Chapter 1

The Least Cost Superreplicating Portfolio for Short Puts and Calls in The Boyle–Vorst Model with Transaction Costs

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Since Black and Scholes (1973) introduced their option-pricing model in frictionless markets, many authors have attempted to develop models incorporating transaction costs. The groundwork of modeling the effects of transaction costs was done by Leland (1985). The Leland model was put into a binomial setting by Boyle and Vorst (1992). Even when the market is arbitrage-free and a given contingent claim has a unique replicating portfolio, there may exist superreplicating portfolios of lower cost. However, it is known that there is no superreplicating portfolio for long calls and puts of lower cost than the replicating portfolio. Nevertheless, this is not true for short calls and puts. As the negative of the cost of the least cost superreplicating portfolios for such a position is a lower bound for the call or put price, it is important to determine this least cost. In this paper, we consider two-period binomial models and show that, for a special class of claims including short call and put options, there are just four possibilities so that the least cost superreplicating portfolios can be easily calculated for such positions. Also we show that, in general, the least cost superreplicating portfolio is path-dependent.

Keywords: Option pricing; transaction costs; binomial model; superreplicating.

1. Introduction

Since Black and Scholes (1973) introduced their option-pricing model in frictionless markets, many authors have attempted to develop models incorporating transaction costs. The groundwork of modeling the effects of transaction costs was done by Leland (1985). The Leland model was put into a binomial setting by Boyle and Vorst (1992). They derived self-financing strategies that

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perfectly replicate the final payoffs to long and short positions in put and call options, assuming proportional transaction costs on trades in the stocks and no transaction costs on trades in the bonds. Recently, Palmer (2001a) clarified the conditions under which there is a unique replicating strategy in the Boyle–Vorst model for an arbitrary contingent claim. Actually, following Stettner (1997) and Rutkowski (1998), Palmer worked in the framework of asymmetric proportional transaction costs, which includes not only the model of Boyle and Vorst, but also the slightly different model of Bensaid, Lesne, Pages, and Scheinkman (1992). For other recent contributions to this subject, see Perrakis and Lefoll (1997, 2000), Reiss (1999), and Chiang and Sheu (2004). A survey of some related results is given in Whalley and Wilmott (1997).

In arbitrage-free markets in the presence of transaction costs, even when a contingent claim has a unique replicating portfolio, there may exist a lower cost superreplicating portfolio. Nevertheless, Bensaid *et al.* (1992) gave conditions under which the cost of the replicating portfolio does not exceed the cost of any superreplicating portfolio. These results were generalized by Stettner (1997) and Rutkowski (1998) to the case of asymmetric transaction costs. Palmer (2001b) provided a further slight generalization. These results have the consequence that there is no superreplicating portfolio for long calls and puts of lower cost than the replicating portfolio. However, this is not true for short calls and puts. As the negative of the cost of the least cost superreplicating portfolios for such a position is a lower bound for the call or put price, it is important to determine this least cost. Recently, in Chen, Palmer, and Sheu (2004), we determined the least cost superreplicating portfolios for general contingent claims in one-period models and showed that there are only finitely many possibilities for the least cost super replicating portfolios of a general two-period contingent claims. Our result narrows down the search for a least cost superreplicating portfolio to a finite number of possibilities. However, the number of possibilities for the least cost superreplicating portfolios is still large. In this paper, we consider a restricted class of claims for which the number of possibilities can be reduced to a manageable number.

In Section 2, we review some basic results for general n-period models. We also quote two results from Chen *et al.* (2006) about the number of replicating portfolios and the least cost superreplicating portfolios for any contingent claim in a one-period binomial model. In Section 3, we recall the results of Chen *et al.* (2006) for the least cost superreplicating portfolios of a general two-period contingent claim. In Section 4, we show that for a special class of claims including short call and put options there are just four possibilities so that the least cost superreplicating portfolios can be easily calculated for such positions. In Section 5, we show that, in general, the least cost superreplicating portfolio is path-dependent.

2. Preliminaries

We consider an n-period binomial model of a financial market with two securities: a risky asset, referred to as a stock, and a risk-free investment, called a bond. If the stock price now is S, then at the end of the next period it is either Su or Sd, where $0 < d < u$. The bond yields a constant rate of return r over each time period meaning that a dollar now is worth $R = 1 + r$ after one period.

