

# Contents

<i>Preface</i>	vii
<i>Glossary of Frequently Used Symbols</i>	xi
1. Introduction	1
1.1 Manifolds and Related Geometrical Structures . . . . .	1
1.1.1 Geometrical Atlas . . . . .	6
1.1.2 Topological Manifolds . . . . .	8
1.1.2.1 Topological manifolds without boundary . . . . .	10
1.1.2.2 Topological manifolds with boundary . . . . .	10
1.1.2.3 Properties of topological manifolds . . . . .	10
1.1.3 Differentiable Manifolds . . . . .	12
1.1.4 Tangent and Cotangent Bundles of Manifolds . . . . .	14
1.1.4.1 Tangent Bundle of a Smooth Manifold . . . . .	14
1.1.4.2 Cotangent Bundle of a Smooth Manifold . . . . .	15
1.1.4.3 Fibre-, Tensor-, and Jet-Bundles . . . . .	15
1.1.5 Riemannian Manifolds: Configuration Spaces for Lagrangian Mechanics . . . . .	16
1.1.5.1 Riemann Surfaces . . . . .	17
1.1.5.2 Riemannian Geometry . . . . .	19
1.1.5.3 APPLICATION: Lagrangian Mechanics . . . . .	22
1.1.5.4 Finsler manifolds . . . . .	25
1.1.6 Symplectic Manifolds: Phase-Spaces for Hamiltonian Mechanics . . . . .	25

1.1.7	Lie Groups . . . . .	28
1.1.7.1	APPLICATION: Physical Examples of Lie Groups . . . . .	30
1.1.8	APPLICATION: A Bird View on Modern Physics . . . . .	31
1.1.8.1	Three Pillars of 20th Century Physics . . . . .	31
1.1.8.2	String Theory in ‘Plain English’ . . . . .	33
1.2	APPLICATION: Paradigm of Differential–Geometric Modelling of Dynamical Systems . . . . .	46
2.	Technical Preliminaries: Tensors, Actions and Functors . . . . .	51
2.1	Tensors: Local Machinery of Differential Geometry . . . . .	51
2.1.1	Transformation of Coordinates and Elementary Tensors . . . . .	51
2.1.1.1	Transformation of Coordinates . . . . .	52
2.1.1.2	Scalar Invariants . . . . .	53
2.1.1.3	Vectors and Covectors . . . . .	53
2.1.1.4	Second–Order Tensors . . . . .	54
2.1.1.5	Higher–Order Tensors . . . . .	56
2.1.1.6	Tensor Symmetry . . . . .	57
2.1.2	Euclidean Tensors . . . . .	58
2.1.2.1	Basis Vectors and the Metric Tensor in $\mathbb{R}^n$ . . . . .	58
2.1.2.2	Tensor Products in $\mathbb{R}^n$ . . . . .	59
2.1.3	Covariant Differentiation . . . . .	60
2.1.3.1	Christoffel’s Symbols . . . . .	60
2.1.3.2	Geodesics . . . . .	61
2.1.3.3	Covariant Derivative . . . . .	61
2.1.3.4	Covariant Form of Differential Operators . . . . .	62
2.1.3.5	Absolute Derivative . . . . .	63
2.1.3.6	3D Curve Geometry: Frenet–Serret Formulae . . . . .	64
2.1.3.7	Mechanical Acceleration and Force . . . . .	64
2.1.4	APPLICATION: Covariant Mechanics . . . . .	65
2.1.4.1	Riemannian Curvature Tensor . . . . .	70
2.1.4.2	Exterior Differential Forms . . . . .	71
2.1.4.3	The Covariant Force Law . . . . .	76
2.1.5	APPLICATION: Nonlinear Fluid Dynamics . . . . .	78
2.1.5.1	Continuity Equation . . . . .	78

2.1.5.2	Forces Acting on a Fluid . . . . .	80
2.1.5.3	Constitutive and Dynamical Equations . . . . .	81
2.1.5.4	Navier–Stokes Equations . . . . .	82
2.2	Actions: The Core Machinery of Modern Physics . . . . .	83
2.3	Functors: Global Machinery of Modern Mathematics . . . . .	87
2.3.1	Maps . . . . .	88
2.3.1.1	Notes from Set Theory . . . . .	88
2.3.1.2	Notes From Calculus . . . . .	89
2.3.1.3	Maps . . . . .	89
2.3.1.4	Algebra of Maps . . . . .	89
2.3.1.5	Compositions of Maps . . . . .	90
2.3.1.6	The Chain Rule . . . . .	90
2.3.1.7	Integration and Change of Variables . . . . .	90
2.3.1.8	Notes from General Topology . . . . .	91
2.3.1.9	Topological Space . . . . .	92
2.3.1.10	Homotopy . . . . .	93
2.3.1.11	Commutative Diagrams . . . . .	95
2.3.1.12	Groups and Related Algebraic Structures . . . . .	97
2.3.2	Categories . . . . .	102
2.3.3	Functors . . . . .	105
2.3.4	Natural Transformations . . . . .	108
2.3.4.1	Compositions of Natural Transformations . . . . .	109
2.3.4.2	Dinatural Transformations . . . . .	109
2.3.5	Limits and Colimits . . . . .	111
2.3.6	Adjunction . . . . .	111
2.3.7	Abelian Categorical Algebra . . . . .	113
2.3.8	$n$ –Categories . . . . .	116
2.3.8.1	Generalization of ‘Small’ Categories . . . . .	117
2.3.8.2	Topological Structure of $n$ –Categories . . . . .	121
2.3.8.3	Homotopy Theory and Related $n$ –Categories . . . . .	121
2.3.8.4	Categorification . . . . .	123
2.3.9	APPLICATION: $n$ –Categorical Framework for Higher Gauge Fields . . . . .	124
2.3.10	APPLICATION: Natural Geometrical Structures . . . . .	128
2.3.11	Ultimate Conceptual Machines: Weak $n$ –Categories . . . . .	132

