GEOMETRY AND INTEGRATION BY PARTS ON $H \setminus \mathrm{Diff}(S^1)$

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We study various tensor fields on the Lie algebra $\mathrm{diff}(S^1)$ and we give their expressions in the trigonometrical basis. We define a bounded operator Ψ on $\mathrm{diff}(S^1)$ modulo $\mathrm{su}(1,1)$. With Ψ, we obtain an integration by parts formula on $H \setminus \mathrm{Diff}(S^1)$.

0. Introduction

In [1], the Levi-Civita connection on $H \setminus \mathrm{Diff}(S^1)$, the quotient space $\mathrm{Diff}(S^1)$ modulo the homographic transformations has been explicited in real and complex coordinates and the existence of the parallel transport on $H \setminus \mathrm{Diff}(S^1)$ has been established. With an extension to the infinite dimensional case of the Fang-Malliavin structure equations (see [12]), integration by parts formulae have been obtained on $H \setminus \mathrm{Diff}(S^1)$. In the first part, we deepen the study of the geometry of tensor fields on $\mathrm{diff}(S^1)$ as started in [6], [7], [8], [15], [16], [4], [1], [14]. Both manifolds $H \setminus \mathrm{Diff}(S^1)$ and $\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$ have a structure of Kählerian manifold. In [6], [7], [8], [15], [16], [4], [1], [14], the Ricci tensor has been proved to be a diagonal operator in the trigonometrical basis. It is a multiple of the metric tensor and these manifolds are Einstein manifolds. The bracket on the Lie algebra $\mathrm{diff}(S^1)$ is defined by

$$[u,v] = uv' - u'v \quad \text{for} \quad u \in \mathrm{diff}(S^1), \quad v \in \mathrm{diff}(S^1) \tag{1}$$

and the Hilbert transform is linear and it is given on trigonometric functions by

$$J\cos k\theta = \sin k\theta \quad \text{and} \quad J\sin k\theta = -\cos k\theta \quad \text{for } k \geq 1, \quad J\,1 = 0. \tag{2}$$

In [1], it has been proved in particular that with the Levi-Civita tensor field Γ, the operators $\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2$ are diagonal operators in the trigonometrical basis. In this work, in relation with the Levi-Civita tensor field on $H \setminus \mathrm{Diff}(S^1)$, we study tensor fields $\Gamma(u) : \mathrm{diff}(S^1) \to \mathrm{diff}(S^1)$ which (i) commute with J, (ii) are torsionless, (iii) for $p \geq 1$, the operators $\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2$ are diagonal operators in the trigonometrical basis

$(e_p)_{p\geq 1} = \{\cos k\theta, \sin k\theta\}_{k\geq 1}$; we resume these three conditions as

$$\begin{array}{ll}\text{(i)} & \Gamma(u)v - \Gamma(v)u = [u,v] \text{ (torsionless condition)} \\ \text{(ii)} & \Gamma(u)Jv = J\Gamma(u)v\end{array} \quad \text{for } u,v \in \text{diff}(S^1), \tag{3}$$

$$\text{(iii)} \quad \begin{cases}[\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2]\sin k\theta = \lambda_{p,k}\sin k\theta, \\ [\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2]\cos k\theta = \lambda_{p,k}\cos k\theta.\end{cases} \tag{4}$$

Let

$$\alpha(k) = ak^3 + bk \quad \text{where } a > 0; \tag{5}$$

then

$$\begin{gathered}(m+j)\alpha(m-j) + (j-2m)\alpha(j) + (2j-m)\alpha(m) = 0, \\ (2j+m)\alpha(m) = (m-j)\alpha(m+j) + (2m+j)\alpha(j).\end{gathered} \tag{6}$$

We shall take $\alpha(k) = k^3 - k$. It satisfies $(k+2)\alpha(k) = (k-1)\alpha(k+1)$. Conversely, it is remarked in section 3 of the present work that the condition (6) on $\alpha(k)$ implies that $\alpha(k)$ is of the form (5) up to a constant term. On $\text{diff}(S^1)$, let $(\ |\ )$ be a pseudo metric defined by the conditions, for $k \geq 1$, $(\cos k\theta \mid \cos k\theta) = (\sin k\theta \mid \sin k\theta) = \alpha(k)$, and $\{\cos k\theta, \sin k\theta\}_{k\geq 0}$ are orthogonal vectors on $\text{diff}(S^1)$. The interest of this metric is due first to its close relationship and well adaptedness to the trigonometrical basis, secondly to its remarkable properties. In particular the second fundamental two-form is closed. In [3], it has been proved that the metric $\alpha(k) = k^3 - k$ is the unique one (up to the multiplication by a constant) such that $Ad(h)$ is unitary for any homographic transformation h. In the following, we extend the construction of diagonal operators associated to the Levi-Civita tensor field, (see [1], section 2) to more general tensor fields Γ. We study the tensor fields Γ satisfying (4) and such that

$$\begin{aligned}\Gamma(\cos p\theta)\sin k\theta = \beta_p(k)\cos(p+k)\theta + \gamma_p(k)\,1_{p\geq k+1}\cos(p-k)\theta \\ + \mu_p(k)\,1_{k\geq p+1}\cos(k-p)\theta + a_k\,\delta_k^p\cos k\theta\end{aligned} \tag{7}$$

where δ_k^p is the Kronecker symbol. We obtain that, for $p \geq 1$, the operator $\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2$ is diagonal in the trigonometric basis $\{\cos k\theta, \sin k\theta\}_{k\geq 1}$ if and only if $\gamma_p(k)\,1_{p\geq k+1} = 0$. In the same way, the operator

$$\Phi_p : u \to [\Gamma(\cos p\theta)u, \cos p\theta] + [\Gamma(\sin p\theta)u, \sin p\theta] \tag{8}$$

is diagonal in the trigonometric basis $\{\cos k\theta, \sin k\theta\}_{k\geq 1}$ if and only if $\gamma_p(k)\,1_{p\geq k+1} = 0$. In that case, $\mathbf{Z}_p$ defined by $\mathbf{Z}_p = \Phi_p + \Gamma(\cos p\theta)^2 +$

$\Gamma(\sin p\theta)^2$ is diagonal and $\mathbf{Z}_p \sin k\theta = \lambda_p^Z(k) \sin k\theta$, $\mathbf{Z}_p \cos k\theta = \lambda_p^Z(k) \cos k\theta$ with

$$\lambda_p^Z(k) = -1_{k \geq p+1}\,(p+k)\left[\frac{1}{2}(2p-k) + \beta_p(k-p)\right] - k\, a_k \delta_k^p. \tag{9}$$

If we assume that $\beta_p(k) = \frac{(2p+k)\alpha(k)}{2\alpha(p+k)}$, this value of $\beta_p(k)$ gives the Levi-Civita connection, then the trace

$$2\sum_{p\geq 0} \lambda_p^Z(k) = -\frac{13}{6} - \frac{2k(a_k-1)}{\alpha(k)} \tag{10}$$

is finite. In particular, when Γ is the Levi-Civita tensor field, we study the operator Ψ defined by

$$\Psi = \sum_j [\Gamma(\epsilon_j) + \phi(\epsilon_j)]\,\Gamma(\epsilon_j) \tag{11}$$

where for h, $u \in \mathrm{diff}(S^1)$,

$$\phi(\epsilon_j)h = [\epsilon_j, h]. \tag{12}$$

The operator Ψ is bounded on $\mathrm{diff}(S^1)$. For more general Γ satisfying (3), we obtain more bounded operators on $\mathrm{diff}(S^1)$ with finite trace as (10).

