} = \rho_{r,\nu}(p = 0)$, and the parameters K_ν are given by,

$$K_\nu = \sqrt{\frac{\pi v_F + \frac{1}{2}(g_{4||} \pm g_{4\perp}) - \frac{1}{2}(g_{2||} \pm g_{2\perp})}{\pi v_F + \frac{1}{2}(g_{4||} \pm g_{4\perp}) + \frac{1}{2}(g_{2||} \pm g_{2\perp})}} . \quad (1.6)$$

The operators of the charge and spin degrees of freedom commute among themselves and are separately conserved, as said above. Moreover, they propagate with different velocities leading to an effective separation in real space.

The non-existence of fermionic excitations implies that there are no quasiparticles at the Fermi surface. The residue of the pole of the Green function is zero. The density of states vanishes at the Fermi surface and therefore we find a spectral weight reduction near that surface. There is ample experimental evidence for these unusual properties. Using photoemission experiments it was found[7] that there is no Fermi edge in the dispersion relation in the two quasi-one-dimensional compounds $K_{0.3}MoO_3$ and $(TaSe_4)_2I$. Also, spin-charge separation was observed[8] in the material $SrCuO_2$. Further evidence for spin-charge separation was found in the organic conductor TTF-TCNQ[9] and it was further established that the experimental results are not compatible with standard band theory and that the interaction/correlation effects are determinant. Further unconventional behavior was observed in the Mott insulators $SrCuO_2$ and Sr_2CuO_3 whose dispersion relations are not consistent with band theory.[10,11]

The diagonalization of the Luttinger model shows that the Hamiltonian and its excitations are described by bosonic modes. However, the calculation of arbitrary correlation functions requires products of fermionic operators. The full solution of the model including all its correlation functions involves the representation of the fermionic operators in terms of the bosonic operators. This procedure is called bosonization.[12]

The charge density and the current, $j_s(x)$, satisfy the following commutator,

$$[\rho_s(x), j_s(x')] = -\frac{i}{\pi}\frac{\partial}{\partial x}\delta(x-x')\,, \tag{1.7}$$

where $j_s(x) =: \rho_{+,s}(x) - \rho_{-,s}(x)$:. Let us then consider two conjugate fields, $\Phi_s(x)$ and $\Pi_s(x)$, satisfying the canonical commutation relation $[\Phi_s(x), \Pi_s(x')] = i\delta(x-x')$. Identifying the operators $\rho_s(x) = -\frac{1}{\pi}\frac{\partial}{\partial x}\Phi_s(x)$ and $j_s(x) = \Pi_s(x)$, satisfies the commutation relation. This result suggests that we may represent the density by a bosonic field of the form $\frac{\partial \Phi_s(x)}{\partial x} = -\pi\rho_s(x)$. The introduction of a fermion at site x creates a kink (soliton) of amplitude π in the bosonic field. The introduction of one particle at point x implies that the rest of the particles have to adjust to accept the new particle at x. We may therefore express the fermionic operator in the form of a translation operator (plus a phase needed to yield the anti-commutation relations),[13]

$$\psi_{r,s}(x) \sim \lim_{\alpha\to 0}\frac{1}{2\pi\alpha}e^{irk_F x - ir\Phi_{r,s}(x) + i\pi\int_{-\infty}^{x} dz \Pi_{r,s}(z)}\,. \tag{1.8}$$

The Luttinger model may also be expressed as $H = H_\rho + H_\sigma$, where,

$$H_\nu = \int dx \left(\frac{\pi u_\nu K_\nu}{2}\Pi_\nu^2 + \frac{u_\nu}{2\pi K_\nu}(\partial_x \phi_\nu)^2 \right)\,, \tag{1.9}$$

with $\nu = \rho, \sigma$. Such a transformation allows the calculation of all the correlation functions of the Luttinger model (gaussian model). The correlation functions are characterized by critical exponents which are non-universal. The calculation of the correlation functions is reduced in this context to averages over a gaussian distribution. For instance, within bosonization it can be shown that,[14]

$$< n(x)n(0) >= \frac{K_\rho}{(\pi x)^2} + A_1\frac{\cos(2k_F x)}{x^{1+K_\rho}}\frac{1}{\ln^{-3/2}(x)} + A_2\frac{\cos(4k_F x)}{x^{4K_\rho}}\,, \tag{1.10}$$

and

$$< \vec{S}(x)\cdot\vec{S}(0) >= \frac{1}{(\pi x)^2} + B_1\frac{\cos(2k_F x)}{x^{1+K_\rho}}\frac{1}{\ln^{1/2}(x)}\,. \tag{1.11}$$

Also, the quantity K_ρ determines the singularity of the momentum distribution,

$$n_k \sim \frac{1}{2} - \frac{k-k_F}{|k-k_F|}|k-k_F|^\alpha\,, \tag{1.12}$$

where $\alpha = (K_\rho + 1/K_\rho - 2)/4$ is the exponent of the single-particle density of states $N(\omega) \sim |\omega|^\alpha$ as well.

One needs to determine the parameter K_ρ for each specific model. We may take several routes. A possible way is to note that the coefficient u_ρ/K_ρ in the Hamiltonian is proportional to the variation of the ground state energy with respect to the particle number, since the gradient of the phase field ϕ_ρ is proportional to the density,

$$\frac{1}{L}\frac{\partial^2 E_0(n)}{\partial n^2} = \frac{\pi}{2}\frac{u_\rho}{K_\rho}\,. \tag{1.13}$$

Once the ground state energy for instance for the Hubbard model may be obtained from the exact solution via the Bethe ansatz, we may calculate the parameter K_ρ and therefore the critical exponents. It turns out that $1/2 < K_\rho < 1$. For large on-site repulsion, U, $K_\rho = 1/2$ and $\alpha = 1/8$. We will return to this point ahead. Very similar results can be obtained for the closely related Tomonaga model.

1.1.6. *Tomonaga-Luttinger liquids*

Even though the Tomonaga and Luttinger models are very simplified, they describe in a qualitatively correct way the low energy properties of many interacting one-dimensional systems. A broadly used nomenclature classifying interacting one-dimensional systems whose low-energy behavior falls in the universality class of the Tomonaga and Luttinger models is that of a Tomonaga-Luttinger liquid. One of the consequences of this universality is that the critical behavior of the Tomonaga-Luttinger liquids is determined by the critical exponents of the Tomonaga and Luttinger models (however, the values of the exponents depend on the parameters of each specific model). Another class of one-dimensional systems is the Luther-Emery class[15] which groups systems with gaps in the spectrum. In particular, for instance the addition of the backscattering term to the Luttinger model leads in some regimes (for an attractive interaction) to a gap in the spin excitations. The Umklapp term leads to a gap in the charge excitations.