3.	Applied Manifold Geometry	137
3.1	Introduction	137
3.1.1	Intuition behind Einstein's Geometrodynamics	138
3.1.2	Einstein's Geometrodynamics in Brief	142
3.2	Intuition Behind the Manifold Concept	143
3.3	Definition of a Differentiable Manifold	145
3.4	Smooth Maps between Smooth Manifolds	147
3.4.1	Intuition behind Topological Invariants of Manifolds	148
3.5	(Co)Tangent Bundles of Smooth Manifolds	150
3.5.1	Tangent Bundle and Lagrangian Dynamics	150
3.5.1.1	Intuition behind a Tangent Bundle	150
3.5.1.2	Definition of a Tangent Bundle	150
3.5.2	Cotangent Bundle and Hamiltonian Dynamics	153
3.5.2.1	Definition of a Cotangent Bundle	153
3.5.3	APPLICATION: Command/Control in Human-Robot Interactions	154
3.5.4	APPLICATION: Generalized Bidirectional Associative Memory	157
3.6	Tensor Fields on Smooth Manifolds	163
3.6.1	Tensor Bundle	163
3.6.1.1	Pull-Back and Push-Forward	164
3.6.1.2	Dynamical Evolution and Flow	165
3.6.1.3	Vector-Fields and Their Flows	167
3.6.1.4	Vector-Fields on $M$	167
3.6.1.5	Integral Curves as Dynamical Trajectories	168
3.6.1.6	Dynamical Flows on $M$	172
3.6.1.7	Categories of ODEs	173
3.6.2	Differential Forms on Smooth Manifolds	174
3.6.2.1	1-Forms on $M$	174
3.6.2.2	$k$ -Forms on $M$	176
3.6.2.3	Exterior Differential Systems	179
3.6.3	Exterior Derivative and (Co)Homology	180
3.6.3.1	Intuition behind Cohomology	182
3.6.3.2	Intuition behind Homology	183
3.6.3.3	De Rham Complex and Homotopy Operators	185

3.6.3.4	Stokes Theorem and de Rham Cohomology . . . . .	186
3.6.3.5	Euler–Poincaré Characteristics of $M$ . . .	188
3.6.3.6	Duality of Chains and Forms on $M$ . . .	188
3.6.3.7	Hodge Star Operator and Harmonic Forms . . . . .	190
3.7	Lie Derivatives on Smooth Manifolds . . . . .	192
3.7.1	Lie Derivative Operating on Functions . . . . .	192
3.7.2	Lie Derivative of Vector Fields . . . . .	194
3.7.3	Time Derivative of the Evolution Operator . . . . .	197
3.7.4	Lie Derivative of Differential Forms . . . . .	197
3.7.5	Lie Derivative of Various Tensor Fields . . . . .	198
3.7.6	APPLICATION: Lie–Derivative Neurodynamics . . .	200
3.7.7	Lie Algebras . . . . .	202
3.8	Lie Groups and Associated Lie Algebras . . . . .	202
3.8.1	Definition of a Lie Group . . . . .	203
3.8.2	Actions of Lie Groups on Smooth Manifolds . . . . .	207
3.8.3	Basic Dynamical Lie Groups . . . . .	210
3.8.3.1	Galilei Group . . . . .	210
3.8.3.2	General Linear Group . . . . .	211
3.8.4	APPLICATION: Lie Groups in Biodynamics . . . . .	212
3.8.4.1	Lie Groups of Joint Rotations . . . . .	212
3.8.4.2	Euclidean Groups of Total Joint Motions . . . . .	216
3.8.4.3	Group Structure of Biodynamical Manifold . . . . .	222
3.8.5	APPLICATION: Dynamical Games on $SE(n)$ –Groups . . . . .	227
3.8.5.1	Configuration Models for Planar Vehicles . . . . .	227
3.8.5.2	Two–Vehicles Conflict Resolution Manoeuvres . . . . .	228
3.8.5.3	Symplectic Reduction and Dynamical Games on $SE(2)$ . . . . .	230
3.8.5.4	Nash Solutions for Multi–Vehicle Manoeuvres . . . . .	233
3.8.6	Classical Lie Theory . . . . .	235
3.8.6.1	Basic Tables of Lie Groups and their Lie Algebras . . . . .	236

