A STABILITY CRITERION FOR DELAY DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECTS

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In this paper, we prove that if a delay differential equation with impulse effects of the form

$$x'(t) = A(t)x(t) + B(t)x(t-\tau) \quad , \; t \neq \theta_i,$$

$$\Delta x(\theta_i) = C_i x(\theta_i) + D_i x(\theta_{i-j}), \quad i \in \mathbb{N},$$

verifies a Perron condition then its trivial solution is uniformly asymptotically stable.

Keywords: Impulse; Delay; Adjoint; Perron; Uniform asymptotic stability.

1. Introduction and Preliminaries

Delay differential equations with impulse effects can suitably model various evolutionary processes that exhibit both delay and impulse characteristics. In particular, they provide a natural description of the motion of several real world processes which, on one hand, depends on the processes history that often turns out to be the cause of phenomena substantially affecting the motion and, on other hand, is subject to short time perturbations whose duration is almost negligible. Such processes are often investigated in various fields of science and technology, such as physics, population dynamics, ecology, biological systems, optimal control, etc., see Refs. 1–11 and reference quoted therein.

It is well known in the theory of ordinary differential equations (see e.g. Ref. 12 [p. 120]) that if for every continuous function $f(t)$ bounded on $[0, \infty)$, the solution of the equation

$$x'(t) = A(t)x(t) + f(t),$$

satisfying $x(0) = 0$ is bounded on $[0, \infty)$, then the trivial solution of the corresponding homogeneous equation

APPLIED ANALYSIS AND DIFFERENTIAL EQUATIONS

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$$x'(t) = A(t)x(t)$$

is uniformly asymptotically stable. This result is referred as Perron theorem Ref. 13. Later, Perron theorem has been extended to delay differential equations Ref. 12 [p. 371]. Indeed, it was shown that if for every continuous function $f(t)$ bounded on $[0, \infty)$, the solution of the equation

$$x'(t) = A(t)x(t) + B(t)x(t-\tau) + f(t),\ t > 0$$

satisfying $x(t) = 0$ for $t \in [-\tau, 0]$ is bounded on $[0, \infty)$, then the trivial solution of the equation

$$x'(t) = A(t)x(t) + B(t)x(t-\tau),$$

is uniformly asymptotically stable. For more related materials, see the papers Refs. 14,15.

In this paper, we carry out the above result to a type of linear delay differential equations with impulse effects. Indeed, we consider equation of the form

$$\begin{aligned} &x'(t) = A(t)x(t) + B(t)x(t-\tau), && t \neq \theta_i, \\ &\Delta x(\theta_i) := x(\theta_i^+) - x(\theta_i) = C_i x(\theta_i) + D_i x(\theta_{i-j}), && i \in \mathbb{N}, \end{aligned} \tag{1}$$

and show that its trivial solution is uniformly asymptotically stable under a Perron condition. Our equation differs from the previous ones, see also Refs. 16–19, not only it is more general but also it allows delay terms in the impulse conditions. Such impulse conditions are more natural for delay differential equations.

With regard to equation (4) it is assumed that

(i) A and B are $n \times n$ continuous bounded matrices, $\tau > 0$ is a positive real number;

(ii) C_i and D_i are $n \times n$ bounded matrices, $j \in \mathbb{N}$ is fixed;

(iii) $\{\theta_i\}$ is an increasing sequence of real numbers with $\lim_{i \to \infty} \theta_i = \infty$.

We also assume that $\det(I + C_i) \neq 0$ and that there exist a positive real numbers ρ and ν such that $\|D_i\| \leq \rho$ and $\|(I + C_i)^{-1}\| \leq \rho$ and $\theta_i - \theta_{i-j} \leq \nu$ for all $i \in \mathbb{N}$. $\| \cdot \|$ denotes any matrix norm.

Definition 1.1. Equation (4) is said to verify Perron condition if for every continuous bounded on $[0, \infty)$ function $f(t)$ and every bounded sequence

β_i the solution of

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)x(t-\tau) + f(t), \quad t \neq \theta_i, \\ \Delta x(\theta_i) &= C_i x(\theta_i) + D_i x(\theta_{i-j}) + \beta_i, \quad i \in \mathbb{N}, \end{aligned} \tag{2}$$

satisfying $x(t) = 0$ for $t \in [-\tau, 0]$ is bounded on $t \in [0, \infty)$.