We assume that, on one hand, proportional transaction costs are incurred when shares of the risky asset are traded but, on the other hand, that trading in riskless bonds is cost-free. More precisely, we assume that when the stock price is S, buying one share incurs a transaction cost of λS and that selling one share incurs a transaction cost of μS, where

$$\lambda \geq 0, \qquad 0 \leq \mu < 1.$$

As is usual, we assume throughout this paper that there are no transaction costs when a portfolio is established at time 0. For no arbitrage consideration, we also assume that

$$d < R < u.$$

Let us denote by $\phi = \{(\Delta_i, B_i),\ i = 0, 1, 2, \ldots, n\}$, a (self-financing) portfolio where Δ_i stands for the number of shares and B_i the number of bonds held at time i. Under our assumption, it is natural that the initial value or cost of the portfolio ϕ is $\Delta_0 S_0 + B_0$.

A contingent claim is a two-dimensional random variable $X = (g, h)$ where g represents the number of shares and h the value of bonds held at time n. We say that a portfolio $\phi = \{(\Delta_i, B_i),\ i = 0, 1, 2, \ldots, n\}$ replicates the claim X that is settled by delivery if it is self-financing and $\Delta_n = g$ and $B_n = h$. We say a self-financing portfolio ϕ is a *superreplicating portfolio* for a contingent claim $X = (g, h)$ settled by delivery at time n if at time n we have $\Delta_n \geq g$ and $B_n \geq h$. An upper arbitrage bound for the price at time 0 of a claim

$X = (g, h)$ is given by the cost of a least cost superreplicating portfolio for a long position in the claim X. A lower arbitrage bound for the price of X at time 0 is given by the negative of the cost of a least cost superreplicating portfolio for a short position in the claim X. As pointed out by several authors, in some circumstances, it is possible to find a portfolio which ultimately dominates a given contingent claim and costs less than a portfolio that replicates the claim. Of course, there are circumstances in which no superreplicating portfolio costs less than a replicating portfolio. Theorems 1 and 2 given in Palmer (2001a) generalize results of Bensaid *et al.* (1992), Stettner (1997), and Rutkowski (1998).

Theorem 1. *Suppose that*

$$d(1+\lambda) \leq R(1-\mu) \leq R(1+\lambda) \leq u(1-\mu).$$

Then, for any contingent claim, there is a unique replicating portfolio and no superreplicating portfolio costs less than the replicating portfolio.

Theorem 2. *Consider a contingent claim in an n-period binomial model with holdings (g_j, h_j) when the terminal stock price is $S_0 u^j d^{n-j}$. If these terminal holdings satisfy*

$$g_{j+1} \geq g_j,$$

$$(g_j - g_{j+1})Su^{j+1}d^{n-j-1}(1+\lambda) + h_j - h_{j+1} \leq 0,$$

and

$$(g_j - g_{j+1})Su^j d^{n-j}(1+\mu) + h_j - h_{j+1} \geq 0,$$

for $j = 0, 1, \ldots, n-1$, then there is a unique replicating portfolio for such a contingent claim and no superreplicating portfolio costs less than the replicating portfolio.

Clearly long positions in calls and puts satisfy these conditions in Theorem 2. However, short positions in calls and puts do not satisfy these conditions.

Consider a contingent claim in a one-period model with holdings $(\Delta_\mathrm{u}, B_\mathrm{u})$ in the up state and $(\Delta_\mathrm{d}, B_\mathrm{d})$ in the down state. Let

$$a_\mathrm{u} = \begin{cases} (\Delta_\mathrm{d} - \Delta_\mathrm{u})Su(1+\lambda) + B_\mathrm{d} - B_\mathrm{u} & \text{if } \Delta_\mathrm{u} \geq \Delta_\mathrm{d}, \\ (\Delta_\mathrm{d} - \Delta_\mathrm{u})Su(1-\mu) + B_\mathrm{d} - B_\mathrm{u} & \text{if } \Delta_u < \Delta_\mathrm{d}, \end{cases}$$

and

$$a_\mathrm{d} = \begin{cases} (\Delta_\mathrm{d} - \Delta_\mathrm{u})Sd(1-\mu) + B_\mathrm{d} - B_\mathrm{u} & \text{if } \Delta_\mathrm{u} \geq \Delta_\mathrm{d}, \\ (\Delta_\mathrm{d} - \Delta_\mathrm{u})Sd(1+\lambda) + B_\mathrm{d} - B_\mathrm{u} & \text{if } \Delta_\mathrm{u} < \Delta_\mathrm{d}. \end{cases}$$

Theorems 3 and 4 are quoted from Chen *et al.* (2004).