The second part is stochastic analysis. With Ψ and an adaptation of Fang's integration by parts on loop groups [11], we establish the following integration by parts formula for the Levi-Civita connection on $H \setminus \mathrm{Diff}(S^1)$. Consider the canonical Brownian motion $x(t)$ on $\mathrm{diff}(S^1)$, see [17], [2], [13]. Let $\Omega(\bullet)$ be the stochastic parallel transport of the frames above $H \setminus \mathrm{Diff}(S^1)$. See [1] for the existence of $\Omega(\bullet)$. We have the SDE

$$dz(t) = \sum_\alpha \Omega_\alpha(t) * dx_\alpha(t)\,; \qquad d\Omega_\alpha^k(t) + \Gamma(\Omega_\alpha(t), *dz(t), \epsilon_k) = 0 \quad \forall\alpha, \forall k\,. \tag{13}$$

A continuous map $t \to \gamma(t)$ from $[0, +\infty[$ to $\mathcal{H} \setminus \mathrm{Diff}(S^1)$ is denoted $\gamma(\bullet)$. We call X the Wiener space of such continuous maps. For $h \in V$ and $\mu \in R$,

$$(\exp(\mu\, h))(\theta) = \theta + \mu h(\theta) + \cdots \tag{14}$$

We have

$$\begin{aligned}&\exp(-\mu\Omega(t)h)\, o\, d\, \exp(\mu\Omega(t)h)\\ &\quad = (I - \mu\Omega(t)h)\, o\, \mu\, d\Omega(t)h \,+\, \text{terms in } \mu^j, \quad j \geq 2\\ &\quad = \mu\, d\Omega(t)\, h + \text{terms in } \mu^j, \quad j \geq 2.\end{aligned} \tag{15}$$

In the same way,

$$d\,\exp(\mu\Omega(t)h)\,o\,\exp(-\mu\Omega(t)h) = \mu d\Omega(t)h + \text{terms in } \mu^j, \quad j \geq 2. \tag{16}$$

We consider the process

$$\gamma_\mu(t) = \exp(\mu\Omega(t)h)\,o\,\gamma(t). \tag{17}$$

For $F : X \to R$, we put

$$D_h^\Gamma F(\gamma(\bullet)) = \frac{d}{d\mu}\bigg|_{\mu=0} F(\exp(\mu\Omega(\bullet)h)o\gamma(\bullet)). \tag{18}$$

Denoting $(\ |\ )$ the metric on V, then

$$E[(D_h^\Gamma F)(\gamma(\bullet))] = E\left[F(\gamma(\bullet))\int_0^1 (\Psi\,\Omega(t)h \mid o\,dx(t))\right]. \tag{19}$$

Part I. Geometry on trigonometric functions

1. Bracket, metric and structure constants

Let δ_i^j be the Kronecker symbol, with the bracket (1), we have

$$\begin{cases} 2[\cos k\theta, \cos p\theta] = \sum_{q\geq 1}[(k-p)\delta_q^{k+p} + (k+p)\delta_q^{k-p} - (k+p)\delta_q^{p-k}]\sin q\theta, \\ 2[\sin k\theta, \sin p\theta] = \sum_{q\geq 1}[(p-k)\delta_q^{p+k} + (k+p)\delta_q^{k-p} - (k+p)\delta_q^{p-k}]\sin q\theta, \\ 2[\cos k\theta, \sin p\theta] = \sum_{q\geq 1}[(p-k)\delta_q^{k+p} + (k+p)\delta_q^{k-p} + (k+p)\delta_q^{p-k}]\cos q\theta \\ \qquad\qquad + 2k\delta_k^p. \end{cases} \tag{20}$$

In complex coordinates, for $m, n \in Z$, $[e^{im\theta}, e^{in\theta}] = i\,(n-m)e^{i(m+n)\theta}$. From (20),

$$\begin{cases} (i) \quad [\sin k\theta, \sin p\theta] - [\cos k\theta, \cos p\theta] = (p-k)\sin(k+p)\theta, \\ (ii) \quad [\cos k\theta, \sin p\theta] + [\sin k\theta, \cos p\theta] = (p-k)\cos(k+p)\theta. \end{cases} \tag{21}$$

We take $\alpha(k) = k^3 - k$. For $j \geq 2$, we put

$$c_j = \frac{\cos j\theta}{\sqrt{\alpha(j)}}, \quad s_j = \frac{\sin j\theta}{\sqrt{\alpha(j)}}. \tag{22}$$

For $j \geq 2$, $k \geq 2$,

$$\begin{cases} 2[c_j, c_k] = \sum_{p\geq 2} K^p_{jk} s_p + \left[\dfrac{\delta_j^{k+1}(2k+1)}{\sqrt{\alpha(k)}\sqrt{\alpha(k+1)}} - \dfrac{\delta_j^{k-1}(2k-1)}{\sqrt{\alpha(k)}\sqrt{\alpha(k-1)}} \right] \sin\theta, \\ 2[s_j, s_k] = \sum_{p\geq 2} S^p_{jk} s_p + \left[\dfrac{\delta_j^{k+1}(2k+1)}{\sqrt{\alpha(k)}\sqrt{\alpha(k+1)}} - \dfrac{\delta_j^{k-1}(2k-1)}{\sqrt{\alpha(k)}\sqrt{\alpha(k-1)}} \right] \sin\theta, \\ 2[s_j, c_k] = \sum_{p\geq 2} (SC)^p_{jk} c_p \\ \qquad\qquad - \left[\dfrac{\delta_j^{k+1}(2k+1)}{\sqrt{\alpha(k)}\sqrt{\alpha(k+1)}} + \dfrac{\delta_j^{k-1}(2k-1)}{\sqrt{\alpha(k)}\sqrt{\alpha(k-1)}} \right] \cos\theta + 2k\delta_k^j, \end{cases}$$

where K^p_{jk}, S^p_{jk} are antisymmetric in (j,k),

$$\begin{aligned} K^p_{jk} &= \frac{(j-k)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}} \delta_p^{j+k} + \frac{(2k+p)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}} \delta_j^{k+p} - \frac{(2j+p)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}} \delta_k^{j+p}, \\ S^p_{jk} &= \frac{(k-j)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}} \delta_p^{j+k} + \frac{(2k+p)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}} \delta_j^{k+p} - \frac{(2j+p)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}} \delta_k^{j+p}, \\ (SC)^p_{jk} &= \frac{(k-j)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}} \delta_p^{j+k} - \frac{(2k+p)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}} \delta_j^{k+p} - \frac{(2j+p)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}} \delta_k^{j+p}. \end{aligned}$$

2. Tensor fields on diff(S^1), their expressions in the trigonometrical basis

The Hilbert transform J possesses the Nijenhuis property with respect to the Lie bracket (1). For u, $v \in V$,

$$[Ju, Jv] - [u,v] = J([u, Jv] + [Ju, v]). \tag{23}$$

We define the following tensor fields, for u, $v \in \mathrm{diff}(S^1)$, see [4],

$$\begin{cases} E(u,v) = [Ju, Jv] - [u,v], \\ F(u,v) = [u, Jv] + [Ju, v], \end{cases} \qquad \begin{cases} G(u,v) = [Ju, Jv] + [u,v], \\ H(u,v) = [u, Jv] - [Ju, v]. \end{cases} \tag{24}$$

We remark that E, F, G are antisymmetric in u, v, whereas H is symmetric in u, v. On the other hand, (23) is the same as

$$E(u,v) - J\,F(u,v) = 0. \tag{25}$$

We obtain

$$E(Ju,Jv) = -E(u,v) \quad \text{and} \quad \begin{aligned} H(u,v) &= G(u,Jv) = -G(Ju,v),\\ [u,v] &= -\frac{1}{2}(E(u,v) - G(u,v)),\\ J[Ju,v] &= \frac{1}{2}(E(u,v) - JH(u,v)). \end{aligned} \tag{26}$$

