Chapter 1

Introduction

Newton's second law of motion, formulated over three centuries ago, forms the backbone of classical physics and continues to describe most of what we observe in the world around us. It is easily stated. One defines the primary inertial coordinate system as a system that is at rest with respect to the fixed stars. The second law then states that in this inertial frame the rate of change of momentum of a point object with inertial mass m_i and momentum $m_i\vec{v}$ is given by the applied force $\vec{F}$

$$\frac{d}{dt}(m_i\vec{v}\,) = \vec{F} \qquad ; \text{ Newton's second law} \tag{1.1}$$

Furthermore, any frame moving with constant velocity relative to the primary inertial coordinate system is again an inertial frame in which Newton's second law holds. In all these frames, it is assumed that there is a single, universal time t. Of course, three-dimensional space here is euclidian, obeying all of Euclid's postulates.

Newton's law of gravitation states that the force between a point object with gravitational mass m_g and another object of mass M separated by a displacement vector $\vec{r}$ is given by

$$\vec{F} = -\frac{GMm_g}{r^2}\frac{\vec{r}}{r} \qquad ; \text{ Newton's law of gravitation} \tag{1.2}$$

Here M is either a point mass, or one is outside of a spherically symmetric distributed mass with $\vec{r}$ referring to its center, and G is Newton's constant.

If these two expressions are equated, and it is assumed that $m_i = m_g$ with both being constant, then *that mass cancels* and one obtains

$$\frac{d\vec{v}}{dt} = -\frac{GM}{r^2}\frac{\vec{r}}{r} \tag{1.3}$$

This expression implies that all massive objects with *any* $m_i = m_g$ move the same way in the gravitational field of the mass M, a result that accords with what we observe in the world around us and what Galileo confirmed long ago in his celebrated experiment dropping various objects from the leaning tower of Pisa.

At the beginning of the 20th century, two fundamental modifications of classical physics were discovered. The first was quantum mechanics, which describes very different behavior in the microscopic domain, yet reduces to Newton's laws in the appropriate limit. We shall have very little to say about quantum mechanics in this text. The second was Einstein's theory of special relativity [Einstein (1905)]. The Michelson-Morley experiment searched for a shift in the fringes of an interferometer moving with various velocities with respect to the primary inertial coordinate system. This experiment ultimately implied that the speed of light c is the same in any inertial frame, a most amazing and non-intuitive result at complete variance with how one adds velocities in classical physics. Lorentz had discovered an algebraic transformation that left the form of the wave equation for light invariant. It was Einstein's genius to give that transformation a physical interpretation and place the transformed coordinates in a one-to-one correspondence with what is actually observed in various inertial frames. The consequences of this association are profound: time is relative and varies from frame to frame; length is also relative; a particle's mass depends on its velocity; there is a relation between energy and mass $E = mc^2$, and so on.

The four-dimensional space $(x^1, x^2, x^3, x^4) \equiv (\vec{x}, ct)$ in which these coordinate transformations take place is no longer euclidian. If one writes an infinitesimal physical displacement in this space as $d\mathbf{s}$, then the square of this displacement, the invariant interval $(d\mathbf{s})^2 = d\mathbf{s} \cdot d\mathbf{s}$, is given by

$$(d\mathbf{s})^2 = \sum_{\mu=1}^{4} \sum_{\nu=1}^{4} g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{1.4}$$

The quantity $g_{\mu\nu}$ is known as the *metric*. In chapter 6 of this book we review the basic principles of Einstein's theory of special relativity and many of its implications, which have been repeatedly confirmed experimentally over the intervening years.

It was again Einstein's genius to realize two profound implications of the above results.[1] The first is the implication of the *equivalence principle* that $m_i = m_g$

$$m_i = m_g \qquad ; \text{ equivalence principle} \tag{1.5}$$

The first mass m_i governs the acceleration of an object with respect to the fixed stars, and the second mass m_g determines the strength of the gravitational force. Why should these things have anything to do with each other? It is a consequence of the equivalence principle that all particles follow the same trajectory in a gravitational field independent of their mass. Thus the trajectory of a particle in such a field is determined only by the *geometry* of the field. This observation provided one of the key points of departure for general relativity.