1.2. Hubbard model

A model whose low-energy physics is in the class of the Tomonaga-Luttinger liquids is the repulsive Hubbard model, away from half-filling. On the other hand, the attractive Hubbard model has a gap in the spin excitations and is in the class of the Luther-Emery model. The model is described by the Hamiltonian,

$$\hat{H} = -t\sum_{j,\sigma}\left(c^\dagger_{j,\sigma}c_{j+1,\sigma} + c^\dagger_{j+1,\sigma}c_{j,\sigma}\right) + U\hat{D} - \frac{U}{2}N + \frac{U}{4}N_a\,,$$
$$\hat{D} = \sum_j c^\dagger_{j,\uparrow}c_{j,\uparrow}c^\dagger_{j,\downarrow}c_{j,\downarrow}\,, \tag{1.14}$$

describing a tight-binding model for N electrons with nearest-neighbor amplitude t where electrons of opposite spins interact with each other via a local repulsive

Coulomb potential, U, on a lattice with N_a sites. In this model only states with $l = 0$ are taken into account and therefore in each lattice site the maximal number of electrons is 2. $\hat{D}$ is the double occupancy operator. In general, we use units of Planck constant $\hbar$ and lattice constant a such that $\hbar = a = 1$. The model has $[SO(4) \times U(1)]/Z_2$ global symmetry and (if N_a is even) commutes with the six generators of the spin and eta-spin $SU(2)$ algebras and the generator of a hidden symmetry which is half of the number operator of sites singly occupied by "rotated electrons".[16] The spin generators and the eta-spin generators are given by $\hat{S}^z_s = -\frac{1}{2}[\hat{N}_\uparrow - \hat{N}_\downarrow]$, $\hat{S}^\dagger_s = \sum_j c^\dagger_{j,\downarrow} c_{j,\uparrow}$ and $\hat{S}_s = \sum_j c^\dagger_{j,\uparrow} c_{j,\downarrow}$; $\hat{S}^z_c = -\frac{1}{2}[N_a - \hat{N}]$, $\hat{S}^\dagger_c = \sum_j (-1)^j c^\dagger_{j,\downarrow} c^\dagger_{j,\uparrow}$ and $\hat{S}_c = \sum_j (-1)^j c_{j,\uparrow} c_{j,\downarrow}$. We call S_c (and S_s) the η-spin (and spin) value of an energy eigenstate and S^z_c (and S^z_s) its η-spin (and spin) projection.

1.2.1. *Bethe ansatz solution*

The Hubbard model has been solved exactly via the Bethe ansatz.[17] That solution refers to a subspace spanned by the lowest-weight states (LWSs) of both the η-spin and spin algebras. The latter states are such that $S_\alpha = -S^z_\alpha$ where $\alpha = c, s$. Within the thermodynamic limit the solution involves degrees of freedom that correspond to different rapidity branches. In addition to a $c0$ charge-momentum rapidity, there are sets of $\alpha\nu$ rapidities. The general rapidity branch label $\alpha\nu$ is such that $\alpha = c, s$ and $\nu = 0, 1, 2, ...$ for $\alpha = c$ and $\nu = 1, 2, ...$ for $\alpha = s$. The $c\nu$ and $s\nu$ rapidities are associated with the charge and spin degrees of freedom, respectively. For $\nu > 0$ the $c\nu$ and $s\nu$ rapidities may be associated with charge 2ν-holon and spin 2ν-spinon composite objects, respectively, where holons and spinons are elementary "particles" which carry η-spin $1/2$ and spin $1/2$, respectively.[28]

For electronic densities $n \leq 1$, the ground state has finite occupancies for the charge $c0$ and spin $s1$ branches only, reinforcing the idea that in that system there is a separation of degrees of freedom. However, the corresponding quantum objects called pseudoparticles in Refs. 27,28 are not independent and have residual interactions. In the $U \to \infty$ limit the equations that determine the residual interactions between those pseudoparticles and corresponding degrees of freedom decouple (but the spin degrees of freedom affect the charge degrees of freedom through a boundary condition term).

The Bethe ansatz equations are given by,

$$q_j = q(k_j) = k_j + \frac{2}{N_a}\sum_{\alpha\nu\neq c0}\sum_{j'=1}^{N^*_{\alpha\nu}} N^R_{\alpha\nu}(\Lambda_{\alpha\nu,\,j'})\arctan\Big(\frac{\sin(k_j)-\Lambda_{\alpha\nu,\,j'}}{\nu U/4t}\Big)\,; \quad j=1,2,...,N_a\,, \tag{1.15}$$

$$\begin{aligned} q_j &= q(\Lambda_{\alpha\nu,\,j}) \\ &= k_{\alpha\nu,\,j} - (\delta_{\alpha,c}-\delta_{\alpha,s})\frac{2}{N_a}\sum_{j'=1}^{N_a} N^R_{c0}(k_{j'})\arctan\Big(\frac{\Lambda_{\alpha\nu,\,j}-\sin(k_{j'})}{\nu U/4t}\Big) \\ &- \frac{1}{N_a}\sum_{\nu'=1}^{N_a/2}\sum_{j'=1}^{N^*_{\alpha\nu'}} N^R_{\alpha\nu'}(\Lambda_{\alpha\nu',\,j'})\Theta_{\nu,\,\nu'}\Big(\frac{\Lambda_{\alpha\nu,\,j}-\Lambda_{\alpha\nu',\,j'}}{U/4t}\Big)\,; \\ & j=1,2,...,N^*_{\alpha\nu}\,; \quad \alpha\nu\neq c0\,, \end{aligned} \tag{1.16}$$

where $k_{\alpha\nu,\,j} = \delta_{\alpha,c}2\,\mathrm{Re}\,\{\arcsin(\Lambda_{c\nu,\,j}+i\nu U/4t)\}$ with $j=1,2,...,N^*_{\alpha\nu}$ for $\alpha\nu\neq c0$ and the value of the number $N^*_{\alpha\nu}\leq N_a$ is defined by Eqs. (B.6) and (B.7) of Ref. 28. The occupied and unoccupied values k_j of the charge-momentum rapidity and the occupied and unoccupied values $\Lambda_{\alpha\nu,\,j}$ of the $\alpha\nu$ rapidities of a given energy eigenstate are determined by these equations which are valid for large values of N_a and N and were first introduced by Takahashi.[19] Here we wrote them in functional form in terms of the distribution functions $N^R_{c0}(k_j)$ and $N^R_{\alpha\nu}(\Lambda_{\alpha\nu,\,j})$, whose occupancies are well defined for each state. The function $\Theta_{\nu,\,\nu'}(x)$ is given in Eq. (B.5) of Ref. 28. The equations (1.15) and (1.16) include the discrete bare-momentum values q_j of the form $q_j=[2\pi/N_a]\,I^{c0}_j$ and $q_j=[2\pi/N_a]\,I^{\alpha\nu}_j$ for $\alpha\nu\neq c0$ where the numbers I^{c0}_j and $I^{\alpha\nu}_j$ with $j=1,2,...,N_a$ and $j=1,2,...,N^*_{\alpha\nu}$, respectively, are the quantum numbers whose occupancy configurations describe the energy eigenstates. The latter numbers are integers or half-odd integers as a result of the following boundary conditions,

$$e^{iq_j\,N_a} = (e^{i\pi})^{[\sum_{\alpha=c,\,s}\sum_{\nu=1}^{\infty}N_{\alpha\nu}]}\,, \tag{1.17}$$

in the case of the $c0$ branch and,

$$e^{iq_j\,N_a} = (e^{i\pi})^{[1+N^*_{\alpha\nu}]} = (e^{i\pi})^{[1+N_{c0}+N_{\alpha\nu}]}\,; \qquad \alpha=c,s\,, \quad \nu=1,2,...\,, \tag{1.18}$$