We put

$$A(u,v) = J[u,v] - [Ju,v] \quad \text{and} \quad B(u,v) = J[u,v] - [u,Jv]. \tag{27}$$

Since $J^2 = -1$, we have when $u \neq v$,

$$\begin{aligned} A(u,v) &= JA(Ju,v)\\ A(u,Jv) + A(Ju,v) &= 0 \end{aligned} \quad \text{and} \quad B(u,v) = JB(u,Jv). \tag{28}$$

A and B are neither symmetric nor antisymmetric in u, v, the decomposition in symmetric and antisymmetric part is given by

$$\begin{cases} 2A(u,v) = JG(u,v) + H(u,v),\\ 2B(u,v) = JG(u,v) - H(u,v). \end{cases} \tag{29}$$

We express these different tensor fields in the trigonometrical basis, we have

$$\begin{aligned} F(\cos k\theta, \cos p\theta) &= (p-k)\cos(p+k)\theta,\\ F(\cos k\theta, \sin p\theta) &= (p-k)\sin(p+k)\theta, \end{aligned} \quad \text{and} \quad \begin{aligned} F(Ju,Jv) &= -F(u,v),\\ JF(u,v) &= F(u,Jv). \end{aligned} \tag{30}$$

For $j \geq 1$, $k \geq 1$,

$$\begin{aligned} H(\sin j\theta, \cos k\theta) &= (k+j)\sin(j-k)\theta,\\ H(\cos j\theta, \cos k\theta) &= (k+j)\cos(j-k)\theta, \end{aligned} \quad \text{and} \quad H(Ju,Jv) = H(u,v). \tag{31}$$

The symmetric tensor

$$Q(u,v) = \frac{1}{2}\, J\, H(u,v) = \frac{1}{2} J([u,Jv] + [v,Ju]) \tag{32}$$

has been found by [14]. For $j \geq 1$, $k \geq 1$,

$$\begin{aligned} 2Q(\sin j\theta, \cos k\theta) &= -(1_{j\geq k} - 1_{k\geq j})(k+j)\cos(k-j)\theta\\ 2Q(\cos j\theta, \cos k\theta) &= (1_{j\geq k} - 1_{k\geq j})(k+j)\sin(j-k)\theta \end{aligned} \tag{33}$$

$$\text{and } Q(Ju,Jv) = Q(u,v).$$

In the same way,

$$\begin{cases} G(\sin j\theta, \cos k\theta) = -(k+j)\cos(k-j)\theta \\ G(\cos j\theta, \cos k\theta) = (k+j)\sin(j-k)\theta \end{cases} \quad \text{and} \quad G(Ju, Jv) = G(u,v) \tag{34}$$

and

$$P(u,v) = \frac{1}{2} JG(u,v) = Q(Ju,v) = \frac{1}{2} J([Ju,Jv] + [u,v]) \tag{35}$$

satisfies

$$\begin{aligned} &2P(\sin j\theta, \cos k\theta) = (1_{j\geq k} - 1_{k\geq j})(k+j)\sin(k-j)\theta, \\ &2P(\cos j\theta, \cos k\theta) = -(1_{j\geq k} - 1_{k\geq j})(k+j)\cos(j-k)\theta, \\ &P(Ju,Jv) = P(u,v). \end{aligned} \tag{36}$$

For A, B,

$$\begin{cases} A(\sin j\theta, \cos k\theta) = -1_{k\geq j}(k+j)\sin(k-j)\theta \\ A(\cos j\theta, \cos k\theta) = 1_{k\geq j}(k+j)\cos(k-j)\theta \end{cases} \quad \text{and } A(Ju,Jv) = A(u,v), \tag{37}$$

$$\begin{cases} B(\sin j\theta, \cos k\theta) = -1_{j\geq k}(k+j)\sin(j-k)\theta \\ B(\cos j\theta, \cos k\theta) = -1_{j\geq k}(k+j)\cos(k-j)\theta \end{cases} \quad \text{and } B(Ju,Jv) = B(u,v). \tag{38}$$

Let

$$D(u,v) = \frac{1}{2}([Ju,Jv] + [u,v]) = \frac{1}{2}G(u,v); \tag{39}$$

then

$$\begin{cases} Q(u,v) + D(u,v) = [u,v] + J[u,Jv] = -JB(u,v), \\ Q(u,v) - D(u,v) = -[u,v] - J[Ju,v] = JA(u,v). \end{cases} \tag{40}$$

On the other hand, let

$$\{u,v\} = \frac{1}{2}([u,Jv] + [Ju,v]); \tag{41}$$

then

$$\begin{aligned} 2\{\cos k\theta, \cos p\theta\} &= (p-k)\cos(p+k)\theta, \\ 2\{\sin k\theta, \sin p\theta\} &= (k-p)\cos(p+k)\theta, \\ 2\{\cos k\theta, \sin p\theta\} &= (p-k)\sin(p+k)\theta. \end{aligned} \tag{42}$$

3. The fundamental two-form and the metric

The identities (6) give the closure condition of the symplectic form on $\mathrm{diff}(S^1)$. It is the same as

$$(m-n)\alpha(p)+(p-n)\alpha(m)+(m-p)\alpha(n)=0 \quad \text{with} \quad m+n+p=0 \tag{43}$$

or equivalently

$$\det\begin{pmatrix}1 & x & \alpha(x)\\ 1 & y & \alpha(y)\\ 1 & -(x+y) & \alpha(-(x+y))\end{pmatrix}=0. \tag{44}$$

We look for solutions of (44) and assume that α has no singularity at zero. The function $\alpha(x)=\lambda x+\mu$ is a solution of (44). We may assume that $\alpha(0)=0$ and $\alpha'(0)=0$. With $y=0$ in (44), we obtain that α is an odd function. In (44), we take the derivative with respect to y, and we put $y=0$ and $\alpha'(0)=0$; we obtain $-3\alpha(x)+x\alpha'(x)=0$, thus $\alpha(x)=bx^3$.

With (6), we obtain (see [4]),

$$([u,v]\mid Jw)+([w,u]\mid Jv)+([v,w]\mid Ju)=0 \tag{45}$$

and for the fundamental two-form Φ

$$\Phi(u,v)=(u\mid Jv) \tag{46}$$

it gives

$$\Phi([u,v],w)+\Phi([w,u],v)+\Phi([v,w],u)=0. \tag{47}$$

4. The Levi-Civita connection on $H\setminus\mathrm{Diff}(S^1)$

Let $u\in V$, $v\in\mathrm{diff}(S^1)$, $w\in V$, we define the Levi-Civita tensor field (see [1]) $\Gamma_1(v)u$ with

$$2(\Gamma_1(v)u\mid w)=([w,v]\mid u)+([w,u]\mid v)-([u,v]\mid w)) \tag{48}$$

and we put

$$2(\Lambda_1(v)u\mid w)=-2(\Gamma_1(u)v\mid w)=([v,w]\mid u)+([u,w]\mid v)+([v,u]\mid w) \tag{48'}$$

then

$$\Gamma_1(u)v+[v,u]=\Gamma_1(v)u. \tag{49}$$

We have

$$\Gamma_1(v)Ju = J\Gamma_1(v)u \quad \text{and} \quad (\Gamma_1(v)u \mid w) = -(u \mid \Gamma_1(v)w). \tag{50}$$

Both Γ_1 and Λ_1 are torsionless. The expression of Γ_1 in the trigonometrical basis has been given in [1]. If $v = 1$, $v = \cos\theta$, $v = \sin\theta$, then $2(\Gamma_1(v)u \mid w) = ([w, v] \mid u) - ([u, v] \mid w)$, thus $2(\Gamma_1(1)u \mid w) = -(w' \mid u) + (u' \mid w) = 2(u' \mid w)$.