Theorem 3. *Consider a contingent claim in a one-period model with holdings* $(\Delta_\mathrm{u}, B_\mathrm{u})$ *in the up state and* $(\Delta_\mathrm{d}, B_\mathrm{d})$ *in the down state. Then the contingent claim has a unique replicating portfolio if and only if it satisfies one of the following conditions*:

(a) $\Delta_\mathrm{u} \geq \Delta_\mathrm{d}$,
(b) $\Delta_\mathrm{u} < \Delta_\mathrm{d}$, $d(1+\lambda) < u(1-\mu)$,
(c) $\Delta_\mathrm{u} < \Delta_\mathrm{d}$, $d(1+\lambda) \geq u(1-\mu)$, $a_\mathrm{u}a_\mathrm{d} > 0$.

The following theorem determines the least cost superreplicating portfolios for any contingent claims in a one-period binomial model.

Theorem 4. *Consider a contingent claim in a one-period model with holdings* $(\Delta_\mathrm{u}, B_\mathrm{u})$ *in the up state and* $(\Delta_\mathrm{d}, B_\mathrm{d})$ *in the down state.*

(a) *When the replicating portfolio is unique*, *it is a least cost superreplicating portfolio unless* $R > u(1-\mu)$, $a_\mathrm{d} < 0$ *when* $(\Delta_\mathrm{u}, B_\mathrm{u}/R)$ *are the holdings in a least cost superreplicating portfolio*, *or if* $R < d(1+\lambda)$, $a_\mathrm{u} > 0$ *when* $(\Delta_\mathrm{d}, B_\mathrm{d}/R)$ *are the holdings in a least cost superreplicating portfolio.*
(b) *When the replicating portfolio is not unique, it is necessary that* $\Delta_\mathrm{u} < \Delta_\mathrm{d}$, $d(1+\lambda) \geq u(1-\mu)$. *Moreover, we have*:
   (i) *If* $R \geq d(1+\lambda)$, *there exists at least one replicating portfolio with share holdings* Δ *satisfying* $\Delta \leq \Delta_\mathrm{u}$ *and all such replicating portfolios are least cost superreplicating portfolios.*
   (ii) *If* $d(1+\lambda) \geq R \geq u(1-\mu)$, *there exists at least one replicating portfolio with share holdings* Δ *satisfying* $\Delta_\mathrm{u} \leq \Delta \leq \Delta_\mathrm{d}$ *and all such replicating portfolios are least cost superreplicating portfolios.*

(iii) *If $R \leq u(1-\mu)$, there exists at least one replicating portfolio with share holdings Δ satisfying $\Delta \geq \Delta_d$ and all such replicating portfolios are least cost superreplicating portfolios.*

Remark 1. As mentioned in the Remarks after Theorem 4.1 in Chen *et al.* (2006), the cost $C(\Delta_u, B_u, \Delta_d, B_d)$ of the least cost superreplicating portfolio is a continuous function which is linear in any region in the $(\Delta_u, B_u, \Delta_d, B_d)$ space where $\Delta_u - \Delta_d$, a_u, and a_d are one-signed. In Chen *et al.* (2006), we proved Theorem 4 by considering the contingent claim according to the following cases:

$$\begin{array}{lll}
\text{Case 1}: & \Delta_u \geq \Delta_d, & a_d \geq a_u > 0 \\
\text{Case 2}: & \Delta_u \geq \Delta_d, & a_d \geq 0 \geq a_u \\
\text{Case 3}: & \Delta_u \geq \Delta_d, & a_u \leq a_d < 0 \\
\text{Case 4}: & \Delta_u < \Delta_d, & u(1-\mu) > d(1+\lambda), \quad a_u > a_d > 0 \\
\text{Case 5}: & \Delta_u < \Delta_d, & u(1-\mu) > d(1+\lambda), \quad a_u \geq 0 \geq a_d \\
\text{Case 6}: & \Delta_u < \Delta_d, & u(1-\mu) > d(1+\lambda), \quad a_d < a_u < 0 \\
\text{Case 7}: & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), \quad a_d > a_u > 0 \\
\text{Case 8}: & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), \quad a_u < a_d < 0 \\
\text{Case 9}: & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), \quad a_u = 0 < a_d \\
\text{Case 10}: & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), \quad a_d = 0 > a_u \\
\text{Case 11}: & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), \quad a_d > 0 > a_u \\
\text{Case 12}: & \Delta_u < \Delta_d, & u(1-\mu) = d(1+\lambda), \quad a_u = a_d > 0 \\
\text{Case 13}: & \Delta_u < \Delta_d, & u(1-\mu) = d(1+\lambda), \quad a_u = a_d < 0 \\
\text{Case 14}: & \Delta_u < \Delta_d, & u(1-\mu) = d(1+\lambda), \quad a_u = a_d = 0.
\end{array}$$