	3.8.6.2	Representations of Lie groups . . . . .	239
	3.8.6.3	Root Systems and Dynkin Diagrams . . . . .	240
	3.8.6.4	Simple and Semisimple Lie Groups and Algebras . . . . .	245
3.9		Lie Symmetries and Prolongations on Manifolds . . . . .	247
	3.9.1	Lie Symmetry Groups . . . . .	247
	3.9.1.1	Exponentiation of Vector Fields on $M$ . . . . .	247
	3.9.1.2	Lie Symmetry Groups and General DEs . . . . .	249
	3.9.2	Prolongations . . . . .	250
	3.9.2.1	Prolongations of Functions . . . . .	250
	3.9.2.2	Prolongations of Differential Equations . . . . .	251
	3.9.2.3	Prolongations of Group Actions . . . . .	252
	3.9.2.4	Prolongations of Vector Fields . . . . .	253
	3.9.2.5	General Prolongation Formula . . . . .	254
	3.9.3	Generalized Lie Symmetries . . . . .	256
	3.9.3.1	Noether Symmetries . . . . .	257
	3.9.4	APPLICATION: Biophysical PDEs . . . . .	261
	3.9.4.1	The Heat Equation . . . . .	261
	3.9.4.2	The Kortevég–De Vries Equation . . . . .	262
	3.9.5	Lie–Invariant Geometric Objects . . . . .	262
	3.9.5.1	Robot Kinematics . . . . .	262
	3.9.5.2	Maurer–Cartan 1–Forms . . . . .	264
	3.9.5.3	General Structure of Integrable One–Forms . . . . .	265
	3.9.5.4	Lax Integrable Dynamical Systems . . . . .	267
	3.9.5.5	APPLICATION: Burgers Dynamical System . . . . .	268
3.10		Riemannian Manifolds and Their Applications . . . . .	271
	3.10.1	Local Riemannian Geometry . . . . .	271
	3.10.1.1	Riemannian Metric on $M$ . . . . .	272
	3.10.1.2	Geodesics on $M$ . . . . .	277
	3.10.1.3	Riemannian Curvature on $M$ . . . . .	278
	3.10.2	Global Riemannian Geometry . . . . .	281
	3.10.2.1	The Second Variation Formula . . . . .	281
	3.10.2.2	Gauss–Bonnet Formula . . . . .	284
	3.10.2.3	Ricci Flow on $M$ . . . . .	285
	3.10.2.4	Structure Equations on $M$ . . . . .	287
	3.10.3	APPLICATION: Autonomous Lagrangian Dynamics . . . . .	289

3.10.3.1	Basis of Lagrangian Dynamics . . . . .	289
3.10.3.2	Lagrange–Poincaré Dynamics . . . . .	291
3.10.4	Core Application: Search for Quantum Gravity . . . . .	292
3.10.4.1	What is Quantum Gravity? . . . . .	292
3.10.4.2	Main Approaches to Quantum Gravity . . . . .	293
3.10.4.3	Traditional Approaches to Quantum Gravity . . . . .	300
3.10.4.4	New Approaches to Quantum Gravity . . . . .	304
3.10.4.5	Black Hole Entropy . . . . .	310
3.10.5	Basics of Morse and (Co)Bordism Theories . . . . .	311
3.10.5.1	Morse Theory on Smooth Manifolds . . . . .	311
3.10.5.2	(Co)Bordism Theory on Smooth Manifolds . . . . .	314
3.11	Finsler Manifolds and Their Applications . . . . .	316
3.11.1	Definition of a Finsler Manifold . . . . .	316
3.11.2	Energy Functional, Variations and Extrema . . . . .	317
3.11.3	APPLICATION: Finsler–Lagrangian Field Theory . . . . .	321
3.11.4	Riemann–Finsler Approach to Information Geometry . . . . .	323
3.11.4.1	Model Specification and Parameter Estimation . . . . .	323
3.11.4.2	Model Evaluation and Testing . . . . .	324
3.11.4.3	Quantitative Criteria . . . . .	324
3.11.4.4	Selection Among Different Models . . . . .	327
3.11.4.5	Riemannian Geometry of Minimum Description Length . . . . .	330
3.11.4.6	Finsler Approach to Information Geometry . . . . .	333
3.12	Symplectic Manifolds and Their Applications . . . . .	335
3.12.1	Symplectic Algebra . . . . .	335
3.12.2	Symplectic Geometry . . . . .	336
3.12.3	APPLICATION: Autonomous Hamiltonian Mechanics . . . . .	338
3.12.3.1	Basics of Hamiltonian Mechanics . . . . .	338
3.12.3.2	Library of Basic Hamiltonian Systems . . . . .	351
3.12.3.3	Hamilton–Poisson Mechanics . . . . .	361
3.12.3.4	Completely Integrable Hamiltonian Systems . . . . .	363