By a solution of (6) on an interval J, we mean a function x defined on J such that x is continuous on J except possibly at $\theta_i \in J$ for $i \in \mathbb{N}$, where $x(\theta_i^+) : \lim_{t \to \theta_i^+} x(t)$ and $x(\theta_i^-) := \lim_{t \to \theta_i^-} x(t)$ exist, $x(\theta_i^-) := x(\theta_i)$, and that x satisfies (6) on J. Clearly, if $f \equiv 0$ and $\beta_i = 0$ for all $i \in \mathbb{N}$ then (6) reduces to (4).

Let $PLC([-\tau, 0], \mathbb{R}^n)$ denote the set of piecewise left continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ having a finite number of discontinuity points of the first kind. Under the above conditions, one can easily show that for given $\sigma \geq 0$ and $\phi \in PLC([-\tau, 0], \mathbb{R}^n)$ there is a unique solution $x(t)$ of (6) such that

$$x(t + \sigma) = \phi(t), \quad t \in [-\tau, 0]. \tag{3}$$

2. Preparatory Lemmas

The following lemmas, see Ref. 12 for delay differential equations without impulse effects, are essential in proving the main result of this paper. Lemma 4.1 is needed to define an adjoint equation of (4), Lemma 2.2 provides representation of solutions, and Lemma 2.3 is concerned with the boundedness of fundamental matrices of (4).

Consider the equation

$$\begin{aligned} &y'(t) = -A^T(t)y(t) - B^T(t+\tau)y(t+\tau), && t \neq \theta_i, \\ &\Delta y(\theta_i) - (I + C_i^T)^{-1} C_i^T y(\theta_i) - (I + C_i^T)^{-1} D_{i+j}^T y(\theta_{i+j}^+), && i \in \mathbb{N}. \end{aligned} \tag{4}$$

We claim that equation (9) is an adjoint of (4) with respect to a function resembles the one used by Halanay in Ref. 12 [p. 371]. It turns out that this function has the form

$$\begin{aligned} < y(t), x(t) > = y^T(t)x(t) + \int_t^{t+\tau} y^T(s)B(s)x(s-\tau)\mathrm{d}s \\ + \sum_{n(t) \leq k < n(t)+j} y^T(\theta_k^+) D_k x(\theta_{k-j}), \end{aligned} \tag{5}$$

where

$$n(t) = \min\{i \in \mathbb{N} : \theta_i \geq t\}.$$

Lemma 2.1. *If $x(t)$ is a solution of* (4) *and $y(t)$ is a solution of* (9) *then*

$$\langle y(t), x(t)\rangle = c = constant,$$

where $\langle\ ,\ \rangle$ is defined by (10).

Proof. Let $t \in (\theta_i, \theta_{i+1})$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle y(t), x(t)\rangle &= -y^T(t)A(t)x(t) - y^T(t+\tau)B(t+\tau)x(t)\\ &\quad + y^T(t+\tau)B(t+\tau)x(t) - y^T(t)B(t)x(t-\tau)\\ &\quad + y^T(t)A(t)x(t) + y^T(t)B(t)x(t-\tau) = 0,\end{aligned}$$

and hence $\langle y(t), x(t)\rangle = c_i =$ constant for $t \in (\theta_i, \theta_{i+1})$. We may claim that $c_i = c$ for $i \in \mathbb{N}$. Indeed, since

$$c_{i+1} - c_i = \Delta\langle y(t), x(t)\rangle\,|_{t=\theta_i},$$

by (10) we have

$$\begin{aligned}c_{i+1} - c_i &= y^T(\theta_i^+)x(\theta_i^+) - y^T(\theta_i)x(\theta_i)\\ &\quad + \sum_{n(\theta_i^+)\le k<n(\theta_i^+)+j} y^T(\theta_k^+)D_k x(\theta_{k-j})\\ &\quad - \sum_{n(\theta_i)\le k<n(\theta_i)+j} y^T(\theta_k^+)D_k x(\theta_{k-j}).\end{aligned}$$