The second key insight was that the world in which we live, at least in free space and moving with uniform velocity, is not a nice four-dimensional *euclidian* space, but a rather mysterious *Minkowski* space with the indefinite metric of Eq. (1.4). Is there any reason that four-dimensional space might not have a more general structure? That it might also be *curved* rather than flat? For example, a particle can move without friction on a flat two-dimensional surface, in which case the trajectory is simply a straight line. That surface might also be curved and distorted, in which case the trajectories can become very involved. Indeed, the mechanics problem of a particle moving without friction on an arbitrarily shaped two-dimensional surface will form the paradigm for all the physics and mathematics we subsequently do in this book. We shall show that the trajectories in this case are just the *geodesics* on the surface — the curves of minimum (or stationary) physical distance. Thus they are entirely determined by the geometry of the surface!

Is it possible that the presence of mass (and energy) as a source can produce a curved four-dimensional space-time such that the equation for the geodesics in this space just reproduces Eq. (1.3) in the appropriate limit? If so, one would have a unified description of both Newton's second law and his universal law of gravitation, two cornerstones of classical physics. It is just this problem that is solved by Einstein's theory of general relativity [Einstein (1916)].

[1] These results are now universally presented to students at the introductory physics level. How many other such results are there that have profound implications currently beyond our grasp??

In order to formulate the theory, one has to have in his or her arsenal the mathematical tools describing a curved space in higher dimensions. This is the theory of *riemannian geometry.* In chapter 3, we first present a straightforward discussion of curvilinear coordinates in higher dimensional euclidian space, including the notion of vectors and tensors. In chapter 5, this discussion is generalized to discuss curved spaces, starting from our paradigm of a particle moving without friction on a curved two-dimensional surface. The only assumption made in this book about the reader's background is that he or she is familiar with classical lagrangian mechanics as presented, for example, in [Fetter and Walecka (2003)]. Beyond that, the reader should find the material in this text self-contained.[2]

The great power of classical lagrangian mechanics, which can be derived from Hamilton's variational principle, is that it is freed from any particular choice of coordinates. Thus in the problem of the particle moving on an arbitrary two-dimensional surface, discussed in detail in chapters 2 and 4, one can introduce *any* set of linearly independent generalized coordinates (q^1, q^2) on the surface with which to describe the particle's location and subsequent motion. We will assume that there is a unique, flat, *tangent plane* at each point on the surface. We make use of the fact that an infinitesimal physical displacement in the surface is identical to that in the tangent plane, and we then demonstrate how each choice of generalized coordinates carries with it an associated *metric* on the surface. We proceed to derive the lagrangian for particle motion on the surface and the corresponding set of Lagrange's equations for (q^1, q^2). These form a set of two coupled, second-order, non-linear differential equations in the time, with the surface entering only through the *affine connection*, a first-order, nonlinear differential form in the metric.[3] We then show that the equations of motion are identical to those for the *geodesics* on the surface. The observation that the information on the intrinsic structure of the surface must somehow be contained in the affine connection forms the basis for our development of riemannian geometry in chapter 5. In that chapter, we show how the *Riemann tensor*, a first-order, nonlinear differential form in the affine connection, characterizes the curvature of the space.

[2]Some familiarity with the rest of classical mechanics, as presented in [Fetter and Walecka (2003)], as well as with special relativity and other aspects of modern physics, as presented, for example, in [Ohanian (1995)], will make this book more meaningful. We do also assume the reader is familiar with vector calculus and linear algebra.

[3]We shall use the terminology "first-order, nonlinear differential form in the metric" to indicate an expression that is nonlinear in the indicated quantity and contains derivatives up through the stated order.

We subsequently, in chapter 5, proceed to prove all of the results on riemannian geometry that we use in the rest of the book.[4] Readers anxious to get to the "meat" of special and general relativity starting in chapter 6 may want to just accept the results that come after the introduction of the Riemann curvature tensor, and come back to the proof of those results at their leisure.