$$\Gamma_1(1)u = u', \tag{51}$$

$$\begin{cases} 2\Gamma_1(\cos\theta)\cos k\theta = -1_{k\geq 3} \times (k+1)\sin(k-1)\theta - (k-1)\sin(k+1)\theta, \\ 2\Gamma_1(\cos\theta)\sin k\theta = 1_{k\geq 3} \times (k+1)\cos(k-1)\theta + (k-1)\cos(k+1)\theta, \end{cases} \tag{52}$$

$$\begin{cases} 2\Gamma_1(\sin\theta)\sin k\theta = -1_{k\geq 3} \times (k+1)\sin(k-1)\theta + (k-1)\sin(k+1)\theta, \\ 2\Gamma_1(\sin\theta)\cos k\theta = -1_{k\geq 3} \times (k+1)\cos(k-1)\theta + (k-1)\cos(k+1)\theta, \end{cases} \tag{53}$$

and, for $p \geq 2$,

$$\begin{cases} 2\Gamma_1(\cos p\theta)(\sin k\theta) \\ \quad = (p+k)1_{k\geq p+2}\cos(k-p)\theta + \dfrac{(2p+k)\alpha(k)}{\alpha(p+k)}\cos(p+k)\theta, \\ 2\Gamma_1(\cos p\theta)(\cos k\theta) \\ \quad = -(p+k)1_{k\geq p+2}\sin(k-p)\theta - \dfrac{(2p+k)\alpha(k)}{\alpha(p+k)}\sin(p+k)\theta, \end{cases} \tag{54}$$

$$\begin{cases} 2\Gamma_1(\sin p\theta)(\sin k\theta) \\ \quad = -(p+k)1_{k\geq p+2}\sin(k-p)\theta + \dfrac{(2p+k)\alpha(k)}{\alpha(p+k)}\sin(p+k)\theta, \\ 2\Gamma_1(\sin p\theta)(\cos k\theta) \\ \quad = -(p+k)1_{k\geq p+2}\cos(k-p)\theta + \dfrac{(2p+k)\alpha(k)}{\alpha(p+k)}\cos(p+k)\theta. \end{cases} \tag{55}$$

We put (see [14])

$$\Lambda_2(v)u = \Lambda_1(v)u - Q(u, v) \quad \text{and} \quad \Gamma_2(v)u = -\Lambda_2(u)v. \tag{56}$$

Thus

$$\Gamma_2(v)u = \Gamma_1(v)u + Q(u, v); \tag{57}$$

we have

$$\begin{aligned} 2(\Lambda_2(v)u \mid w) &= ([w, Jv] \mid Ju) + ([w, Ju] \mid Jv) - ([u, v] \mid w), \\ 2(\Gamma_2(v)u \mid w) &= -2(\Lambda_2(u)v \mid w) \\ &= ([Jv, w] \mid Ju) + ([Ju, w] \mid Jv) - ([u, v] \mid w). \end{aligned} \tag{58}$$

We obtain

$$\Lambda_2(v)Ju = J\Lambda_2(v)u \tag{59}$$

and both Λ_2 and Γ_2 are torsionless. For $p \geq 2$,

$$\begin{cases} 2\Lambda_2(\cos p\theta)(\sin k\theta) \\ \quad = (p+k)1_{k\geq p+2}\cos(k-p)\theta - \dfrac{(2k+p)\alpha(p)}{\alpha(p+k)}\cos(p+k)\theta, \\ 2\Lambda_2(\cos p\theta)(\cos k\theta) \\ \quad = -(p+k)1_{k\geq p+2}\sin(k-p)\theta + \dfrac{(2k+p)\alpha(p)}{\alpha(p+k)}\sin(p+k)\theta, \end{cases} \tag{60}$$

$$\begin{cases} 2\Lambda_2(\sin p\theta)(\sin k\theta) \\ \quad = -(p+k)1_{k\geq p+2}\sin(k-p)\theta - \dfrac{(2k+p)\alpha(p)}{\alpha(p+k)}\sin(p+k)\theta, \\ 2\Lambda_2(\sin p\theta)(\cos k\theta) \\ \quad = -(p+k)1_{k\geq p+2}\cos(k-p)\theta - \dfrac{(2k+p)\alpha(p)}{\alpha(p+k)}\cos(p+k)\theta. \end{cases} \tag{61}$$

Proof of (60)–(61). As for (54)–(55), we have

$$\begin{aligned} &4(\Lambda_2(\cos p\theta)\sin k\theta \mid \cos m\theta) \\ &\quad = 2(-[\cos m\theta, \sin p\theta] \mid \cos k\theta) - 2([\cos m\theta, \cos k\theta] \mid \sin p\theta) \\ &\qquad - 2([\sin k\theta, \cos p\theta] \mid \cos m\theta) \\ &\quad = (m-p)\,\alpha(k)\delta_k^{m+p} - (p+m)\,\alpha(k)\delta_p^{m+k} - (m+p)\,\alpha(k)\delta_m^{k+p} \\ &\qquad + (k+m)\,\alpha(p)\delta_k^{m+p} + (k-m)\,\alpha(p)\delta_p^{m+k} - (k+m)\,\alpha(p)\delta_m^{k+p} \\ &\qquad + (k+p)\,\alpha(m)\delta_k^{m+p} + (k+p)\,\alpha(m)\delta_p^{m+k} + (k-p)\,\alpha(m)\delta_m^{k+p}. \end{aligned}$$

□

5. Commuting with the Hilbert transform, torsionless and antisymmetry

We prove that Γ_1 of last section is characterized by torsionless condition, commutation with J and antisymmetry condition.

We put ourselves on $\mathcal{G} = \operatorname{diff}(S^1)$. *Then having discussed the properties of* Γ *on* $\operatorname{diff}(S^1)$, *we take the orthogonal projection* $\pi : \operatorname{diff}(S^1) \to V$ *and we define, for* $v \in \operatorname{diff}(S^1)$, *the operator*

$$\Gamma_{\text{proj}}(v) = \pi \circ \Gamma(v)_{|V} \tag{62}$$

where $\Gamma(v)_{|V}$ *denotes the restriction of* $\Gamma(v)$ *to the subspace* V. In fact, the metric on the linear subspace V of $\mathrm{diff}(S^1)$ will determine the curvature of the quotient space.

Torsionless condition and commutation with J on $\mathrm{diff}(S^1)$. In the following lemma, we characterize Γ when

$$\begin{aligned} &\text{(i) } \Gamma(u)v - \Gamma(v)u = [u,v] \quad \text{(torsionless condition)} \\ &\text{(ii) } \Gamma(u)Jv = J\Gamma(u)v \end{aligned} \qquad \text{for } v \in \mathrm{diff}(S^1). \tag{63}$$

Notice that when $u = 1$ and v is in the subspace of $\mathcal{G}$ generated by $\{\cos k\theta, \sin k\theta\}_{k\geq 1}$, then with $\Gamma(1)u = u'$, $\Gamma(v)(1) = 0$ and $\Gamma(1)Jv = (Jv)' = Jv'$, the conditions (63) are satisfied. We consider the case where u and v are in the subspace generated by $\{\cos k\theta, \sin k\theta\}_{k\geq 1}$. Since $[\cos p\theta, \sin k\theta]$ is expressed in terms of cos, we put for p, $k \geq 1$,

$$\begin{aligned} \Gamma(\cos p\theta)\sin k\theta = \beta_p(k)\cos(p+k)\theta &+ \gamma_p(k)1_{p\geq k+1}\cos(p-k)\theta \\ &+ \mu_p(k)1_{k\geq p+1}\cos(k-p)\theta + a_k\delta_k^p \cos k\theta. \end{aligned} \tag{64}$$