It follows from Theorem 3 that in Cases 1–8, 12, and 13, there is a unique replicating portfolio (Δ, B) which, as in Chen *et al.* (2004), has cost given by

$$C = \Delta S + B = \frac{p}{R}[\Delta_u S\overline{u} + B_u] + \frac{1-p}{R}[\Delta_d S\overline{d} + B_d]$$

where

$$\overline{u} = \begin{cases} u(1+\lambda) & \text{if } a_d \geq 0, \\ u(1-\mu) & \text{if } a_d < 0, \end{cases} \qquad \overline{d} = \begin{cases} d(1+\lambda) & \text{if } a_u \geq 0, \\ d(1-\mu) & \text{if } a_u < 0, \end{cases} \qquad p = \frac{R-\overline{d}}{\overline{u}-\overline{d}}.$$

3. General Contingent Claims in the Two-Period Case

In this section, we recall some results of Chen *et al.* (2006) for a general two-period contingent claim with terminal holdings $\{(\Delta_{uu}, B_{uu}), (\Delta_{ud}, B_{ud}),$

$(\Delta_{\text{dd}}, \ B_{\text{dd}})\}$. Write

$$b_{\text{u}}(\Delta_{\text{u}}) = \max\{B_{\text{uu}} + e(\Delta_{\text{u}} - \Delta_{\text{uu}})Su^2, \quad B_{\text{ud}} + e(\Delta_{\text{u}} - \Delta_{\text{ud}})Sud\}$$

and

$$b_{\text{d}}(\Delta_{\text{d}}) = \max\{B_{\text{ud}} + e(\Delta_{\text{d}} - \Delta_{\text{ud}})Sud, \quad B_{\text{dd}} + e(\Delta_{\text{d}} - \Delta_{\text{dd}})Sd^2\},$$

where $e(\Delta) = -\Delta + \mu\Delta^+ + \lambda\Delta^-$. The significance of these two quantities is that $(\Delta_{\text{u}}, B_{\text{u}})$ is a superreplicating portfolio for the one-period claim $\{(\Delta_{\text{uu}}, B_{\text{uu}}), (\Delta_{\text{ud}}, B_{\text{ud}})\}$ with initial stock price Su if and only if $B_{\text{u}} \geq b_{\text{u}}(\Delta_{\text{u}})/R$ and $(\Delta_{\text{d}}, B_{\text{d}})$ is a superreplicating portfolio for the one-period claim $\{(\Delta_{\text{ud}}, B_{\text{ud}}), (\Delta_{\text{dd}}, B_{\text{dd}})\}$ with initial stock price Sd if and only if $B_{\text{d}} \geq b_{\text{d}}(\Delta_{\text{d}})/R$. Denote by $C(\Delta_{\text{u}}, \Delta_{\text{d}})$ the least cost of superreplicating portfolios for the one-period contingent claim $\{(\Delta_{\text{u}}, b_u(\Delta_{\text{u}})/R), (\Delta_{\text{d}}, b_{\text{d}}(\Delta_{\text{d}})/R)\}$ with initial stock price S. Then it was proved in Chen *et al.* (2004) that the infimum of the cost of a superreplicating portfolio for the two-period contingent claim $\{(\Delta_{\text{uu}}, B_{\text{uu}}), (\Delta_{\text{ud}}, B_{\text{ud}}), (\Delta_{\text{dd}}, B_{\text{dd}})\}$ is equal to the infimum over $(\Delta_{\text{u}}, \Delta_{\text{d}})$ of $C(\Delta_{\text{u}}, \Delta_{\text{d}})$. Theorem 5 shows that we need only consider the function $C(\Delta_{\text{u}}, \Delta_{\text{d}})$ in a certain rectangle in the $(\Delta_{\text{u}}, \Delta_{\text{d}})$-plane.