Since $n(\theta_i^+) = i+1$ and $n(\theta_i) = i$, we have

$$\begin{aligned}c_{i+1} - c_i = y^T(\theta_i^+)x(\theta_i^+) - y^T(\theta_i)x(\theta_i) - y^T(\theta_i^+)D_i x(\theta_{i-j})\\ + y^T(\theta_{i+j}^+)D_{i+j}x(\theta_i).\end{aligned}$$

Using the impulse conditions in (4)

$$x(\theta_i^+) = (I + C_i)x(\theta_i) + D_i x(\theta_{i-j})$$

and the impulse conditions in (9)

$$y^T(\theta_{i+j}^+)D_{i+j} = y^T(\theta_i) - y^T(\theta_i^+) - y^T(\theta_i^+)C_i$$

we deduce that $c_{i+1} - c_i = 0$ for all $i \in \mathbb{N}$ and thus $\langle y(t), x(t)\rangle = c$. □

Remark 2.1. It is easy to verify also that the adjoint of (9) is (4), i.e they are mutually adjoint of each other.

Definition 2.1. A matrix solution $X(t,\alpha)$ of (4) satisfying $X(\alpha,\alpha) = I$ and $X(t,\alpha) = 0$ for $t < \alpha$ is called a fundamental matrix of (4).

Definition 2.2. A matrix solution $Y(t,\alpha)$ of (9) satisfying $Y(\alpha,\alpha) = I$ and $Y(t,\alpha) = 0$ for $t > \alpha$ is said to be a fundamental matrix of (9).

Lemma 2.2. *Let $X(t,\alpha)$ be a fundamental matrix of* (4) *and $\sigma \geq 0$ a real number. If $x(t)$ is a solution of* (6)*, then*

$$\begin{aligned} x(t) &= X(t,\sigma)x(\sigma) \\ &\quad + \int_{\sigma-\tau}^{\sigma} X(t,\alpha+\tau)B(\alpha+\tau)x(\alpha)\mathrm{d}\alpha + \int_{\sigma}^{t} X(t,\alpha)f(\alpha)\mathrm{d}\alpha \\ &\quad + \sum_{n(\sigma)-j\leq k<n(\sigma)} X(t,\theta_{k+j}^{+})D_{k+j}x(\theta_k) + \sum_{n(\sigma)\leq k<n(t)} X(t,\theta_k^{+})\beta_k. \quad (6) \end{aligned}$$

Proof. Multiplying the differential equation in (6) by the matrix $Y^T(\alpha,t)$ and integrating with respect to α from σ to t, we obtain

$$\begin{aligned} x(t) &= Y^T(\sigma,t)x(\sigma) - \int_{\sigma}^{t} Y^T(\alpha+\tau,t)B(\alpha+\tau)x(\alpha)\mathrm{d}\alpha \\ &\quad + \int_{\sigma}^{t} Y^T(\alpha,t)B(\alpha)x(\alpha-\tau)\mathrm{d}\alpha + \int_{\sigma}^{t} Y^T(\alpha,t)f(\alpha)\mathrm{d}\alpha \\ &\quad + \sum_{n(\sigma)\leq k<n(t)} \Big[Y^T(\theta_k^{+},t)x(\theta_k^{+}) - Y^T(\theta_k,t)x(\theta_k)\Big]. \end{aligned}$$

Replacing α by $\alpha+\tau$ in the second integral and using the impulse conditions in (6) and (9), we have

$$\begin{aligned} x(t) &= Y^T(\sigma,t)x(\sigma) + \int_{\sigma-\tau}^{\sigma} Y^T(\alpha+\tau,t)B(\alpha+\tau)x(\alpha)\mathrm{d}\alpha \\ &\quad + \sum_{n(\sigma)-j\leq k<n(\sigma)} Y^T(\theta_{k+j}^{+},t)D_{k+j}x(\theta_k) \\ &\quad + \int_{\sigma}^{t} Y^T(\alpha,t)f(\alpha)\mathrm{d}\alpha + \sum_{n(\sigma)\leq k<n(t)} Y^T(\theta_k^{+},t)\beta_k. \end{aligned} \quad (7)$$