for the $\alpha\nu\neq c0$ branches. Thus, for $\alpha\nu\neq c0$ the quantum numbers $I^{\alpha\nu}_j$ are integers (half-odd integers), if $N^*_{\alpha\nu}$ is odd (even). On the other hand, the quantum numbers I^{c0}_j are integers (half-odd integers), if $\frac{N_a}{2}-\sum_{\alpha\nu\neq c0}N_{\alpha\nu}$ is odd (even). There is for the $c0$ branch (and the $\alpha\nu\neq c0$ branches) of all energy eigenstates a one-to-one correspondence between the discrete bare-momentum value q_j and the discrete charge-momentum rapidity value k_j (and discrete $\alpha\nu$ rapidity value $\Lambda_{\alpha\nu,\,j}$) with the same value for the index j such that $j = 1, ..., N_a$ (and $j = 1, ..., N^*_{\alpha\nu}$). That correspondence is such that there is no level crossing between the set of N_a $c0$ pseudoparticle discrete bare-momentum values $\{q_j\}$ (and $N^*_{\alpha\nu}$ $\alpha\nu \neq c0$ pseudoparticle discrete bare-momentum values $\{q_j\}$) and the set of charge momentum rapidity values $\{k_j\}$ (and $\alpha\nu$ rapidity values $\{\Lambda_{\alpha\nu,\,j}\}$). This means that if $q_j > q_{j'}$ for the $c0$ branch (and for a $\alpha\nu \neq c0$ branch), then $k_j > k_{j'}$ (and $\Lambda_{\alpha\nu,\,j} > \Lambda_{\alpha\nu,\,j'}$) for the same values of j and j', respectively.

The occupancies of the bare-momentum values q_j obey a Pauli principle, *i.e.* a discrete bare-momentum value q_j can either be unoccupied or singly occupied. Such occupancies can be described by bare-momentum distribution functions $N_{\alpha\nu}(q_j)$. Moreover, there is also a one-to-one correspondence between the occupied discrete bare-momentum values q_j and the occupied discrete charge-momentum rapidity values k_j or discrete $\alpha\nu$ rapidity values $\Lambda_{\alpha\nu,\,j}$, such that $j = 1, ..., N_{c0}$ or $j = 1, ..., N_{\alpha\nu}$, respectively. That correspondence is behind the equalities $N_{c0}(q_j) = N^R_{c0}(k_j)$ and $N_{\alpha\nu}(q_j) = N^R_{\alpha\nu}(\Lambda_{\alpha\nu,\,j})$ for the same values of the index j. The bare-momentum distribution functions read $N_{\alpha\nu}(q_j) = 1$ for occupied discrete bare-momentum values q_j and $N_{\alpha\nu}(q_j) = 0$ for unoccupied discrete bare-momentum values q_j. The pseudoparticle representation of Ref. 28 corresponds to the description of the energy eigenstates in terms of the discrete bare-momentum q_j occupancy configurations, instead of the charge-momentum rapidity k_j and $\alpha\nu$ rapidity $\Lambda_{\alpha\nu,\,j}$ occupancy configurations. Thus, the above distributions $N_{\alpha\nu}(q_j)$ are the $\alpha\nu$ bare-momentum pseudoparticle distribution functions. These functions are for all energy eigenstates the eigenvalues of the following pseudoparticle bare-momentum distribution function operators,

$$\hat{N}_{\alpha\nu}(q_j) = b^\dagger_{q_j\,,\,\alpha\nu}\, b_{q_j\,,\,\alpha\nu}\,. \tag{1.19}$$

Here the operator $b^\dagger_{q_j\,,\,\alpha\nu}$ (and $b_{q_j\,,\,\alpha\nu}$) creates (and annihilates) a $\alpha\nu$ pseudoparticle of bare-momentum q_j. Each LWS of both the η-spin and spin algebras is uniquely specified by the values of $[N_a - N]$, $[N_\uparrow - N_\downarrow]$, and the set of bare-momentum distribution functions $\{N_{\alpha\nu}(q_j)\}$ such that $\nu = 0, 1, 2, ...$ for $\alpha = c$ and $\nu = 1, 2, ...$ for $\alpha = s$ and $j = 1, ..., N^*_{\alpha\nu}$.

One finds by straightforward manipulation of the Bethe-ansatz equations (1.15) and (1.16) that the spacings $[k_{j+1} - k_j]$, $[k_{c\nu,\,j+1} - k_{c\nu,\,j}]$, and $[\Lambda_{\alpha\nu,\,j+1} - \Lambda_{\alpha\nu,\,j}]$ depend on the value of j and are given by,

$$k_{j+1} - k_j = \frac{2\pi}{L}\frac{1}{2\pi\rho(k_j)}\,; \qquad k_{c\nu,\,j+1} - k_{c\nu,\,j} = \frac{2\pi}{L}\frac{1}{2\pi\rho_{c\nu}(\Lambda_{c\nu,\,j})}\,;$$

$$\Lambda_{\alpha\nu,\,j+1} - \Lambda_{\alpha\nu,\,j} = \frac{2\pi}{L}\frac{1}{2\pi\sigma_{\alpha\nu}(\Lambda_{\alpha\nu,\,j})}\,. \tag{1.20}$$

Here $2\pi\rho_{c\nu}(\Lambda) = \frac{1}{2}\,2\pi\sigma_{c\nu}(\Lambda)\,\mathrm{Re}\sqrt{1-(\Lambda+i\nu U/4t)^2}$ and the functionals $2\pi\rho(k)$ and $2\pi\sigma_{\alpha\nu}(\Lambda)$ are the solutions of well defined coupled integral equations. In contrast, the bare-momentum spacing is independent of j and given by $[q_{j+1}-q_j] = 2\pi/L$ and hence one can replace q_j by a continuous bare-momentum q so that the charge-momentum rapidity $k_j = k(q_j)$ and the $\alpha\nu$ rapidity $\Lambda_{\alpha\nu,\,j} = \Lambda_{\alpha\nu}(q_j)$ are for each energy eigenstate described by functions of q, $k(q)$ and $\Lambda_{\alpha\nu}(q)$, respectively. The Takahashi's equations (1.15) and (1.16) are expressed in terms of the corresponding bare-momentum pseudoparticle distribution functions as follows,

$$q = k(q) + \frac{1}{\pi}\sum_{\alpha\nu\neq c0}\int_{-q_{\alpha\nu}}^{q_{\alpha\nu}} dq'\,N_{\alpha\nu}(q')\,\arctan\Big(\frac{\sin(k(q))-\Lambda_{\alpha\nu}(q')}{\nu U/4t}\Big)\,, \tag{1.21}$$

$$\begin{aligned} q &= k_{\alpha\nu}(q) - (\delta_{\alpha,c}-\delta_{\alpha,s})\frac{1}{\pi}\int_{q_c^-}^{q_c^+} dq'\,N_{c0}(q')\,\arctan\Big(\frac{\Lambda_{\alpha\nu}(q)-\sin(k(q'))}{\nu U/4t}\Big) \\ &- \frac{1}{2\pi}\sum_{\nu'=1}^{N_a/2}\int_{-q_{\alpha\nu}}^{q_{\alpha\nu}} dq'\,N_{\alpha\nu'}(q')\Theta_{\nu,\,\nu'}\Big(\frac{\Lambda_{\alpha\nu}(q)-\Lambda_{\alpha\nu'}(q')}{U/4t}\Big)\,; \qquad \alpha\nu\neq c0\,. \end{aligned} \tag{1.22}$$