Einstein's theory of general relativity is introduced in chapter 7 through a set of three assumptions:

(1) We live in a four-dimensional riemannian space with a local, flat Minkowski tangent space;
(2) The structure of the space is given by the Einstein field equations

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \qquad ; \text{ Einstein equations} \tag{1.6}$$

Here $G^{\mu\nu}$ is the Einstein tensor derived from the Riemann curvature tensor (it is a second-order, nonlinear differential form in the metric), and $T^{\mu\nu}$ is the energy-momentum tensor for the medium under consideration.
(3) Particles follow the geodesics in this space.

The Einstein field equations are solved outside of a spherically symmetric source, and it is shown how this Schwarzschild solution for the metric [Schwarzschild (1916)] then leads to Eq. (1.3) in the appropriate limit. We take this demonstration as the *basic justification of Einstein's theory*, and then proceed to investigate its further consequences.

Since coordinates, and the corresponding metric, by themselves have no meaning, it is often difficult to get at the underlying physical implications of the results obtained in general relativity. The key to our interpretation of the results lies in the *equivalence principle*. As a consequence of this principle, there is always one frame, the *local freely falling frame (LF^3)*, in which one has neither gravity nor inertial forces. This is a frame that is just held and then let go in the local gravitational field. In this frame, for a short time, one has only flat Minkowski space and the laws of special relativity. All other frames, with their corresponding interpretation, can then be obtained through a coordinate transformation from this one. We show that both time dilation and radial length contraction are implied by the Schwarzschild metric, arising now from an *acceleration* of the LF^3 relative

[4]The theory of riemannian geometry is due to the brilliant mathematician G. F. R. Riemann (1826-1866).

to the inertial laboratory frame, rather than from the relative *velocity* of two frames in special relativity.

We proceed to investigate some of the further consequences of Einstein's theory in the subsequent chapters. In chapter 8, we show how one can generate a lagrangian for particle motion from any invariant interval, and we then construct a lagrangian $L(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi})$ for particle motion in the Schwarzschild metric. From this, we calculate the precession of the perihelion of an almost circular orbit as a straightforward exercise in lagrangian mechanics. The precession of the perihelion of the planet mercury forms one of the most important confirmed predictions of general relativity. An index of refraction calculation, relegated to the problems, provides insight into the deflection of light in a gravitational field, another of the classic tests of the theory.

In chapter 9, the frequency shift of a source in the Schwarzschild metric and the redshift of light propagating out of that gravitational field are discussed. In chapter 10, the Tolman-Oppenheimer-Volkoff (TOV) equations for the structure of neutron stars are derived [Tolman (1939); Oppenheimer and Volkoff (1939)], and the solution of these equations using an equation of state derived from the relativistic mean field theory of nuclear matter is presented. Also discussed there is the transition into a black hole, where the nuclear repulsion is insufficient to overcome the gravitational attraction and the star collapses inside of its Schwarzschild radius.[5]

Chapter 11 deals with what is probably the most fascinating aspect of general relativity, at least the one that most readily captures the public's imagination, and that is cosmology — the time evolution of the universe. The Einstein field equations are solved in the case of a uniform (baryonic) matter distribution and flat, time-dependent, three-dimensional space. The resulting Robertson-Walker metric with $k = 0$ [Robertson (1935); Walker (1936)] turns out to describe well most of what is observed today. The corresponding cosmological redshift is discussed as well as the age of the universe and the role of the horizon. An expanding flat rubber sheet provides a useful two-dimensional analogy for the interpretation of the results.

Just as there is electromagnetic radiation in free space propagating with velocity c that follows from Maxwell's equations in E&M, there is gravitational radiation propagating with velocity c that follows from the Einstein

[5]The Schwarzschild radius is that value of the coordinate r at which the radial part of the Schwarzschild metric becomes singular.

field equations. In chapter 12, those equations are linearized in a perturbation of the Minkowski metric, and the wave equation is derived for that perturbation. A plane wave solution to the field equations is found and its properties examined. Here the simultaneous stretching and contracting of a flat, tranverse rubber sheet provides a useful analogy for the interpretation of the results.