Lemma. *Assume that* Γ *satisfies* (63)–(64)*; then*

$$\left\{\begin{aligned} \Gamma(\cos p\theta)\sin k\theta &= \beta_p(k)\cos(p+k)\theta + \gamma_p(k)1_{p\geq k+1}\cos(p-k)\theta \\ &\qquad +\mu_p(k)1_{k\geq p+1}\cos(k-p)\theta + a_k\,\delta_k^p, \\ \Gamma(\cos p\theta)\cos k\theta &= -\beta_p(k)\sin(p+k)\theta - \gamma_p(k)1_{p\geq k+1}\sin(p-k)\theta \\ &\qquad -\mu_p(k)1_{k\geq p+1}\sin(k-p)\theta, \end{aligned}\right. \tag{65a}$$

$$\left\{\begin{aligned} \Gamma(\sin p\theta)\cos k\theta &= \beta_p(k)\cos(p+k)\theta \\ &\qquad +(\mu_p(k) - (p+k))\,1_{k\geq p+1}\cos(k-p)\theta \\ &\qquad +\gamma_p(k)\,1_{p\geq k+1}\cos(p-k)\theta + (a_k - k)\delta_k^p, \\ \Gamma(\sin p\theta)\sin k\theta &= \beta_p(k)\sin(p+k)\theta \\ &\qquad +(\mu_p(k) - (p+k))\,1_{k\geq p+1}\sin(k-p)\theta \\ &\qquad +\gamma_p(k)\,1_{p\geq k+1}\sin(p-k)\theta, \end{aligned}\right. \tag{65b}$$

where

$$\begin{aligned} -\beta_p(k) + \beta_k(p) &= \frac{1}{2}(p-k), \\ -\gamma_p(k) + \mu_k(p) &= \frac{1}{2}(k+p) \quad \textit{for } \; p \geq k+1. \end{aligned} \tag{66}$$

In particular,

$$\textit{if } \; \gamma_p(k)1_{p\geq k+1} = 0, \quad \textit{then } \; \mu_p(k) = \frac{1}{2}(k+p) \;\; \textit{for } \; k \geq p+1. \tag{67}$$

Proof. From (63), and since $\cos k\theta = -J\sin k\theta$, we obtain $\Gamma(\cos p\theta)\cos k\theta$. With (63) (i) (torsionless condition), $\Gamma(\cos p\theta)\cos k\theta = \Gamma(\cos k\theta)\cos p\theta + [\cos p\theta, \cos k\theta]$. This gives the condition (66). With the torsionless condition, and with (64), we calculate

$$\Gamma(\sin p\theta)\cos k\theta = \Gamma(\cos k\theta)\sin p\theta + [\sin p\theta, \cos k\theta].$$

With (63) (ii) (commutation with J), we find $\Gamma(\sin p\theta)\sin k\theta$. The torsionless condition on $\Gamma(\sin p\theta)\sin k\theta$ yields again the conditions (66). ☐

Lemma. *Assume that Γ satisfies the conditions of the previous lemma, i.e., it is given by (65)–(66); then, for $p \geq 1$, the operator $\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2$ is diagonal in the trigonometric basis $\{\cos k\theta, \sin k\theta\}_{k\geq 1}$ if and only if $\gamma_p(k)\,1_{p\geq k+1} = 0$. In that case, we have, for $k \geq 1$,*

$$\begin{aligned}&\left[\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2\right]\sin k\theta\\ &\qquad = -[(2p+k)\beta_p(k) + (p+k)\beta_p(k-p)\,1_{k\geq p+1}]\sin k\theta. \qquad (68)\end{aligned}$$

In particular, if $\Gamma = \Gamma_1$ or $\Gamma = \Lambda_2$, then $\left[\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2\right]$ is a diagonal operator.

Proof.

$$\left\{\begin{aligned}&\Gamma(\cos p\theta)\sin k\theta\\ &\quad = \textstyle\sum_{r\geq 1}\left[\beta_p(k)\delta_r^{p+k} + \gamma_p(k)\delta_r^{p-k} + \mu_p(k)\delta_r^{k-p}\right]\cos r\theta + a_k\delta_k^p,\\ &\Gamma(\cos p\theta)\cos r\theta\\ &\quad = -\textstyle\sum_{j\geq 1}\left[\beta_p(r)\delta_j^{p+r} + \gamma_p(r)\delta_j^{p-r} + \mu_p(r)\delta_j^{r-p}\right]\,\sin j\theta.\end{aligned}\right.$$

Thus

$$\begin{aligned}\Gamma(\cos p\theta)^2\sin k\theta = -\sum_{j\geq 1}\sum_{r\geq 1}&\left[\beta_p(k)\delta_r^{p+k} + \gamma_p(k)\delta_r^{p-k} + \mu_p(k)\delta_r^{k-p}\right]\\ &\times\left[\beta_p(r)\delta_j^{p+r} + \gamma_p(r)\delta_j^{p-r} + \mu_p(r)\delta_j^{r-p}\right]\,\sin j\theta\end{aligned}$$

and

$$\begin{aligned}\Gamma(\sin p\theta)^2\sin k\theta = \sum_{j\geq 1}\sum_{r\geq 1}&\left[\beta_p(k)\delta_r^{p+k} + \gamma_p(k)\delta_r^{p-k} + (\mu_p(k) - (p+k))\delta_r^{k-p}\right]\\ \times&\left[\beta_p(r)\delta_j^{p+r} + \gamma_p(r)\delta_j^{p-r} + (\mu_p(r) - (p+r))\delta_j^{r-p}\right]\,\sin j\theta.\end{aligned}$$

Adding, we find

$$\begin{aligned}
&[\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2] \sin k\theta \\
&\qquad = [k(p+k) - k\mu_p(k) - (p+k)\mu_p(k-p)] 1_{k \geq 2p+1} \sin(k-2p)\theta \\
&\qquad\quad - (p+k)\gamma_p(k-p) 1_{2p \geq k+1} \sin(2p-k)\theta \\
&\qquad\quad - [(2p+k)\beta_p(k) + (p+k)\beta_p(k-p)] \sin k\theta.
\end{aligned}$$

We proceed in the same way with $[\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2] \sin k\theta$,

$$\begin{aligned}
&\big([\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2] \cos k\theta \mid \cos j\theta\big) \\
&\qquad\qquad = \big([\Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2] \sin k\theta \mid \sin j\theta\big).
\end{aligned}$$

□

Corollary 1. *For* $\Gamma = \Gamma_1$, $p \geq 0$, $k \geq 2$,

$$\begin{aligned}
&2\Gamma(\cos p\theta)^2 \sin k\theta + 2\Gamma(\sin p\theta)^2 \sin k\theta \\
&\qquad = -(p+k)^2 \frac{\alpha(k-p)}{\alpha(k)} 1_{k \geq p+2} \sin k\theta - \frac{(2p+k)^2 \alpha(k)}{\alpha(p+k)} \sin k\theta. \qquad (69)
\end{aligned}$$

Moreover $\sum_{p \geq 2} \Gamma\big(\cos p\theta/\sqrt{\alpha(p)}\big)^2 + \Gamma\big(\sin p\theta/\sqrt{\alpha(p)}\big)^2$ *is a diagonal operator. The coefficients on the diagonal are given by*

$$\lambda_k = \frac{1}{2} \sum_{p \geq 2} \left[-(p+k)^2 \frac{\alpha(k-p)}{\alpha(p)\alpha(k)} 1_{k \geq p+2} - \frac{(2p+k)^2 \alpha(k)}{\alpha(p)\alpha(p+k)} \right] \qquad (70)$$

where the series converge.