Here $k_{\alpha\nu}(q) = \delta_{\alpha,c}2\,\mathrm{Re}\,\{\arcsin(\Lambda_{c\nu}(q)+i\nu U/4t)\}$ for $\alpha\nu\neq c0$, the function $\Theta_{\nu,\,\nu'}(x)$ is defined in Eq. (B.5) of Ref. 28, and the limiting bare-momentum values $q_{\alpha\nu}$ and $q_c^{\pm}$ are defined by Eqs. (B.14) and (B.16)-(B.17), respectively, of that reference. For a given energy eigenstate specified by the set of bare-momentum distribution functions $\{N_{\alpha\nu}(q)\}$, the solution of Eqs. (1.21) and (1.22) uniquely defines the set of occupied and unoccupied values of the charge rapidity momentum function $k(q)$ and set of $\alpha\nu$ rapidity functions $\Lambda_{\alpha\nu}(q)$ associated with that state.

For LWSs the energy and the momentum spectra are given by,

$$\begin{aligned} E &= -\frac{t\,N_a}{\pi}\int_{q_c^-}^{q_c^+} dq\,N_{c0}(q)\cos(k(q)) - \frac{U}{2}N + \frac{U}{4}N_a \\ &+ \frac{2t\,N_a}{\pi}\sum_{\nu=1}^{N_a/2}\int_{-q_{c\nu}}^{q_{c\nu}} dq\,N_{c\nu}(q)\,\mathrm{Re}\Big\{\sqrt{1-(\Lambda_{c\nu}(q)+i\nu U/4t)^2}\Big\} \end{aligned} \tag{1.23}$$

and

$$P = \frac{L}{2\pi}\Big\{\int_{q_{c0}^-}^{q_{c0}^+} dq\, N_c(q)\, q + \sum_{\nu=1}^{\infty}\int_{-q_{s\nu}}^{q_{s\nu}} dq\, N_{s\nu}(q)\, q$$
$$+ \sum_{\nu=1}^{\infty}\int_{-q_{c\nu}}^{q_{c\nu}} dq\, N_{c\nu}(q)\, [\frac{\pi}{a}(1+\nu) - q]\Big\}\,, \qquad (1.24)$$

respectively, where when $|P| > \pi$ the value of the momentum should be brought to the first Brillouin zone. The states occupation numbers are not independent. For instance, they have to obey the sum rules,

$$N = N_{c0} + 2\sum_{c\nu\neq c0}\nu N_{c\nu}\,,$$
$$N_\uparrow - N_\downarrow = N_{c0} - 2\sum_{s\nu}\nu N_{s\nu}\,, \qquad (1.25)$$

and the values of the quantum numbers I_j^{c0} and $I_j^{\alpha\nu}$ are contained in intervals such that the corresponding bare-momentum values $q_j = [2\pi/N_a]\, I_j^{c0}$ and $q_j = [2\pi/N_a]\, I_j^{\alpha\nu}$ belong to the ranges $q_c^- \leq q_j \leq q_c^+$ and $-q_{\alpha\nu} \leq q_j \leq q_{\alpha\nu}$, respectively. For the low-energy subspace spanned by states with vanishing occupancies for the sets of numbers $\{N_{s\nu}\} = 0$ and $\{N_{c\nu}\} = 0$ for $\nu > 1$ and $\nu > 0$, respectively, the magnetization provided in Eq. (1.25) simplifies to $N_\uparrow - N_\downarrow = N_{c0} - 2N_{s1}$.

The generators of the LWSs onto the electronic vacuum can be expressed as products of the pseudoparticle operators $b^\dagger_{q,\alpha\nu}$ and all energy eigenstates are also eigenstates of the operators (1.19) whose eigenvalues are the pseudoparticle numbers. The pseudoparticles do not obey fermionic statistics (except for the $c0$ pseudoparticles) but their statistics can be classified according the generalized Pauli principle of Haldane.[33] The pseudoparticle operator anticommutation relations are given by,

$$\{b^\dagger_{q_j,\,\alpha\nu},\, b_{q_{j'},\,\alpha'\nu'}\} = \delta_{\alpha\nu,\,\alpha'\nu'}\, F(q_j,\, q_{j'})\,;$$
$$\{b^\dagger_{q_j,\,\alpha\nu},\, b^\dagger_{q_{j'},\,\alpha'\nu'}\} = \{b_{q_j,\,\alpha\nu},\, b_{q_{j'},\,\alpha'\nu'}\} = 0\,, \qquad (1.26)$$

where

$$F(q_j,\, q_{j'}) = \delta_{q_j,\, q_{j'}}\,, \qquad (1.27)$$

when for $\alpha\nu = \alpha'\nu'$ both the $I_j^{\alpha\nu}$ and $I_{j'}^{\alpha\nu}$ numbers such that $q_j = [2\pi/L]\, I_j^{\alpha\nu}$ and $q_j = [2\pi/L]\, I_{j'}^{\alpha\nu}$, respectively, are integers or half-odd integers and,

$$F(q_j,\, q_{j'}) = \frac{i}{L}\frac{1}{e^{+i(q_j - q_{j'})/2}\,\sin([q_j - q_{j'}]/2)}\,, \qquad (1.28)$$

when for $\alpha\nu = \alpha'\nu'$ the above $I_j^{\alpha\nu}$ numbers are integers (or half-odd integers) and the $I_{j'}^{\alpha\nu}$ numbers are half-odd integers (or integers).

The momentum dependent creation and annihilation operators can be formally defined locally on an effective $\alpha\nu$ lattice, whose lattice constant $a_{\alpha\nu}$ is defined so that the length of such a lattice is $\alpha\nu$ independent and equal to L: $a_{\alpha\nu} = a\frac{N_a}{N^*_{\alpha\nu}}$. Hence the above numbers $N^*_{\alpha\nu}$ are also the number of $\alpha\nu$ lattice sites. The numbers $N^*_{\alpha\nu} \leq N_a$ correspond to the upper and lower bounds on the quantum numbers $I_j^{\alpha\nu}$ for $\alpha\nu \neq c0$ such that $j = 1, ..., N^*_{\alpha\nu}$ of the above Bethe-ansatz equations. Such equations are valid for $N_a >> 1$ within the so called Takahashi string hypothesis[19] and provide naturally the values of the number $N^*_{\alpha\nu} \leq N_a$, which are given in Eqs. (B.6) and (B.7) of Ref. 28. The corresponding numbers $\pm q_{\alpha\nu}$ refer to the largest possible absolute bare-momentum value (the boundaries of the $\alpha\nu$ bare-momentum Brillouin zone). Only for one branch ($c0$-pseudoparticles), does the total number of allowed discrete momenta equals the number of "real" lattice sites N_a.