3.12.3.5	Momentum Map and Symplectic Reduction . . . . .	372
3.12.4	Multisymplectic Geometry . . . . .	374
3.13	APPLICATION: Biodynamics–Robotics . . . . .	375
3.13.1	Muscle–Driven Hamiltonian Biodynamics . . . . .	376
3.13.2	Hamiltonian–Poisson Biodynamical Systems . . . . .	379
3.13.3	Lie–Poisson Neurodynamics Classifier . . . . .	383
3.13.4	Biodynamical Functors . . . . .	384
3.13.4.1	The Covariant Force Functor . . . . .	384
3.13.4.2	Lie–Lagrangian Biodynamical Functor . . . . .	385
3.13.4.3	Lie–Hamiltonian Biodynamical Functor . . . . .	391
3.13.5	Biodynamical Topology . . . . .	401
3.13.5.1	(Co)Chain Complexes in Biodynamics . . . . .	401
3.13.5.2	Morse Theory in Biodynamics . . . . .	405
3.13.5.3	Hodge–De Rham Theory in Biodynamics . . . . .	415
3.13.5.4	Lagrangian–Hamiltonian Duality in Biodynamics . . . . .	419
3.14	Complex and Kähler Manifolds and Their Applications . . . . .	428
3.14.1	Complex Metrics: Hermitian and Kähler . . . . .	431
3.14.2	Calabi–Yau Manifolds . . . . .	436
3.14.3	Special Lagrangian Submanifolds . . . . .	437
3.14.4	Dolbeault Cohomology and Hodge Numbers . . . . .	438
3.15	Conformal Killing–Riemannian Geometry . . . . .	441
3.15.1	Conformal Killing Vector–Fields and Forms on $M$ . . . . .	442
3.15.2	Conformal Killing Tensors and Laplacian Symmetry . . . . .	443
3.15.3	APPLICATION: Killing Vector and Tensor Fields in Mechanics . . . . .	445
3.16	APPLICATION: Lax–Pair Tensors in Gravitation . . . . .	448
3.16.1	Lax–Pair Tensors . . . . .	450
3.16.2	Geometrization of the 3–Particle Open Toda Lattice . . . . .	452
3.16.2.1	Tensorial Lax Representation . . . . .	453
3.16.3	4D Generalizations . . . . .	456
3.16.3.1	Case I . . . . .	456
3.16.3.2	Case II . . . . .	457
3.16.3.3	Energy–Momentum Tensors . . . . .	457

3.17 Applied Unorthodox Geometries . . . . .	458
3.17.1 Noncommutative Geometry . . . . .	458
3.17.1.1 Moyal Product and Noncommutative Algebra . . . . .	458
3.17.1.2 Noncommutative Space–Time Manifolds . . . . .	459
3.17.1.3 Symmetries and Diffeomorphisms on Deformed Spaces . . . . .	462
3.17.1.4 Deformed Diffeomorphisms . . . . .	465
3.17.1.5 Noncommutative Space–Time Geometry . . . . .	467
3.17.1.6 Star–Products and Expanded Einstein–Hilbert Action . . . . .	470
3.17.2 Synthetic Differential Geometry . . . . .	473
3.17.2.1 Distributions . . . . .	474
3.17.2.2 Synthetic Calculus in Euclidean Spaces . . . . .	476
3.17.2.3 Spheres and Balls as Distributions . . . . .	478
3.17.2.4 Stokes Theorem for Unit Sphere . . . . .	480
3.17.2.5 Time Derivatives of Expanding Spheres . . . . .	481
3.17.2.6 The Wave Equation . . . . .	482
4. Applied Bundle Geometry . . . . .	485
4.1 Intuition Behind a Fibre Bundle . . . . .	485
4.2 Definition of a Fibre Bundle . . . . .	486
4.3 Vector and Affine Bundles . . . . .	491
4.3.1 The Second Vector Bundle of the Manifold $M$ . . . . .	495
4.3.2 The Natural Vector Bundle . . . . .	496
4.3.3 Vertical Tangent and Cotangent Bundles . . . . .	498
4.3.3.1 Tangent and Cotangent Bundles Revisited . . . . .	498
4.3.4 Affine Bundles . . . . .	500
4.4 APPLICATION: Semi–Riemannian Geometrical Mechanics . . . . .	501
4.4.1 Vector–Fields and Connections . . . . .	501
4.4.2 Hamiltonian Structures on the Tangent Bundle . . . . .	503
4.5 $K$ –Theory and Its Applications . . . . .	508
4.5.1 Topological $K$ –Theory . . . . .	508
4.5.1.1 Bott Periodicity Theorem . . . . .	509
4.5.2 Algebraic $K$ –Theory . . . . .	510
4.5.3 Chern Classes and Chern Character . . . . .	511
4.5.4 Atiyah’s View on $K$ –Theory . . . . .	515
4.5.5 Atiyah–Singer Index Theorem . . . . .	518