$$\begin{cases} [\Gamma(\cos\theta)^2 + \Gamma(\sin\theta)^2] \sin k\theta = -(k^2-2) \sin k\theta \\ \Gamma(1)^2 \sin k\theta = -k^2 \sin k\theta \end{cases} \qquad \textit{for } k \geq 2. \qquad (71)$$

In the same order of idea, looking for diagonal operators, we have

Lemma. *With the assumptions* (65)–(66) *on* Γ, *the operator*

$$\Phi_p \, : \, u \longrightarrow [\Gamma(\cos p\theta)u, \cos p\theta] + [\Gamma(\sin p\theta)u, \sin p\theta] \qquad (72)$$

is diagonal in the trigonometric basis $\{\cos k\theta, \sin k\theta\}_{k \geq 1}$ *if and only if* $\gamma_p(k)\, 1_{p \geq k+1} = 0$. *In that case we have, for* $k \geq 1$, $\Phi_p(\sin k\theta) = \lambda_p^\Phi(k) \sin k\theta$ *and* $\Phi_p(\cos k\theta) = \lambda_p^\Phi(k) \cos k\theta$, *where*

$$\lambda_p^\Phi(k) = -\frac{1}{2}(p+k)(2p-k)\, 1_{k \geq p+1} + (2p+k)\beta_p(k) - k\, a_k \delta_k^p. \qquad (73)$$

Proof.

$$
\begin{aligned}
&[\Gamma(\cos p\theta)\sin k\theta, \cos p\theta] \\
&\quad = \frac{1}{2}\sum_{r\geq 1, j\geq 1} [\beta_p(k)\delta_r^{p+k} + \gamma_p(k)\delta_r^{p-k} + \mu_p(k)\delta_r^{k-p}] \\
&\qquad\qquad \times [(r-p)\delta_j^{r+p} + (r+p)\delta_j^{r-p} - (r+p)\delta_j^{p-r}]\sin j\theta \\
&\qquad + a_k\,\delta_k^p\,[1, \cos p\theta],
\end{aligned}
$$

$$
\begin{aligned}
&[\Gamma(\sin p\theta)\sin k\theta, \sin p\theta] \\
&\quad = \frac{1}{2}\sum_{r\geq 1, j\geq 1} [\beta_p(k)\delta_r^{p+k} + \gamma_p(k)\delta_r^{p-k} + (\mu_p(k) - (p+k))\delta_r^{k-p}] \\
&\qquad\qquad \times [(p-r)\delta_j^{r+p} + (r+p)\delta_j^{r-p} - (r+p)\delta_j^{p-r}]\sin j\theta.
\end{aligned}
$$

Adding, we see that the terms corresponding to $j = k - 2p$ and $j = 2p - k$ vanish if and only if $\mu_p(k) = \frac{1}{2}(p+k)$. This gives the condition on Φ_p to be diagonal. In that case, we calculate $\lambda_p^{\Phi}(k)$. □

Theorem. *We keep the assumptions* (65)–(66) *on* Γ. *For* $p \geq 1$, *the operator* $\mathbf{Z}_p$ *defined by*

$$\mathbf{Z}_p = \Phi_p + \Gamma(\cos p\theta)^2 + \Gamma(\sin p\theta)^2 \tag{74}$$

is diagonal and $\mathbf{Z}_p \sin k\theta = \lambda_p^Z(k)\sin k\theta$, $\mathbf{Z}_p\cos k\theta = \lambda_p^Z(k)\cos k\theta$ *with*

$$\lambda_p^Z(k) = -1_{k\geq p+1}(p+k)\left[\frac{1}{2}(2p-k) + \beta_p(k-p)\right] - k\,a_k\delta_k^p. \tag{75}$$

Corollary 1. *With* (75), *for* $p \geq 0$, *we assume that* $\beta_p(k) = \frac{(2p+k)\alpha(k)}{2\alpha(p+k)}$ *(notice that this value of* $\beta_p(k)$ *is in* Γ_1*). Then*

$$\lambda_p^Z(k) = \lambda_p^R(k)\alpha(p) \;\; \textit{with} \;\; \lambda_p^R(k) = -1_{k\geq p+1}\times(p+k)\frac{(2k-p)}{2\alpha(k)} - k\,a_k\frac{1}{\alpha(k)}\delta_k^p. \tag{76}$$

Moreover,

$$2\sum_{p\geq 0}\lambda_p^Z(k) = -\frac{13}{6} - \frac{2k(a_k-1)}{\alpha(k)}. \tag{77}$$

Proof. For $p \geq 1$, $\beta_p(k-p) = \frac{(p+k)\alpha(k-p)}{2\alpha(k)}$. To calculate the sum, we remark that

$$\sum_{1\leq p\leq k-1}\frac{(p+k)(2k-p)}{\alpha(k)} = \frac{13}{6}k(k^2-1) - \frac{2k(k-1)}{\alpha(k)};$$

then, we add the two terms corresponding to $p = k$ and $p = 0$. We see that the corresponding operator is bounded and that

$$\text{if } a_k = 1, \quad \text{then } 2\sum_{p\geq 0} \lambda_p^Z(k) = -\frac{13}{6}.$$

We have $[\Gamma(\cos\theta)u, \cos\theta] + [\Gamma(\sin\theta)u, \sin\theta] + \Gamma(\cos\theta)^2 u + \Gamma(\sin\theta)^2 u = 0$ and $[\Gamma(1)u, 1] + \Gamma(1)^2 u = 0$. □

Theorem (of unicity). *Let* Γ *be given by* (64),

$$\Gamma(\cos p\theta)\sin k\theta = \beta_p(k)\cos(p+k)\theta + \gamma_p(k)\, 1_{p\geq k+1}\, \cos(p-k)\theta \\ + \mu_p(k)\, 1_{k\geq p+1} \cos(k-p)\theta + a_k \delta_k^p.$$

Assume that $\Gamma(u)v - \Gamma(v)u = [u,v]$ *and* $\Gamma(u)Jv = J\Gamma(u)v$. *Then the condition*

$$(\Gamma(v)u \mid w) = \epsilon(\Gamma(v)w \mid u) \quad \textit{with } \epsilon^2 = 1 \tag{78}$$

implies that $\epsilon = -1$ *and*

$$\beta_p(k) = \frac{(2p+k)\alpha(k)}{2\alpha(p+k)}, \quad \textit{and for } k \geq p+1,\ \gamma_k(p) = 0,\ \mu_p(k) = \frac{p+k}{2}. \tag{79}$$

Proof. We define $H(u)w$ with $(H(u)w \mid v) = (\Gamma(v)u \mid w)$. We calculate $H(u)w$ in the trigonometric basis,

$$\begin{aligned} H(\cos p\theta)\sin k\theta = &-\frac{\beta_{k-p}(p)\alpha(k)}{\alpha(k-p)}\, 1_{k\geq p+1} \cos(k-p)\theta \\ &- \frac{\gamma_{k+p}(p)\alpha(k)}{\alpha(k+p)} \cos(k+p)\theta \\ &- \frac{\mu_{p-k}(p)\alpha(k)}{\alpha(p-k)}\, 1_{p\geq k+1} \cos(p-k)\theta, \\ H(\sin k\theta)\cos p\theta = &\frac{\beta_{p-k}(k)\alpha(p)}{\alpha(p-k)}\, 1_{p\geq k+1} \cos(p-k)\theta \\ &+ \frac{\gamma_{k+p}(k)\alpha(p)}{\alpha(k+p)} \cos(k+p)\theta \\ &+ \frac{\mu_{k-p}(k)\alpha(p)}{\alpha(k-p)}\, 1_{k\geq p+1} \cos(k-p)\theta. \end{aligned}$$