In the standard Bethe ansatz literature one often uses the charge $c0$ and spin $s1$ rapidity density functions $2\pi\rho(k)$ and $2\pi\sigma_{s1}(\Lambda)$, respectively, appearing in Eq. (1.20),[17] which are the only relevant ones for the above low-energy subspace. For that subspace, they obey the simplified integral equations,

$$2\pi\rho(k) = 1 + \frac{U}{4\pi t}\cos k \int_{-B}^{B} d\Lambda' \frac{2\pi\sigma_{s1}(\Lambda')}{[U/4t]^2 + [\sin k - \Lambda']^2}\,,$$

$$2\pi\sigma_{s1}(\Lambda) = \frac{U}{4\pi t}\int_{-Q}^{Q} dk' \frac{2\pi\rho(k')}{[U/4t]^2 + [\sin k' - \Lambda]^2} - \frac{U}{8\pi t}\int_{-B}^{B} d\Lambda' \frac{2\pi\sigma_{s1}(\Lambda')}{[U/4t]^2 + [(\Lambda - \Lambda')/2]^2}\,.$$

In these equations the cutoff parameters Q and B are defined by,

$$\int_0^Q dk' 2\pi\rho(k') = \pi n = 2k_F\,,$$

$$\int_0^B d\Lambda' 2\pi\sigma_{s1}(\Lambda') = \pi n_{\downarrow} = k_{F\downarrow}\,.$$

The solution of the problem reveals that in general the state of the system is metallic, except at half-filling ($n = N/N_a = 1$) where it constitutes a Mott-Hubbard insulator for any finite value of $U > 0$.[17] The thermodynamics of the model was solved and leads to a low temperature specific heat that is linear in the temperature and to a susceptibility that is finite (as in a Luttinger liquid and Fermi liquid). Except for the one-electron properties, this result suggests an alternative description of the system in a form closer to that of the Landau theory.

From the thermodynamics point of view, the system has similar properties to those of a Fermi liquid but the correlation functions are qualitatively different. In particular, the correlation function of a single particle is qualitatively different, as seen above for the Luttinger liquid. The analysis of the correlation functions of the Hubbard model will reveal that at low energies the model is indeed of the Tomonaga-Luttinger liquid class. However, at finite energies a new description, reviewed in Ref. 1, is necessary.

Below, we consider often the above-mentioned low-energy subspace where the limiting bare-momentum values q_{s1} and $q_c^{\pm}$ defined by Eqs. (B.14) and (B.16)-(B.17) of Ref. 28, respectively, simplify and except for corrections of order $1/L$ can be written as,

$$q_{s1} = k_{F\uparrow}\,; \quad q_c^{\pm} = \pm\pi\,; \quad q_{Fs1} = k_{F\downarrow}\,; \quad q_{Fc0} = 2k_F\,. \tag{1.29}$$

In this equation we also provided the values of the $c0$ and $s1$ Fermi momenta which appear in the ground-state bare-momentum distributions used below.

1.2.2. *Landau liquid description*

As discussed above, the Bethe ansatz solution can be described in terms of a pseudoparticle representation associated with the bare momenta $q_j = [2\pi/N_a]\, I_j^{c0}$ and $q_j = [2\pi/N_a]\, I_j^{\alpha\nu}$ and corresponding quantum numbers I_j^{c0} and $I_j^{\alpha\nu}$ which have a regular distribution, similar to that of the discrete momenta of usual non-interacting fermionic systems. For example, the ground state of the system is obtained considering a symmetrical distribution of the numbers I_j^{c0} around the origin, filling the acessible numbers until a value such that the maximal occupied number is according to Eq. (1.29), $|q| = q_{Fc0} = 2k_F = \pi n$ (the maximal value of $|q|$ is π), defining the Fermi surface of the $c0$ band. In the same way, the numbers I_j^{s1} are distributed in a symmetrical way, such that the maximal occupied number corresponds to the value $|q| = q_{Fs1} = k_{F\downarrow}$ given in Eq. (1.29) (the maximal value is also provided in that equation and reads $q_{s1} = k_{F\uparrow}$ where $k_{F\sigma} = N_\sigma/N$, with $\sigma =\uparrow,\downarrow$). At zero magnetization $q_{Fs1} = q_{s1} = k_F$, the $s1$ bare-momentum band is full and at half-filling the $c0$ band is full as well.

Away from half-filling and at finite magnetization the low-energy excitations around the ground state lead to small deviations relative to the equilibrium distributions of the quantum numbers I_j^{c0}, I_j^{s1}. Those excitations are just particle-hole excitations in the $c0$ and $s1$ bare-momentum bands, like in the usual description of a Fermi liquid, except that the $s1$ pseudoparticles are not strictly fermions and both the $c0$ and $s1$ pseudoparticles do not have a one-to-one correspondence to the electrons, as $U/t \to 0$. Their occupancy configurations correspond to exact energy eigenstates of the many-body system obtained from the Bethe ansatz solution. In the standard rapidity description the excitations are obtained by introducing "holes" or "particles" in the distribution functions of the related charge-momentum rapidities k_j and $\alpha\nu$ rapidities $\Lambda_{\alpha\nu,\,j}$.[18]

Let us now focus our attention on the ground state distributions which, except for corrections of the order of $1/L$, can be written as $N^0_{c0}(q) = \Theta(q_{Fc0} - |q|) = \Theta(2k_F - |q|)$ and $N^0_{s1}(q) = \Theta(q_{Fs1} - |q|) = \Theta(k_{F\downarrow} - |q|)$. Here q is the above pseudoparticle bare momentum. At low energies the excitations are characterized by deviations from the ground-state distributions. Then one may introduce general distributions $N_{c0}(q) = N^0_{c0}(q) + \Delta N_{c0}(q)$ and $N_{s1}(q) = N^0_{s1}(q) + \Delta N_{s1}(q)$. In the limit when the deviations are small the energy of the system may be expanded around the ground-state distributions as follows,

$$
\begin{aligned}
E &= E_0 + E_1 + E_2\,,\\
E_1 &= \frac{L}{2\pi}\left\{\int_{-\pi}^{\pi} dq\Delta N_{c0}(q)\epsilon_{c0}(q) + \int_{-k_{F\uparrow}}^{k_{F\uparrow}} dq\Delta N_{s1}(q)\epsilon_{s1}(q)\right\}\,,\\
E_2 &= \frac{L}{(2\pi)^2}\{\int_{-\pi}^{\pi} dq\int_{-\pi}^{\pi} dq'\Delta N_{c0}(q)\frac{f_{c0\,c0}(q,q')}{2}\Delta N_{c0}(q')\\
&\quad + \int_{-k_{F\uparrow}}^{k_{F\uparrow}} dq\int_{-k_{F\uparrow}}^{k_{F\uparrow}} dq'\Delta N_{s1}(q)\frac{f_{s1\,s1}(q,q')}{2}\Delta N_{s1}(q')\\
&\quad + \int_{-\pi}^{\pi} dq\int_{-k_{F\uparrow}}^{k_{F\uparrow}} dq'\Delta N_{c0}(q)f_{c0\,s1}(q,q')\Delta N_{s1}(q)\}\,,
\end{aligned}
\tag{1.30}
$$

in a way analogous to a Fermi liquid (to simplify, at this stage we only consider the lower energy excitations). This reformulation of the problem has the advantage of a standard band-like interpretation of the excitation spectrum. The energies $\epsilon_{c0}(q)$ and $\epsilon_{s1}(q)$ are the charge $c0$ and spin $s1$ bands and the parameters $f_{c0\,c0}$, $f_{s1\,s1}$, and $f_{c0\,s1} = f_{s1\,c0}$ describe the residual interactions between the pseudoparticles. Even though the formulation is similar to that of a Fermi liquid, the pseudoparticles refer to energy eigenstates that do not decay in time.