4.5.6	The Infinite-Order Case . . . . .	520
4.5.7	Twisted $K$ -Theory and the Verlinde Algebra . . . . .	523
4.5.8	APPLICATION: $K$ -Theory in String Theory . . . . .	526
4.5.8.1	Classification of Ramond-Ramond Fluxes . . . . .	526
4.5.8.2	Classification of $D$ -Branes . . . . .	528
4.6	Principal Bundles . . . . .	529
4.7	Distributions and Foliations on Manifolds . . . . .	533
4.8	APPLICATION: Nonholonomic Mechanics . . . . .	534
4.9	APPLICATION: Geometrical Nonlinear Control . . . . .	537
4.9.1	Introduction to Geometrical Nonlinear Control . . . . .	537
4.9.2	Feedback Linearization . . . . .	539
4.9.3	Nonlinear Controllability . . . . .	547
4.9.4	Geometrical Control of Mechanical Systems . . . . .	554
4.9.4.1	Abstract Control System . . . . .	554
4.9.4.2	Global Controllability of Linear Control Systems . . . . .	555
4.9.4.3	Local Controllability of Affine Control Systems . . . . .	555
4.9.4.4	Lagrangian Control Systems . . . . .	556
4.9.4.5	Lie-Adaptive Control . . . . .	566
4.9.5	Hamiltonian Optimal Control and Maximum Principle . . . . .	567
4.9.5.1	Hamiltonian Control Systems . . . . .	567
4.9.5.2	Pontryagin's Maximum Principle . . . . .	570
4.9.5.3	Affine Control Systems . . . . .	571
4.9.6	Brain-Like Control Functor in Biodynamics . . . . .	573
4.9.6.1	Functor Control Machine . . . . .	574
4.9.6.2	Spinal Control Level . . . . .	576
4.9.6.3	Cerebellar Control Level . . . . .	581
4.9.6.4	Cortical Control Level . . . . .	584
4.9.6.5	Open Liouville Neurodynamics and Biodynamical Self-Similarity . . . . .	587
4.9.7	Brain-Mind Functorial Machines . . . . .	594
4.9.7.1	Neurodynamical 2-Functor . . . . .	594
4.9.7.2	Solitary 'Thought Nets' and 'Emerging Mind' . . . . .	597
4.9.8	Geometroynamics of Human Crowd . . . . .	602
4.9.8.1	Crowd Hypothesis . . . . .	603

4.9.8.2	Geometrodynamics of Individual Agents . . . . .	603
4.9.8.3	Collective Crowd Geometrodynamics . . . . .	605
4.10	Multivector–Fields and Tangent–Valued Forms . . . . .	606
4.11	APPLICATION: Geometrical Quantization . . . . .	614
4.11.1	Quantization of Hamiltonian Mechanics . . . . .	614
4.11.2	Quantization of Relativistic Hamiltonian Mechanics . . . . .	617
4.12	Symplectic Structures on Fiber Bundles . . . . .	624
4.12.1	Hamiltonian Bundles . . . . .	625
4.12.1.1	Characterizing Hamiltonian Bundles . . . . .	625
4.12.1.2	Hamiltonian Structures . . . . .	626
4.12.1.3	Marked Hamiltonian Structures . . . . .	630
4.12.1.4	Stability . . . . .	632
4.12.1.5	Cohomological Splitting . . . . .	632
4.12.1.6	Homological Action of $Ham(M)$ on $M$ . . . . .	634
4.12.1.7	General Symplectic Bundles . . . . .	636
4.12.1.8	Existence of Hamiltonian Structures . . . . .	637
4.12.1.9	Classification of Hamiltonian Structures . . . . .	642
4.12.2	Properties of General Hamiltonian Bundles . . . . .	645
4.12.2.1	Stability . . . . .	645
4.12.2.2	Functorial Properties . . . . .	648
4.12.2.3	Splitting of Rational Cohomology . . . . .	650
4.12.2.4	Hamiltonian Bundles and Gromov–Witten Invariants . . . . .	654
4.12.2.5	Homotopy Reasons for Splitting . . . . .	659
4.12.2.6	Action of the Homology of $(M)$ on $H_*(M)$ . . . . .	661
4.12.2.7	Cohomology of General Symplectic Bundles . . . . .	664
4.13	Clifford Algebras, Spinors and Penrose Twistors . . . . .	666
4.13.1	Clifford Algebras and Modules . . . . .	666
4.13.1.1	The Exterior Algebra . . . . .	669
4.13.1.2	The Spin Group . . . . .	672
4.13.1.3	4D Case . . . . .	672
4.13.2	Spinors . . . . .	675
4.13.2.1	Basic Properties . . . . .	675
4.13.2.2	4D Case . . . . .	677
4.13.2.3	(Anti) Self Duality . . . . .	681