To do this, we consider functions

$$f_{\text{u}}(\Delta_{\text{u}}) = B_{\text{uu}} + e(\Delta_{\text{u}} - \Delta_{\text{uu}})Su^2 - B_{\text{ud}} - e(\Delta_{\text{u}} - \Delta_{\text{ud}})Sud \qquad (1)$$

and

$$f_{\text{d}}(\Delta_{\text{d}}) = B_{\text{ud}} + e(\Delta_{\text{d}} - \Delta_{\text{ud}})Sud - B_{\text{dd}} - e(\Delta_{\text{d}} - \Delta_{\text{dd}})Sd^2. \qquad (2)$$

Note that the values of Δ_{u} satisfying $f_{\text{u}}(\Delta_{\text{u}}) = 0$ are exactly those for which $(\Delta_{\text{u}}, b_{\text{u}}(\Delta_{\text{u}})/R)$ is a replicating portfolio for the contingent claim $\{(\Delta_{\text{uu}}, B_{\text{uu}}), (\Delta_{\text{ud}}, B_{\text{ud}})\}$ with initial stock price Su and the values of Δ_{d} satisfying $f_{\text{d}}(\Delta_{\text{d}}) = 0$ are exactly those for which $(\Delta_{\text{d}}, b_{\text{d}}(\Delta_{\text{d}})/R)$ is a replicating portfolio for the contingent claim $\{(\Delta_{\text{ud}}, B_{\text{ud}}), (\Delta_{\text{dd}}, B_{\text{dd}})\}$ with initial stock price Sd.

Let $[\alpha_{\text{u}}, \beta_{\text{u}}]$ be the smallest closed interval containing all solutions of $f_{\text{u}}(\Delta_{\text{u}}) = 0$ and also Δ_{uu} and Δ_{ud}. Similarly, let $[\alpha_{\text{d}}, \beta_{\text{d}}]$ be the smallest closed interval containing all solutions of $f_{\text{d}}(\Delta_{\text{d}}) = 0$ and also Δ_{ud} and Δ_{dd}.

Let Π be the rectangle in the (Δ_u, Δ_d)-plane given by

$$\Pi = \{(\Delta_u, \Delta_d) : \alpha_u \le \Delta_u \le \beta_u,\ \alpha_d \le \Delta_d \le \beta_d\}.$$

Theorem 5. *For a general two-period contingent claim* $\{(\Delta_{uu}, B_{uu}), (\Delta_{ud}, B_{ud}), (\Delta_{dd}, B_{dd})\}$*, the function* $C(\Delta_u, \Delta_d)$ *takes its minimum in the rectangle* Π *at some point* (Δ_u, Δ_d) *and a least cost superreplicating portfolio for the one-period claim* $\{(\Delta_u, b_u(\Delta_u)/R), (\Delta_d, b_d(\Delta_d)/R)\}$ *with initial stock price S yields a least cost super replicating portfolio for the two-period claim.*

(It is worth noting that in the case $(\Delta_{uu}, B_{uu}) = (\Delta_{ud}, B_{ud})$, there is always a least cost superreplicating portfolio with $\Delta_u = \Delta_{uu}$ because in this case $\alpha_u = \beta_u = \Delta_{uu}$. We consider this special case in more detail in Section 4.)

Consider the two quantities a_u and a_d,

$$a_u = a_u(\Delta_u, \Delta_d) = (\Delta_d - \Delta_u)S\overline{u} + \frac{b_d(\Delta_d) - b_u(\Delta_u)}{R},$$
$$a_d = a_d(\Delta_u, \Delta_d) = (\Delta_d - \Delta_u)S\overline{d} + \frac{b_d(\Delta_d) - b_u(\Delta_u)}{R},$$

where

$$\overline{u} = \begin{cases} u(1+\lambda) & \text{if } \Delta_u \ge \Delta_d, \\ u(1-\mu) & \text{if } \Delta_u < \Delta_d, \end{cases} \qquad \overline{d} = \begin{cases} d(1-\mu) & \text{if } \Delta_u \ge \Delta_d, \\ d(1+\lambda) & \text{if } \Delta_u < \Delta_d. \end{cases}$$

By using the fundamental theorem of linear programing, the following theorem shows that there are only finitely many possibilities for a least cost superreplicating portfolio.