The relation $H(\sin k\theta)\cos p\theta = \epsilon H(\cos p\theta)\sin k\theta$ gives

$$\begin{aligned} \gamma_{p+k}(k)\alpha(p) &= -\epsilon\gamma_{k+p}(p)\alpha(k) \\ \beta_{p-k}(k)\alpha(p) &= -\epsilon\mu_{p-k}(p)\alpha(k) \quad \text{for } p \geq k+1. \end{aligned} \tag{i}$$

In a similar way, we have

$$\begin{aligned} H(\cos p\theta)\cos k\theta = {} & \frac{\beta_{k-p}(p)\alpha(k)}{\alpha(k-p)}\, 1_{k\geq p+1}\, \sin(k-p)\theta \\ & + \frac{\gamma_{k+p}(p)\alpha(k)}{\alpha(k+p)} \sin(k+p)\theta \\ & + \frac{\mu_{p-k}(p)-(2p-k)\alpha(k)}{\alpha(p-k)}\, 1_{p\geq k+1}\, \sin(p-k)\theta. \end{aligned}$$

The condition $H(\cos p\theta)\cos k\theta = \epsilon H(\cos k\theta)\cos p\theta$ gives

$$\begin{aligned} \gamma_{p+k}(p)\alpha(k) &= \epsilon\gamma_{p+k}(k)\alpha(p) \\ \beta_{k-p}(p)\alpha(k) &= \epsilon(\mu_{k-p}(k)-(2k-p))\alpha(p) \quad \text{for } k\geq p+1. \end{aligned} \tag{ii}$$

It implies that $\gamma = 0$. The second equation in (ii) can be written as

$$\beta_{p-k}(k)\alpha(p) = \epsilon(\mu_{p-k}(p)-(2p-k))\,\alpha(k)$$

for $p \geq k+1$. The comparison with (i) gives

$$\mu_k(p) = \frac{k+p}{2} \quad \text{for } p\geq k+1.$$

Replacing in $\beta_k(p)\,\alpha(k+p) = \epsilon(\mu_k(k+p)-(2k+p))\,\alpha(p)$, we find

$$\beta_k(p) = -\epsilon\frac{(2k+p)\alpha(p)}{2\alpha(k+p)}.$$

Then the condition $\beta_k(p)-\beta_p(k) = \frac{1}{2}(p-k)$ determines $\epsilon = -1$. □

Part II. Stochastics on H \ Diff (S^1) and integration by parts formula

We identify the quotient space $\mathrm{diff}(S^1)/\mathrm{su}(1,1)$ with the subspace V of $\mathrm{diff}(S^1)$ generated by $\{\cos k\theta, \sin k\theta\}_{k\geq 2}$. The following is valid when Γ is the Levi-Civita connection. Let $(\epsilon_j)_{j\geq 2}$ be the orthonormal basis of V defined by (22). Let $(x_j(t))_{j\geq 0}$ be independent Brownian motions. We put

$$x(t) = \sum_j \epsilon_j x_j(t). \tag{80}$$

We denote '$*d$' the Stratonovitch differential and 'o' the composition of maps. On $\mathcal{H} \setminus \mathrm{Diff}(S^1)$, let $\gamma(t)$ be the solution of

$$d\gamma(t) = \sum_{j \geq 2} (\epsilon_j \, o \, \gamma(t)) * dx_j(t) \quad \text{with } \gamma(0) = \mathrm{Id}. \tag{81}$$

We denote

$$*dx(t) \, o \, \gamma(t) = \sum_j (\epsilon_j \, o \, \gamma(t)) * dx_j(t). \tag{82}$$

For fixed j, then $\epsilon_j x_j(t)$ is a process in $\mathrm{diff}(S^1)$. More generally, consider random vectors $(y_j(t))_j$ with $y_j(t) \in \mathrm{diff}(S^1)$ for any j. Let

$$y(t) = \sum_{j \geq 2} y_j(t). \tag{83}$$

We put

$$*dy(t) \, o \, \gamma(t) = \sum_j *dy_j(t) \, o \, \gamma(t). \tag{84}$$

For the Levi-Civita connection Γ, for any h in V, consider the parallel transport given by (see [1])

$$d\Omega(t)h = \Gamma(*\, dx(t)) \, \Omega(t) \, h = \sum_j \Gamma(\epsilon_j) \, \Omega(t)h \, *\, dx_j(t). \tag{85}$$

With the notations of (14)–(17), we consider the process

$$\gamma_\mu(t) = \exp(\mu \Omega(t)h) \, o \, \gamma(t). \tag{86}$$

Theorem (Integration by parts). *Let $\Omega(\bullet)h$ be a solution of* (85). *For $F : X \to R$, we define $D_h^\Gamma F(\gamma(\bullet))$ as in* (18), *then* (19) *holds.*

Proof. It is an adaptation of Fang's proof [11]. Since the adaptation is not straightforward, we give the details and we divide the proof into three steps.

Step 1. Consider the process $\gamma_\mu(t) = \exp(\mu \Omega(t)h) \, o \, \gamma(t)$ as in (86). We construct a *tangent process* $\widetilde{\gamma}_\mu(t)$ such that (i), (ii) and (iii) are fulfilled,

$$\begin{aligned}
&\text{(i) } \widetilde{\gamma}_0(t) = \gamma(t), \\
&\text{(ii) } d\widetilde{\gamma}_\mu(t) = (dy^\mu(t) + \mu \, z(t) \, dt) \, o \, \widetilde{\gamma}_\mu(t) \quad \text{with } \widetilde{\gamma}_\mu(0) = \mathrm{Id}, \\
&\text{(iii) } \left.\frac{d}{d\mu}\right|_{\mu=0} \widetilde{\gamma}_\mu = \left.\frac{d}{d\mu}\right|_{\mu=0} \gamma_\mu,
\end{aligned} \tag{87}$$

where $y^\mu(t) = \int_0^t \exp(\mu\Gamma(\Omega(s)h))\,dx(s)$. Since the operator $\Gamma(\omega(s)h)$ is antisymmetric from V to V, then $y^\mu(t)$ is a Brownian motion on $\mathrm{diff}(S^1)$.

Construction of the tangent process $\widetilde{\gamma}_\mu$ and the expressions of $z(s)$ and $dy^\mu(s)$ in (ii).

Proposition 1. *Let*

$$M(t) = \left(\frac{d}{d\mu}\gamma_\mu\right)\Big|_{\mu=0}; \tag{88}$$

then $M(t)$ satisfies

$$dM(t) = {*dy(t)}\,o\,\gamma(t) + {*dx(t)}\,o\,M(t) \tag{89}$$

and the differential of $y(t)$ in Itô's form is

$$dy(t) = \Gamma(\Omega(t)h)\,dx(t) + \Psi\,\Omega(t)h\,dt. \tag{90}$$

Proof. Notice that $dx(t) \in \mathrm{diff}(S^1)$ and $\Gamma(\Omega(t)h)$ is an antisymmetric operator on $\mathrm{diff}(S^1)$. Taking the stochastic derivative of $\gamma_\mu(t)$ in (17), we obtain

$$\begin{aligned} d\gamma_\mu(t) &= *d(\exp(\mu\Omega(t)h))\,o\,\exp(-\mu\Omega(t)h)\,o\,\gamma_\mu(t) \\ &\quad + (\exp(\mu\Omega(t)h))'\,o\,*\,d\gamma(t) \\ &= \big(\mu * d(\Omega(t)h) + \text{ terms in } \mu^j \text{ with } j \geq 2\big)\,o\,\gamma_\mu \\ &\quad + (\exp(\mu\Omega(t)h))'\,o\,*\,d\gamma(t). \end{aligned} \tag{91}$$