4.13.2.4	Hermitian Structure on the Spinors . . .	686
4.13.2.5	Symplectic Structure on the Spinors . . .	689
4.13.3	Penrose Twistor Calculus . . . . .	691
4.13.3.1	Penrose Index Formalism . . . . .	691
4.13.3.2	Twistor Calculus . . . . .	698
4.13.4	APPLICATION: Rovelli's Loop Quantum Gravity .	701
4.13.4.1	Introduction to Loop Quantum Gravity	701
4.13.4.2	Formalism of Loop Quantum Gravity .	708
4.13.4.3	Loop Algebra . . . . .	709
4.13.4.4	Loop Quantum Gravity . . . . .	711
4.13.4.5	Loop States and Spin Network States .	712
4.13.4.6	Diagrammatic Representation of the States . . . . .	715
4.13.4.7	Quantum Operators . . . . .	716
4.13.4.8	Loop v.s. Connection Representation .	717
4.14	APPLICATION: Seiberg–Witten Monopole Field Theory .	718
4.14.1	SUSY Formalism . . . . .	721
4.14.1.1	$N = 2$ Supersymmetry . . . . .	721
4.14.1.2	$N = 2$ Super–Action . . . . .	721
4.14.1.3	Spontaneous Symmetry–Breaking . . .	723
4.14.1.4	Holomorphy and Duality . . . . .	724
4.14.1.5	The SW Prepotential . . . . .	724
4.14.2	Clifford Actions, Dirac Operators and Spinor Bundles . . . . .	725
4.14.2.1	Clifford Algebras and Dirac Operators .	727
4.14.2.2	$Spin$ and $Spin_c$ Structures . . . . .	729
4.14.2.3	Spinor Bundles . . . . .	730
4.14.2.4	The Gauge Group and Its Equations . .	731
4.14.3	Original SW Low Energy Effective Field Action .	732
4.14.4	QED With Matter . . . . .	735
4.14.5	QCD With Matter . . . . .	737
4.14.6	Duality . . . . .	738
4.14.6.1	Witten's Formalism . . . . .	740
4.14.7	Structure of the Moduli Space . . . . .	746
4.14.7.1	Singularity at Infinity . . . . .	746
4.14.7.2	Singularities at Strong Coupling . . . .	747
4.14.7.3	Effects of a Massless Monopole . . . . .	748
4.14.7.4	The Third Singularity . . . . .	749

4.14.7.5	Monopole Condensation and Confinement . . . . .	750
4.14.8	Masses and Periods . . . . .	752
4.14.9	Residues . . . . .	754
4.14.10	SW Monopole Equations and Donaldson Theory . . . . .	757
4.14.10.1	Topological Invariance . . . . .	760
4.14.10.2	Vanishing Theorems . . . . .	762
4.14.10.3	Computation on Kähler Manifolds . . . . .	765
4.14.11	SW Theory and Integrable Systems . . . . .	769
4.14.11.1	$SU(N)$ Elliptic CM System . . . . .	771
4.14.11.2	CM Systems Defined by Lie Algebras . . . . .	772
4.14.11.3	Twisted CM–Systems Defined by Lie Algebras . . . . .	773
4.14.11.4	Scaling Limits of CM–Systems . . . . .	774
4.14.11.5	Lax Pairs for CM–Systems . . . . .	776
4.14.11.6	CM and SW Theory for $SU(N)$ . . . . .	779
4.14.11.7	CM and SW Theory for General Lie Algebra . . . . .	782
4.14.12	SW Theory and WDVV Equations . . . . .	784
4.14.12.1	WDVV Equations . . . . .	784
4.14.12.2	Perturbative SW Prepotentials . . . . .	787
4.14.12.3	Associativity Conditions . . . . .	789
4.14.12.4	SW Theories and Integrable Systems . . . . .	790
4.14.12.5	WDVV Equations in SW Theories . . . . .	793
5.	Applied Jet Geometry . . . . .	797
5.1	Intuition Behind a Jet Space . . . . .	797
5.2	Definition of a 1–Jet Space . . . . .	801
5.3	Connections as Jet Fields . . . . .	806
5.3.1	Principal Connections . . . . .	815
5.4	Definition of a 2–Jet Space . . . . .	818
5.5	Higher–Order Jet Spaces . . . . .	822
5.6	APPLICATION: Jets and Non–Autonomous Dynamics . . . . .	824
5.6.1	Geodesics . . . . .	830
5.6.2	Quadratic Dynamical Equations . . . . .	831
5.6.3	Equation of Free–Motion . . . . .	832
5.6.4	Quadratic Lagrangian and Newtonian Systems . . . . .	833
5.6.5	Jacobi Fields . . . . .	835
5.6.6	Constraints . . . . .	836