It is shown in Ref. 28 that for $\nu > 0$ the $c\nu$ and $s\nu$ pseudoparticles are composite objects: the $c\nu$ (and $s\nu$) pseudoparticles are η-spin singlet 2ν-holon (and spin-singlet 2ν-spinon) composite objects of η-spin $1/2$ ν holons of η-spin projection $1/2$ and ν holons of η-spin projection $-1/2$ (and spin $1/2$ ν spinons of spin projection $1/2$ and ν spinons of spin projection $-1/2$). The η-spin projection $1/2$ (and $-1/2$) holons correspond to rotated-electron unoccupied sites (and doubly-occupied sites). The spinons of spin projection $\pm 1/2$ refer to the spins of the rotated electrons which singly occupied sites.The original non-perturbative electronic problem (the spectral function of the 1D Hubbard model is fully incoherent) becomes "perturbative" in the pseudoparticle basis.[23]

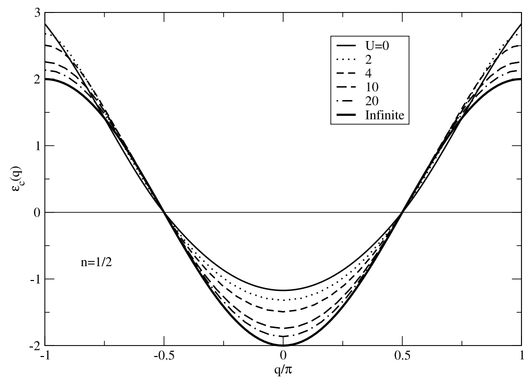

Fig. 1.1. The pseudoparticle energy band $\epsilon_{c0}(q)$ in units of t for density $n = 1/2$ and various values of U/t. Reproduced with permission of the American Physical Society from Ref. 27.

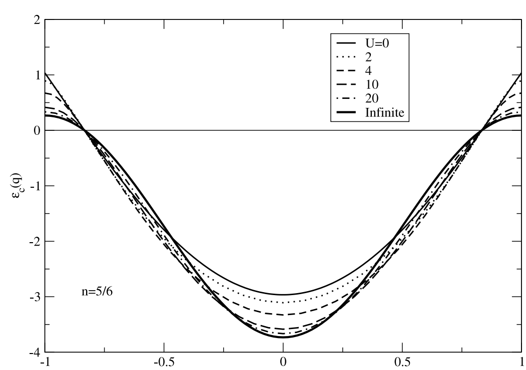

Fig. 1.2. The pseudoparticle energy band $\epsilon_{c0}(q)$ in units of t for density $n = 5/6$ and various values of U/t. Reproduced with permission of the American Physical Society from Ref. 27.

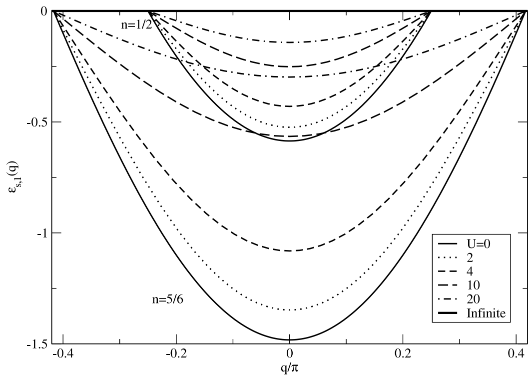

Fig. 1.3. The pseudoparticle energy band $\epsilon_{s1}(q)$ in units of t for density $n = 5/6$ and various values of U/t. Reproduced with permission of the American Physical Society from Ref. 27.

The pseudoparticle band expressions can be expressed in terms of the following integrals,[28]

$$\epsilon_{c0}(q) = \int_{Q}^{k^0(q)} dk' 2t\eta(k')\,,$$
$$\epsilon_{s1}(q) = \int_{B}^{\Lambda^0_{s1}(q)} d\Lambda' 2t\eta_{s1}(\Lambda')\,,$$

where the functions in the upper limits are such that their inverse functions are given by the following integrals,

$$q = \int_{0}^{k^0(q)} dk' 2\pi\rho(k')\,,$$
$$q = \int_{0}^{\Lambda^0_{s1}(q)} d\Lambda' 2\pi\sigma_{s1}(\Lambda')\,. \tag{1.31}$$

At the $c0$ and $s1$ Fermi momenta these functions read $k^0(2k_F) = Q$ and $\Lambda^0_{s1}(k_{F\downarrow}) = B$, respectively. The other distributions involved in the above band expressions obey the integral equations,

$$2t\eta(k) = 2t\sin k + \frac{U}{4\pi t}\cos k\int_{-B}^{B} d\Lambda' \frac{2t\eta_{s1}(\Lambda')}{[U/4t]^2 + [\sin k - \Lambda']^2}\,,$$

$$\begin{aligned} 2t\eta_{s1}(\Lambda) = {} & \frac{U}{4\pi t}\int_{-Q}^{Q} dk' \frac{2t\eta(k')}{[U/4t]^2 + [\sin k' - \Lambda]^2} \\ & - \frac{U}{8\pi t}\int_{-B}^{B} d\Lambda' \frac{2t\eta_{s1}(\Lambda')}{[U/4t]^2 + [(\Lambda - \Lambda')/2]^2}\,. \end{aligned}$$

The velocities associated with the energy bands are given by $v_{c0}(q) = d\epsilon_{c0}(q)/dq$ and $v_{s1}(q) = d\epsilon_{s1}(q)/dq$. The bands are such that $\epsilon_{c0}(2k_F) = 0$ and $\epsilon_{s1}(k_{F\downarrow}) = 0$. The energy bands are shown in Figs. 1.1,1.2, and 1.3.

The pseudoparticle f-functions which describe their residual interactions read,

$$\begin{aligned} f_{c0\,c0}(q,q') = {} & 2\pi v_{c0}(q)\Phi_{c0\,c0}(q,q') + 2\pi v_{c0}(q')\Phi_{c0\,c0}(q',q) \\ & + [2\pi v_{c0}]\sum_{j=\pm1}\Phi_{c0\,c0}(2k_F j,q)\Phi_{c0\,c0}(2k_F j,q') \\ & + [2\pi v_{s1}]\sum_{j=\pm1}\Phi_{s1\,c0}(k_{F\downarrow} j,q)\Phi_{s1\,c0}(k_{F\downarrow} j,q')\,, \end{aligned} \tag{1.32}$$

$$\begin{aligned} f_{s1\,s1}(q,q') = {} & 2\pi v_{s1}(q)\Phi_{s1\,s1}(q,q') + 2\pi v_{s1}(q')\Phi_{s1\,s1}(q',q) \\ & + [2\pi v_{s1}]\sum_{j=\pm1}\Phi_{s1\,s1}(k_{F\downarrow} j,q)\Phi_{s1\,s1}(k_{F\downarrow} j,q') \\ & + [2\pi v_{c0}]\sum_{j=\pm1}\Phi_{c0\,s1}(2k_F j,q)\Phi_{c0\,s1}(2k_F j,q')\,, \end{aligned} \tag{1.33}$$