Since $X(t,\sigma) = Y^T(\sigma,t)$, which can be seen by replacing $x(t)$ by the fundamental matrix $X(t,\sigma)$ in (7) with $f \equiv 0$ and $\beta_i = 0$ for all $i \in \mathbb{N}$, (7) is the same as (6). □

Corollary 2.1. *Let* $X(t,\alpha)$ *be a fundamental matrix of* (4) *and* $Y(t,\alpha)$ *be a fundamental matrix of* (9). *Then*

$$X(t,\alpha) = Y^T(\alpha,t).$$

Lemma 2.3. *If* (4) *verifies Perron condition, then*

$$\|X(t,\alpha)\| < M \textit{ for } t \geq \alpha \geq 0.$$

Proof. We first claim that there exists a constant d such that

$$\int_0^t \|X(t,\alpha)\|d\alpha + \sum_{0\leq m<n(t)} \|X(t,\theta_m^+)\| < d \quad \text{for } t \geq 0. \tag{8}$$

Define the space $\Pi = CB \times S$, where CB is the set of bounded functions $f \in C([0,\infty),\mathbb{R}^n)$ and S is the set of bounded sequences $\beta = \{\beta_m\}$, $\beta_m \in \mathbb{R}^n$, $m \in \mathbb{N}$. The elements are represented by the pair (f,β) supplied by the norm $\|(f,\beta)\| = \sup\limits_{t\in[0,\infty)} \|f(t)\| + \sup\limits_{m\in\mathbb{N}} \|\beta_m\|$. Consider the operator U defined on the Banach space Π by

$$U(f,\beta_m) = \int_0^t X(t,\alpha)f(\alpha)\mathrm{d}\alpha + \sum_{0\leq m<n(t)} X(t,\theta_m^+)\beta_m.$$

We may use the Banach Steinhaus theorem Ref. 20 by employing similar arguments developed in Ref. 12 to arrive at (8).

Now let us consider (9) satisfied by $Y(\alpha,t)$. Integrating both sides from σ to t leads to

$$\begin{aligned} Y^T(\sigma,t) = I + \int_\sigma^t Y^T(\alpha,t)A(\alpha)\mathrm{d}\alpha + \int_\sigma^t Y^T(\alpha+\tau,t)B(\alpha+\tau)\mathrm{d}\alpha \\ - \sum_{n(\sigma)\leq i<n(t)} \Delta Y^T(\theta_i,t). \end{aligned}$$

Observing that $Y^T(\theta_i,t) = Y^T(\theta_i^+,t)(I+C_i) + Y^T(\theta_{i+j}^+,t)D_i$, we obtain

$$\sum_{n(\sigma)\leq i<n(t)} \|\Delta Y^T(\theta_i,t)\| \leq 2\rho \sum_{0\leq i<n(t)} \|Y^T(\theta_i^+,t)\|.$$

It follows that

APPLIED ANALYSIS AND DIFFERENTIAL EQUATIONS

$$\|Y^T(\sigma,t)\| \le 1 + 2\gamma \int_0^t \|Y^T(\alpha,t)\| \mathrm{d}\alpha + 2\rho \sum_{0\le i<n(t)} \|Y^T(\theta_i^+,t)\|,$$

where

$$\gamma = \max\left\{\sup_{t\ge 0}\|A(t)\|, \sup_{t\ge 0}\|B(t)\|\right\}.$$

Replacing $Y^T(\sigma,t)$ by $X(t,\sigma)$ and using inequality (8) result in the desired conclusion. □

3. The Main Result

Theorem 3.1. *If equation* (4) *verifies Perron condition then its trivial solution is uniformly asymptotically stable.*