5.6.7	Time-Dependent Lagrangian Dynamics . . . . .	841
5.6.8	Time-Dependent Hamiltonian Dynamics . . . . .	843
5.6.9	Time-Dependent Constraints . . . . .	848
5.6.10	Lagrangian Constraints . . . . .	849
5.6.11	Quadratic Degenerate Lagrangian Systems . . . . .	852
5.6.12	Time-Dependent Integrable Hamiltonian Systems . . . . .	855
5.6.13	Time-Dependent Action-Angle Coordinates . . . . .	858
5.6.14	Lyapunov Stability . . . . .	860
5.6.15	First-Order Dynamical Equations . . . . .	861
5.6.16	Lyapunov Tensor and Stability . . . . .	863
	5.6.16.1 Lyapunov Tensor . . . . .	863
	5.6.16.2 Lyapunov Stability . . . . .	864
5.7	APPLICATION: Jets and Multi-Time Rheonomic Dynamics . . . . .	868
	5.7.1 Relativistic Rheonomic Lagrangian Spaces . . . . .	870
	5.7.2 Canonical Nonlinear Connections . . . . .	871
	5.7.3 Cartan's Canonical Connections . . . . .	874
	5.7.4 General Nonlinear Connections . . . . .	876
5.8	Jets and Action Principles . . . . .	877
5.9	APPLICATION: Jets and Lagrangian Field Theory . . . . .	883
	5.9.1 Lagrangian Conservation Laws . . . . .	888
	5.9.2 General Covariance Condition . . . . .	893
5.10	APPLICATION: Jets and Hamiltonian Field Theory . . . . .	897
	5.10.1 Covariant Hamiltonian Field Systems . . . . .	899
	5.10.2 Associated Lagrangian and Hamiltonian Systems . . . . .	902
	5.10.3 Evolution Operator . . . . .	904
	5.10.4 Quadratic Degenerate Systems . . . . .	910
5.11	APPLICATION: Gauge Fields on Principal Connections . . . . .	913
	5.11.1 Connection Strength . . . . .	913
	5.11.2 Associated Bundles . . . . .	914
	5.11.3 Classical Gauge Fields . . . . .	915
	5.11.4 Gauge Transformations . . . . .	917
	5.11.5 Lagrangian Gauge Theory . . . . .	919
	5.11.6 Hamiltonian Gauge Theory . . . . .	920
	5.11.7 Gauge Conservation Laws . . . . .	923
	5.11.8 Topological Gauge Theories . . . . .	924
5.12	APPLICATION: Modern Geometrodynamics . . . . .	928
	5.12.1 Stress-Energy-Momentum Tensors . . . . .	928
	5.12.2 Gauge Systems of Gravity and Fermion Fields . . . . .	955

5.12.3	Hawking–Penrose Quantum Gravity and Black Holes . . . . .	963
6.	Geometrical Path Integrals and Their Applications	983
6.1	Intuition Behind a Path Integral . . . . .	984
6.1.1	Classical Probability Concept . . . . .	984
6.1.2	Discrete Random Variable . . . . .	984
6.1.3	Continuous Random Variable . . . . .	984
6.1.4	General Markov Stochastic Dynamics . . . . .	985
6.1.5	Quantum Probability Concept . . . . .	989
6.1.6	Quantum Coherent States . . . . .	991
6.1.7	Dirac’s $\langle bra   ket \rangle$ Transition Amplitude . . . . .	992
6.1.8	Feynman’s Sum-over-Histories . . . . .	994
6.1.9	The Basic Form of a Path Integral . . . . .	996
6.1.10	APPLICATION: Adaptive Path Integral . . . . .	997
6.2	Path Integral History . . . . .	998
6.2.1	Extract from Feynman’s Nobel Lecture . . . . .	998
6.2.2	Lagrangian Path Integral . . . . .	1002
6.2.3	Hamiltonian Path Integral . . . . .	1003
6.2.4	Feynman–Kac Formula . . . . .	1004
6.2.5	Itô Formula . . . . .	1006
6.3	Standard Path–Integral Quantization . . . . .	1006
6.3.1	Canonical versus Path–Integral Quantization . . . . .	1006
6.3.2	APPLICATION: Particles, Sources, Fields and Gauges . . . . .	1011
6.3.2.1	Particles . . . . .	1011
6.3.2.2	Sources . . . . .	1012
6.3.2.3	Fields . . . . .	1013
6.3.2.4	Gauges . . . . .	1013
6.3.3	Riemannian–Symplectic Geometries . . . . .	1014
6.3.4	Euclidean Stochastic Path Integral . . . . .	1016
6.3.5	APPLICATION: Stochastic Optimal Control . . . . .	1020
6.3.5.1	Path–Integral Formalism . . . . .	1021
6.3.5.2	Monte Carlo Sampling . . . . .	1023
6.3.6	APPLICATION: Nonlinear Dynamics of Option Pricing . . . . .	1025
6.3.6.1	Theory and Simulations of Option Pricing . . . . .	1025
6.3.6.2	Option Pricing via Path Integrals . . . . .	1029