On the other hand, because of (81),

$$\begin{aligned} &(\exp(\mu\Omega(t)h))'\,o\,*\,d\gamma(t) = \sum_j (\exp(\mu\Omega(t)h))'\,o\,\epsilon_j\,o\,\gamma(t)\ *\,dx_j(t) \\ &= \sum_j (I + \mu(\Omega(t)h)')\,o\,\epsilon_j\,o\,\exp(-\mu\Omega(t)h)\,o\,\gamma_\mu(t)\ *\,dx_j(t) + \text{terms in } \mu^j, \\ &\hspace{22em} j \geq 2. \end{aligned} \tag{92}$$

We differentiate this last identity with respect to μ and take $\mu = 0$, it gives

$$\begin{aligned} &\frac{d}{d\mu}\Big|_{\mu=0} \big[(\exp(\mu\Omega(t)h))'\,o\,*\,d\gamma(t)\big] \\ &\quad = \sum_j (\Omega(t)h))'\,o\,\epsilon_j - \epsilon_j'\,o\,\Omega(t)h)\,o\,\gamma(t)\ *\,dx_j(t) + \epsilon_j\,o\,M(t)\ *\,dx_j(t) \\ &\quad = \sum_j \big[\Omega(t)h, \epsilon_j\big]\,o\,\gamma(t)\ *\,dx_j(t) + \epsilon_j\,o\,M(t)\ *\,dx_j(t), \end{aligned} \tag{93}$$

The bracket $[\Omega(t)h, \epsilon_j]$ is given by $[\Omega(t)h, \epsilon_j] = (\Omega(t)h))' \, o \, \epsilon_j - \epsilon_j' \, o \, \Omega(t)h$. Notice that in our case the Lie bracket is different from the one in [11]. We deduce that

$$dM(t) = \sum_j \big(\Gamma(\epsilon_j)\Omega(t)h + [\Omega(t)h, \epsilon_j]\big) \, o \, \gamma(t) * dx_j(t) + \sum_j \epsilon_j \, oM(t) * dx_j(t). \tag{94}$$

We put

$$\begin{aligned} dy_j(t) &= \big(\Gamma(\epsilon_j)\Omega(t)h + [\Omega(t)h, \epsilon_j]\big) * dx_j(t) \\ &= \Gamma(\Omega(t)h)\epsilon_j * dx_j(t), \end{aligned} \tag{95}$$

where the last equality is a consequence of

$$\Gamma(\epsilon_j)\Omega(t)h + [\Omega(t)h, \epsilon_j] = \Gamma(\Omega(t)h)\epsilon_j. \tag{96}$$

Compare with (49). The equation (95) is a Stratonovitch equation. Let

$$y(t) = \sum_j y_j(t). \tag{97}$$

We have in Stratonovitch differential

$$dy(t) = \sum_j \Gamma(\Omega(t)h)\epsilon_j * dx_j(t) \tag{98}$$

and we have

$$dM(t) = *dy(t) \, o \, \gamma(t) + *dx(t) \, o \, M(t). \tag{99}$$

From (95), we calculate Itô's stochastic differential of $y(t)$. The Itô contractions are given by

$$\begin{aligned} I_j dt &= \big\langle d(\Gamma(\epsilon_j)\Omega(t)h + [\Omega(t)h, \epsilon_j]), dx_j(t)\big\rangle \\ &= [\Gamma(\epsilon_j)\Gamma(\epsilon_j)\Omega(t)h + \phi(\epsilon_j)\Gamma(\epsilon_j)\Omega(t)h) \, dt \end{aligned} \tag{100}$$

and

$$\sum_j I_j \, dt = \Psi\Omega(t)h \, dt, \tag{101}$$

where Ψ and ϕ are given by (11)–(12). We see that Itô's differential of $y(t)$ is given by (90). This proves Proposition 1. □

Definition. For $t \geq 0$, let Q_t^μ be the solution of

$$dQ_t^\mu = \Gamma(\Omega(t)h)Q_t^\mu \, d\mu \quad \text{with } Q_t^0 = \mathrm{Id}_{\mathrm{diff}(S^1)}. \tag{102}$$

We have

$$\frac{d}{d\mu}\bigg|_{\mu=0} Q_t^\mu = \Gamma(\Omega(t)h). \tag{103}$$

Since $\Gamma(\Omega(t)h)$ is antisymmetric on $\mathrm{diff}(S^1)$, then Q_t^μ is orthogonal with the scalar product. We define

$$y^\mu(t) = \int_0^t Q_s^\mu \, dx(s); \tag{104}$$

then $y^0(t) = x(t)$ and $y^\mu(t)$ is a Brownian motion on $\mathrm{diff}(S^1)$ from the orthogonality of Q_s: the Itô contraction

$$\langle dy^\mu(t), dy^\mu(t)\rangle = \sum_j \|Q_t\epsilon_j\|^2 dt = dt. \tag{105}$$

Proposition 2. *Let* $\widetilde{\gamma}_\mu$ *be the solution of the Stratonovitch equation*

$$d\widetilde{\gamma}_\mu = \big(*dy^\mu(t) + \mu\Psi\Omega(t)h\,dt\big)\, o\, \widetilde{\gamma}_\mu \tag{106}$$

and let

$$\widetilde{M} = \frac{d}{d\mu}\bigg|_{\mu=0} \widetilde{\gamma}_\mu\,; \tag{107}$$

we have $\widetilde{\gamma}_0(t) = \gamma(t)$ *since* $y^0(t) = x(t)$. *Moreover,* $\widetilde{M} = M$.

Proof.

$$d\widetilde{M} = \left(d\left(\frac{d}{d\mu}\bigg|_{\mu=0} y_\mu\right) + \Psi\Omega(t)h\,dt\right) o\, \gamma(t) + (*dx(t))\, o\, \widetilde{M}. \tag{108}$$

We have to verify that

$$dy(t) = d\left(\frac{d}{d\mu}\bigg|_{\mu=0} y_\mu\right) + \Psi\Omega(t)h\,dt \tag{109}$$

or equivalently

$$d\left(\frac{d}{d\mu}\bigg|_{\mu=0} y_\mu\right) = \Gamma(\Omega(t)h)\, dx(t). \tag{110}$$

To verify (110), we differentiate $dy^\mu(t) = Q_t^\mu\, dx(t)$ with respect to μ. It gives

$$d\left(\frac{d}{d\mu}\bigg|_{\mu=0} y_\mu\right) = \left(\frac{d}{d\mu}\bigg|_{\mu=0} Q_t^\mu\right) dx(t).$$

Since $M(t)$ and $\widetilde{M}(t)$ satisfy the same stochastic differential equation, we conclude that $\widetilde{M} = M$. □

Step 2. The Girsanov formula is written for the tangent process $\widetilde{\gamma}_\mu$. Let

$$T_\mu : \gamma \rightarrow \widetilde{\gamma}_\mu$$

and the *density* K_μ

$$K_\mu(\gamma)(t) = \exp\left(-\mu \int_0^t (z_\gamma(s) \mid dy_\mu(t)) - \mu^2 \int_0^t |z_\gamma(s)|^2_{H^{\frac{3}{2}}}\, ds\right)$$

with $z_\gamma(s) = \Psi\Omega(s)h$ and where $\Omega(s)$ is the parallel transport along γ. From Girsanov theorem, for $F : X \rightarrow R$,

$$E[(F o T_\mu) \times K_\mu] = E[F].$$

Step 3. We differentiate with respect to μ the previous formula,

$$E[D^\Gamma_h F] = -E\left[F \frac{d}{d\mu}\bigg|_{\mu=0} K_\mu\right].$$

Since $\frac{d}{d\mu}\big|_{\mu=0} K_\mu = -\int_0^t (z_\gamma(s) \mid dx(s))$, we obtain the integration by parts formula. □

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