Proof. Let $x(t;\sigma,\phi)$ denote the solution of (4) satisfying (3). From Lemma 2.2,

$$x(t;\sigma,\phi) = X(t,\sigma)\phi(0) + \int_{-\tau}^0 X(t,\alpha+\sigma+\tau)B(\alpha+\sigma+\tau)\phi(\alpha)\mathrm{d}\alpha + \sum_{-j\le k<0} X(t,\theta_{k+n(\sigma)+j}^+)D_{k+n(\sigma)+j}\phi(\theta_k).$$

By Lemma 2.3, there exists $M>0$ such that $\|X(t,r)\| < M$. Hence

$$\|x(t;\sigma,\phi)\| \le M(1+\tau\gamma+j\rho)\|\phi\|_0 = M_1\|\phi\|_0,$$

where

$$M_1 = M(1+\tau\gamma+j\rho)$$

and

$$\|\phi\|_0 = \sup_{r\in[-\tau,0]} \|\phi(r)\|.$$

Thus, the zero solution of (4) is uniformly stable.

To complete the proof we need to show that

$$\lim_{t\to\infty} x(t;\sigma,\phi) = 0 \quad \text{uniformly with respect to } \sigma \text{ and } \phi. \tag{9}$$

APPLIED ANALYSIS AND DIFFERENTIAL EQUATIONS

For our purpose, let $\mu \geq \sigma$. It is clear that $x(t) = x(t;\sigma,\phi)$ satisfies

$$x(t;\sigma,\phi) = X(t,\mu)x(\mu;\sigma,\phi) + \int_{\mu-\tau}^{\mu} X(t,\alpha+\tau)B(\alpha+\tau)x(\alpha;\sigma,\phi)\mathrm{d}\alpha$$
$$+ \sum_{n(\mu)-j\leq k<n(\mu)} X(t,\theta_{k+j}^{+})D_{k+j}x(\theta_k;\sigma,\phi).$$

Integrating both sides from σ to t and then changing the order of integrations and the order of summation and the integral, we have

$$\begin{aligned}(t-\sigma)x(t;\sigma,\phi) &= \int_{\sigma-\tau}^{\sigma}\int_{\sigma}^{\alpha+\tau} X(t,\alpha+\tau)B(\alpha+\tau)x(\alpha;\sigma,\phi)\mathrm{d}\mu\mathrm{d}\alpha \\ &+ \int_{\sigma}^{t-\tau}\int_{\alpha}^{\alpha+\tau} X(t,\alpha+\tau)B(\alpha+\tau)x(\alpha;\sigma,\phi)\mathrm{d}\mu\mathrm{d}\alpha \\ &+ \int_{\sigma}^{t} X(t,\mu)x(\mu;\sigma,\phi)\mathrm{d}\mu \\ &+ \sum_{n(\sigma)-j\leq k<n(\sigma)} \int_{\sigma}^{\theta_{k+j}} X(t,\theta_{k+j}^{+})D_{k+j}x(\theta_k;\sigma,\phi)\mathrm{d}\mu \\ &+ \sum_{n(\sigma)\leq k<n(t)-j} \int_{\theta_k}^{\theta_{k+j}} X(t,\theta_{k+j}^{+})D_{k+j}x(\theta_k;\sigma,\phi)\mathrm{d}\mu.\end{aligned}$$

We easily see from above that

$$\begin{aligned}(t-\sigma)\|x(t;\sigma,\phi)\| &\leq \gamma\tau^2 MM_1\|\phi\|_0 + j\rho\nu MM_1\|\phi\|_0 \\ &+ M_1\max\{\tau\gamma,\nu\rho,1\}\|\phi\|_0\Big[\int_0^t \|X(t,s)\|\mathrm{d}s \\ &+ \sum_{0\leq r<n(t)} \|X(t,\theta_r)\|\Big]. \end{aligned} \tag{10}$$

In view of (8), the right side of (10) is bounded. Hence

$$\|x(t;\sigma,\phi)\| \leq \frac{M_2}{t-\sigma}\|\phi\|_0, \tag{11}$$

where M_2 is chosen so that

$$M_2 < MM_1(\gamma\tau^2 + j\rho\nu) + M_1\max\{\tau\gamma_2,\nu\rho,1\}d.$$

Obviously, (9) follows from (11). □

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