$$\begin{aligned} f_{c0\,s1}(q,q') = {} & 2\pi v_{c0}(q)\Phi_{c0\,s1}(q,q') + 2\pi v_{s1}(q')\Phi_{s1\,c0}(q',q) \\ & + [2\pi v_{c0}]\sum_{j=\pm1}\Phi_{c0\,c0}(2k_F j,q)\Phi_{c0\,s1}(2k_F j,q') \\ & + [2\pi v_{s1}]\sum_{j=\pm1}\Phi_{s1\,s1}(k_{F\downarrow} j,q)\Phi_{s1\,c0}(k_{F\downarrow} j,q')\,, \end{aligned} \tag{1.34}$$

where $v_{c0} = v_{c0}(2k_F)$ and $v_{s1} = v_{s1}(k_{F\downarrow})$. While the f functions are associated with the residual interactions of the pseudoparticles, the functions Φ are the phase shifts, in units of π, of the collisions between the corresponding pseudofermions. The latter objects are introduced in the following chapter, Ref. 1. The phase shifts appearing in the above f-function expressions are functions of the two momentum values. Alternatively, one can define phase shifts which depend on the corresponding two rapidity values. The two types of phase shifts are related according to

$$\Phi_{c0\,c0}(q,q') = \bar{\Phi}_{c0\,c0}(4t\sin k^0(q)/U, 4t\sin k^0(q')/U),$$
$$\Phi_{c0\,s1}(q,q') = \bar{\Phi}_{c0\,s1}(4t\sin k^0(q)/U, 4t\Lambda^0_{s1}(q')/U),$$
$$\Phi_{s1\,s1}(q,q') = \bar{\Phi}_{s1\,s1}(4t\Lambda^0_{s1}(q)/U, 4t\Lambda^0_{s1}(q')/U),$$
$$\Phi_{s1\,c0}(q,q') = \bar{\Phi}_{s1\,c0}(4t\Lambda^0_{s1}(q)/U, 4t\sin k^0(q')/U).$$

The rapidity phase shifts satisfy the integral equations,

$$\begin{aligned}
\bar{\Phi}_{c0\,c0}(r,r') &= \frac{1}{\pi}\int_{-y_0}^{y_0} dr''\,\frac{\bar{\Phi}_{s1\,c0}(r,r')}{1+(r-r'')^2},\\
\bar{\Phi}_{c0\,s1}(r,r') &= -\frac{1}{\pi}\text{arc }\tan(r-r') + \frac{1}{\pi}\int_{-y_0}^{y_0} dr''\,\frac{\bar{\Phi}_{s1\,s1}(r'',r')}{1+(r-r'')^2},\\
\bar{\Phi}_{s1\,c0}(r,r') &= -\frac{1}{\pi}\text{arc }\tan(r-r') + \frac{1}{\pi}\int_{-y_0}^{y_0} dr''\,G(r,r'')\bar{\Phi}_{s1\,c0}(r,r')\,,\\
\bar{\Phi}_{s1\,s1}(r,r') &= \frac{1}{\pi}\text{arc }\tan(\frac{r-r'}{2}) - \frac{1}{\pi^2}\int_{-x_0}^{x_0} dr''\,\frac{\text{arc }\tan(r''-r')}{1+(r-r'')^2}\\
&\quad + \int_{-y_0}^{y_0} dr'' G(r,r'')\bar{\Phi}_{s1\,s1}(r'',r')\,. \qquad (1.35)
\end{aligned}$$

Here $x_0 = 4t\sin Q/U$, $y_0 = 4tB/U$, and the kernel $G(r,r')$ is given by,

$$G(r,r') = -\frac{1}{2\pi}\left[\frac{1}{1+((r-r')/2)^2}\right]\left[1-\frac{1}{2}\left(t(r)+t(r')+\frac{l(r)-l(r')}{r-r'}\right)\right], \qquad (1.36)$$

where

$$t(r) = \frac{1}{\pi}\left[\text{arc }\tan(r+x_0) - \text{arc }\tan(r-x_0)\right],$$
$$l(r) = \frac{1}{\pi}\left[\ln(1+(r+x_0)^2) - \ln(1+(r-x_0)^2)\right].$$

The following phase-shift parameters play an important role in the quantum-liquid physics,

$$\zeta^i_{c0\,c0} = 1 + \bar{\Phi}_{c0\,c0}(x_0,x_0) + (-1)^i\bar{\Phi}_{c0\,c0}(x_0,-x_0)\,,$$
$$\zeta^i_{c0\,s1} = \bar{\Phi}_{c0\,s1}(x_0,y_0) + (-1)^i\bar{\Phi}_{c0\,s1}(x_0,-y_0)\,,$$
$$\zeta^i_{s1\,c0} = \bar{\Phi}_{s1\,c0}(y_0,x_0) + (-1)^i\bar{\Phi}_{s1\,c0}(y_0,-x_0)\,,$$
$$\zeta^i_{s1\,s1} = 1 + \bar{\Phi}_{s1\,s1}(y_0,y_0) + (-1)^i\bar{\Phi}_{s1\,s1}(y_0,-y_0)\,, \quad i=0,1\,.$$

These parameters can be written as $\zeta^i_{c0\,c0} = \zeta^i_{c0\,c0}(x_0)$, $\zeta^i_{c0\,s1} = \zeta^i_{c0\,s1}(x_0)$, $\zeta^i_{s1\,c0} = \zeta^i_{s1\,c0}(y_0)$, and $\zeta^i_{s1\,s1} = \zeta^i_{s1\,s1}(y_0)$ where the functions on the right-hand side of these equations are defined as follows,

$$\zeta^1_{c0\,c0}(r) = 1 + \frac{1}{\pi}\int_{-y_0}^{y_0} dr''\,\frac{\zeta^1_{s1\,c0}(r'')}{1+(r-r'')^2}\,,$$
$$\zeta^1_{c0\,s1}(r) = \frac{1}{\pi}\int_{-y_0}^{y_0} dr''\,\frac{\zeta^1_{s1\,s1}(r'')}{1+(r-r'')^2}\,,$$
$$\zeta^1_{s1\,c0}(r) = t(r) + \int_{-y_0}^{y_0} dr'' G(r,r'')\zeta^1_{s1\,c0}(r'')\,,$$
$$\zeta^1_{s1\,s1}(r) = 1 + \int_{-y_0}^{y_0} dr'' G(r,r'')\zeta^1_{s1\,s1}(r'')\,. \tag{1.37}$$