6.3.6.3	Continuum Limit and American Options . . . . .	1035
6.3.7	APPLICATION: Nonlinear Dynamics of Complex Nets . . . . .	1036
6.3.7.1	Continuum Limit of the Kuramoto Net . . . . .	1037
6.3.7.2	Path–Integral Approach to Complex Nets . . . . .	1038
6.3.8	APPLICATION: Dissipative Quantum Brain Model . . . . .	1039
6.3.9	APPLICATION: Cerebellum as a Neural Path–Integral . . . . .	1043
6.3.9.1	Spinal Autogenetic Reflex Control . . . . .	1045
6.3.9.2	Cerebellum – the Comparator . . . . .	1047
6.3.9.3	Hamiltonian Action and Neural Path Integral . . . . .	1049
6.3.10	Path Integrals via Jets: Perturbative Quantum Fields . . . . .	1050
6.4	Sum over Geometries and Topologies . . . . .	1055
6.4.1	Simplicial Quantum Geometry . . . . .	1057
6.4.2	Discrete Gravitational Path Integrals . . . . .	1059
6.4.3	Regge Calculus . . . . .	1061
6.4.4	Lorentzian Path Integral . . . . .	1064
6.4.5	APPLICATION: Topological Phase Transitions and Hamiltonian Chaos . . . . .	1069
6.4.5.1	Phase Transitions in Hamiltonian Systems . . . . .	1069
6.4.5.2	Geometry of the Largest Lyapunov Exponent . . . . .	1072
6.4.5.3	Euler Characteristics of Hamiltonian Systems . . . . .	1075
6.4.6	APPLICATION: Force–Field Psychodynamics . . . . .	1079
6.4.6.1	Motivational Cognition in the Life Space Foam . . . . .	1080
6.5	APPLICATION: Witten’s TQFT, SW–Monopoles and Strings . . . . .	1097
6.5.1	Topological Quantum Field Theory . . . . .	1097
6.5.2	Seiberg–Witten Theory and TQFT . . . . .	1103
6.5.2.1	SW Invariants and Monopole Equations . . . . .	1103
6.5.2.2	Topological Lagrangian . . . . .	1105
6.5.2.3	Quantum Field Theory . . . . .	1107

6.5.2.4	Dimensional Reduction and 3D Field Theory . . . . .	1112
6.5.2.5	Geometrical Interpretation . . . . .	1115
6.5.3	TQFTs Associated with SW–Monopoles . . . . .	1118
6.5.3.1	Dimensional Reduction . . . . .	1122
6.5.3.2	TQFTs of 3D Monopoles . . . . .	1124
6.5.3.3	Non–Abelian Case . . . . .	1135
6.5.4	Stringy Actions and Amplitudes . . . . .	1138
6.5.4.1	Strings . . . . .	1139
6.5.4.2	Interactions . . . . .	1140
6.5.4.3	Loop Expansion – Topology of Closed Surfaces . . . . .	1141
6.5.5	Transition Amplitudes for Strings . . . . .	1143
6.5.6	Weyl Invariance and Vertex Operator Formulation . . . . .	1146
6.5.7	More General Actions . . . . .	1146
6.5.8	Transition Amplitude for a Single Point Particle . . . . .	1147
6.5.9	Witten’s Open String Field Theory . . . . .	1148
6.5.9.1	Operator Formulation of String Field Theory . . . . .	1149
6.5.9.2	Open Strings in Constant $B$ –Field Background . . . . .	1151
6.5.9.3	Construction of Overlap Vertices . . . . .	1154
6.5.9.4	Transformation of String Fields . . . . .	1164
6.6	APPLICATION: Dynamics of Strings and Branes . . . . .	1168
6.6.1	A Relativistic Particle . . . . .	1169
6.6.2	A String . . . . .	1171
6.6.3	A Brane . . . . .	1173
6.6.4	String Dynamics . . . . .	1175
6.6.5	Brane Dynamics . . . . .	1177
6.7	APPLICATION: Topological String Theory . . . . .	1180
6.7.1	Quantum Geometry Framework . . . . .	1180
6.7.2	Green–Schwarz Bosonic Strings and Branes . . . . .	1181
6.7.3	Calabi–Yau Manifolds, Orbifolds and Mirror Symmetry . . . . .	1186
6.7.4	More on Topological Field Theories . . . . .	1189
6.7.5	Topological Strings . . . . .	1204
6.7.6	Geometrical Transitions . . . . .	1221
6.7.7	Topological Strings and Black Hole Attractors . . . . .	1225

6.8	APPLICATION: Advanced Geometry and Topology of String Theory . . . . .	1232
6.8.1	String Theory and Noncommutative Geometry . .	1232
6.8.1.1	Noncommutative Gauge Theory . . . .	1233
6.8.1.2	Open Strings in the Presence of Constant $B$ -Field . . . . .	1235
6.8.2	$K$ -Theory Classification of Strings . . . . .	1241
	<i>Bibliography</i>	1253
	<i>Index</i>	1295