A few special topics in general relativity are explored in chapter 13. First, the Einstein field equations in free space are derived from the Einstein-Hilbert lagrangian density and Hamilton's principle in continuum mechanics. What is varied is the metric at each point in space-time. It is tempting to just add a constant to that lagrangian density, as was done by Einstein, and the implications of that cosmological constant are explored. Once one has a lagrangian for the gravitational field, it is straightforward to augment that lagrangian with the contribution of additional matter fields, and we here explore the consequences of adding a scalar field.

We then look at the solution to the Einstein field equations for a cosmology where three-dimensional space, while homogeneous and isotropic, is also curved. The resulting metric is known as the Robertson-Walker metric with $k \neq 0$. The corresponding Friedmann equation [Friedmann (1922)] makes it appear extremely puzzling that the $k = 0$ solution should be relevant today, since with any traditional equation of state, the system diverges from that solution with time. The answer to this puzzle is given by the theory of inflation [Guth (2000)], whereby the development of a scalar field through spontaneous symmetry breaking and an initially unstable field configuration converges to the appropriate flatness, and provides as well the energy to create an associated hot plasma of particles and antiparticles.

Chapter 14 contains an extensive set of problems, some for each chapter. The goal of the problems is not to stump the reader, but to enhance and extend the coverage. Most of the problems guide the reader with steps, and give the answers. Many important concepts and applications are covered in the problems, for example:

- applications of special relativity
- geodesic equations in the Schwarzschild metric
- applications of the precession of the perihelion
- index of refraction for propagation of light in the Schwarzschild metric
- frequency and wavelength of light in the Schwarzschild metric
- numerical solution of the TOV equations
- thermodynamics of black holes

- cosmic microwave background
- time evolution from the “hot big bang”
- detection of gravitational waves
- source of gravitational waves
- solution of the Friedmann equation with various equations of state
- dynamics of a scalar field

The reader is strongly encouraged to work as many of the problems as possible in order to gain some confidence in this subject.

This book is meant as an *introduction* to general relativity. There are many good references for further study, for example, [Landau and Lifshitz (1975); Wald (1984); Shutz (1985); Hughston and Tod (1990); Kolb and Turner (1990); Linde (1990); Peebles (1993); Ohanian and Ruffini (1994); Peacock (1999); Taylor and Wheeler (2000); Hartle (2002); Foster and Nightengale (2006)]. It is hoped that the present book will allow a student to read with a deeper understanding and sense of appreciation the two classic texts [Weinberg (1972)] and [Misner, Thorne, and Wheeler (1973)].

Ours might truly be called the golden age of general relativity and cosmology. Satellite observations continue to push the boundaries of our understanding. The Cosmic Background Explorer (COBE) and Wilkinson Microwave Anisotropy Probe (WMAP) examine the isotropy and homogeneity of the cosmic microwave background left over from the big bang and give us a window back in time [COBE (2006); WMAP (2006)]. The Laser Interferometer Gravitational-Wave Observatory (LIGO) provides a powerful detector to search for gravitational radiation [LIGO (2006)]. The Chandra X-ray observatory provides important information on events yielding radiation in the high-energy end of the spectrum [Chandra (2006)]. The Hubble Space Telescope provides amazing visual images from the far outer regions of our universe [Hubble (2006)]. The Gravity Probe B experiment tests the general relativity prediction for the precession of a gyroscope in orbit around the earth [Einstein (2006)].

It is indeed a fascinating and dynamic time, where a wealth of truly impressive data continues to challenge and refine our understanding of general relativity.[6] It is important for every physicist to have a working knowledge of this subject.

[6] See, for example, [Sky and Telescope (2006)]. The author has found this publication to be an excellent source of information at the “semi-lay level” for keeping up with modern developments in general relativity.