The parameters corresponding to the symetrical linear combination of the phase shifts are obtained as the inverse of the transpose of the matrix whose entries are the antisymmetrical parameters given here. The point is that the above phase-shift parameters are the elementary pieces of other quantities which play the same role as the Landau parameters of Fermi liquid theory. Such pseudoparticle Landau parameters are given by,

$$v_{c0} + F^i_{c0\,c0} = v_{c0}[\zeta^i_{c0\,c0}]^2 + v_{s1}[\zeta^i_{s1\,c0}]^2\,,$$
$$v_{s1} + F^i_{s1\,s1} = v_{s1}[\zeta^i_{s1\,s1}]^2 + v_{c0}[\zeta^i_{c0\,s1}]^2\,,$$
$$F^i_{c0\,s1} = F^i_{s1\,c0} = v_{c0}\zeta^i_{c0\,c0}\zeta^i_{c0\,s1} + v_{s1}\zeta^i_{s1\,s1}\zeta^i_{s1\,c0}\,, \quad i = 0,1\,. \tag{1.38}$$

The pseudoparticle Landau parameters can be defined in a way similar to that of the Fermi-liquid theory quasiparticles,

$$F^i_{c0\,c0} = \frac{1}{2\pi}\sum_{j=\pm1}(j)^i f_{c0\,c0}(2k_F, j2k_F)\,,$$
$$F^i_{s1\,s1} = \frac{1}{2\pi}\sum_{j=\pm1}(j)^i f_{s1\,s1}(k_{F\downarrow}, jk_{F\downarrow})\,,$$
$$F^i_{c0\,s1} = F^i_{s1\,c0} = \frac{1}{2\pi}\sum_{j=\pm1}(j)^i f_{c0\,s1}(2k_F, jk_{F\downarrow})$$
$$= \frac{1}{2\pi}\sum_{j=\pm1}(j)^i f_{s1\,c0}(k_{F\downarrow}, j2k_F)\,. \tag{1.39}$$

1.2.3. *Low-temperature thermodynamics*

Many low-energy quantities of the one-dimensional Hubbard model can be expressed in terms of the phase-shift parameters and related pseudoparticle Landau parameters. As in a Fermi liquid the low-temperature specific heat does not depend on such parameters and only involves the c0 and s1 pseudoparticle Fermi velocities. For electronic densities in the range $n < 1$ and spin densities $m > 0$ the specific heat reads,[22]

$$\frac{c_V}{N_a} = \left(\frac{k_B^2 \pi}{3}\right)\left(\frac{1}{v_{c0}} + \frac{1}{v_{s1}}\right) T \, . \tag{1.40}$$

This result is obtained considering the first-order momentum distribution deviation contributions to the energy when expressed in terms of low-temperature Fermi-Dirac distributions, for both the $c0$ pseudoparticles and $s1$ pseudoparticles. The energy deviation is expressed in terms of the deviations,[22]

$$\begin{aligned} \Delta N_{c0}(q) &= N_{c0}(q) - \Theta(2k_F - |q|) \, , \\ \Delta N_{s1}(q) &= N_{s1}(q) - \Theta(k_{F,\downarrow} - |q|) \, , \end{aligned} \tag{1.41}$$

where $N_{c0}(q)$ and $N_{s1}(q)$ are the Fermi-Dirac distributions.

Also, the static charge and spin susceptibilities may be obtained in a way similar to that of a Fermi liquid.[24] The magnetic susceptibility at zero temperature and spin density $m = 0$ was obtained first by Shiba.[20] Here we follow the procedure of Ref. 24 and present the expressions derived in that reference for $m > 0$. For most cases the $m \to 0$ limit of the obtained expressions provides the corresponding $m = 0$ expression. The basic procedure corresponds to using expressions for the chemical potential and the magnetic field given by,[24]

$$\begin{aligned} \mu(n) &= \frac{U}{2} - \epsilon_{c0}^0(2k_F) - \frac{1}{2}\epsilon_{s1}^0(k_{F\downarrow}) \, , \\ H(m) &= -\frac{\epsilon_{s1}^0(k_{F\downarrow})}{2} \, , \end{aligned} \tag{1.42}$$

where $\epsilon_{c0}(q) = \epsilon_{c0}^0(q) + \mu - \mu_0 H$ and $\epsilon_{s1}(q) = \epsilon_{s1}^0(q) + 2\mu_0 H$. The charge susceptibility is then expressed as,

$$\chi_c|_{H,m} = -\frac{1}{n^2}\frac{1}{\partial\mu(n)/\partial n|_{H,m}} \, , \tag{1.43}$$

and the spin susceptibility may be expressed as,

$$\chi_s|_{\mu,n} = \frac{2\mu_0}{\partial H(m)/\partial m|_{\mu,n}} \, . \tag{1.44}$$

It was obtained that these quantities can be written in terms of the above phase-shift parameters as follows,[24]

$$\chi_c|_H = \frac{1}{\pi n^2}\left(\frac{(\zeta^1_{c0\,c0})^2}{v_{c0}} + \frac{(\zeta^1_{s1\,c0})^2}{v_{s1}}\right),$$

$$\chi_s|_\mu = \frac{\mu_0^2}{\pi}\left(\frac{(\zeta^1_{c0\,c0} - 2\zeta^1_{c0\,s1})^2}{v_{c0}} + \frac{(\zeta^1_{s1\,c0} - 2\zeta^1_{s1\,s1})^2}{v_{s1}}\right),$$

$$\chi_c|_m = \frac{1}{\pi n^2}\left(\frac{1}{v_{c0}(\zeta^0_{c0\,c0} + \zeta^0_{c0\,s1}/2)^2 + v_{s1}(\zeta^0_{s1\,c0} + \zeta^0_{s1\,s1}/2)^2}\right),$$

$$\chi_s|_n = \frac{\mu_0^2}{\pi}\left(\frac{1}{v_{c0}(\zeta^0_{c0\,s1}/2)^2 + v_{s1}(\zeta^0_{s1\,s1}/2)^2}\right). \tag{1.45}$$

The dependence of these thermodynamic quantities on the various parameters is discussed in Ref. 24.

Alternatively, the above given charge and spin susceptibilities can be expressed in terms of the Landau parameters provided in Eqs. (1.38) and (1.39), as in Fermi liquid theory.

1.3. Summary

In this chapter we have briefly reviewed several schemes used in the description of the unusual properties of low-dimensional correlated systems. A hint on these properties is provided by the Tomonaga and Luttinger models where bosonization techniques allow the solution at low energies.

We have devoted most of our attention to the one-dimensional Hubbard model whose low-energy physics can, in spite of the lack of Fermi liquid behavior, be described by a functional theory in terms of pseudoparticle bare-momentum distributions, which resembles that of Fermi liquid theory. Except that in the limit of zero interaction the pseudoparticles do not map onto electrons, and for $U > 0$ the one-electron spectral function is fully incoherent, the low-temperature thermodynamics and the low-energy charge and spin susceptibilities can be derived as in a Fermi liquid. However, it has proven exceedingly difficult in the past to obtain information on correlation functions via the exact Bethe ansatz solution, if we do not restrict to the asymptotic regime in space and time. The calculation of correlation functions at general momentum and frequency is a complex problem that has only been solved recently, as shown in the following chapter.

In the ensuing companion chapter we review a transformation which maps the pseudoparticles considered here onto non-interacting pseudofermions. That enables the evaluation of matrix elements between energy eigenstates and the construction of a pseudofermion dynamical theory. Such a theory provides expressions for finite-energy correlation and spectral functions.

Acknowledgments

We thank the support of FCT under the grant POCTI/FIS/58133/2004 and that of the ESF Science Programme INSTANS 2005-2010.

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