Theorem 6. *For a general two-period contingent claim with terminal holdings* $\{(\Delta_{uu}, B_{uu}), (\Delta_{ud}, B_{ud}), (\Delta_{dd}, B_{dd})\}$*, there always exists a least cost superreplicating portfolio with initial holdings* (Δ, B) *and holdings* (Δ_u, B_u), (Δ_d, B_d) *at the end of the first period which represent a least cost superreplicating portfolio for the one-period claim* $\{(\Delta_u, b_u(\Delta_u)/R), (\Delta_d, b_d(\Delta_u)/R)\}$ *and such that at least two distinct conditions from the following list are satisfied*:

$$\Delta_u = \Delta_{uu}, \qquad \Delta_u = \Delta_{ud}, \qquad \Delta_d = \Delta_{ud}, \qquad \Delta_d = \Delta_{dd}, \qquad \Delta_u = \Delta_d,$$
$$a_u(\Delta_u, \Delta_d) = 0, \qquad a_d(\Delta_u, \Delta_d) = 0,$$
$$f_u(\Delta_u) = 0, \qquad f_d(\Delta_d) = 0.$$

Note that the condition $a_{\mathrm{u}}(\Delta_{\mathrm{u}}, \Delta_{\mathrm{d}}) = 0$ means that $(\Delta_{\mathrm{d}}, b_{\mathrm{d}}(\Delta_{\mathrm{d}})/R^2)$ is a replicating portfolio for the contingent claim $\{(\Delta_{\mathrm{u}}, b_{\mathrm{u}}(\Delta_{\mathrm{u}})/R), (\Delta_{\mathrm{d}}, b_{\mathrm{d}}(\Delta_{\mathrm{d}})/R)\}$ with initial stock price S. Likewise, the condition $a_{\mathrm{d}}(\Delta_{\mathrm{u}}, \Delta_{\mathrm{d}}) = 0$ means that $(\Delta_{\mathrm{u}}, b_{\mathrm{u}}(\Delta_{\mathrm{u}})/R^2)$ is a replicating portfolio for the contingent claim $\{(\Delta_{\mathrm{u}}, b_{\mathrm{u}}(\Delta_{\mathrm{u}})/R), (\Delta_{\mathrm{d}}, b_{\mathrm{d}}(\Delta_{\mathrm{d}})/R)\}$. We note again that the values of Δ_{u} satisfying $f_{\mathrm{u}}(\Delta_{\mathrm{u}}) = 0$ are exactly those for which $(\Delta_{\mathrm{u}}, b_{\mathrm{u}}(\Delta_{\mathrm{u}})/R)$ is a replicating portfolio for the contingent claim $\{(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}), (\Delta_{\mathrm{ud}}, B_{\mathrm{ud}})\}$ with initial stock price Su and the values of Δ_{d} satisfying $f_{\mathrm{d}}(\Delta_{\mathrm{d}}) = 0$ are exactly those for which $(\Delta_{\mathrm{d}}, b_{\mathrm{d}}(\Delta_{\mathrm{d}})/R)$ is a replicating portfolio for the contingent claim $\{(\Delta_{\mathrm{ud}}, B_{\mathrm{ud}}), (\Delta_{\mathrm{dd}}, B_{\mathrm{dd}})\}$ with initial stock price Sd.

Theorem 6 narrows down the search for a least cost superreplicating portfolio to a finite number of possibilities. However, the number of possibilities is still quite large. In the following section, we consider a restricted class of claims for which the number of possibilities can be reduced to a manageable number.

4. Least Cost Superreplicating Portfolios for Short Puts and Calls in the Two-Period Case

In this section, we determine the initial holdings of the least cost superreplicating portfolios for a claim in the two-period model with

$$\Delta_{\mathrm{uu}} = \Delta_{\mathrm{ud}} < \Delta_{\mathrm{dd}}, \qquad B_{\mathrm{uu}} = B_{\mathrm{ud}}. \tag{3}$$

This includes short calls and puts with the exercise price between Sud and Sd^2. Note that we could treat the case $\Delta_{\mathrm{uu}} < \Delta_{\mathrm{ud}} = \Delta_{\mathrm{dd}}$, $B_{\mathrm{ud}} = B_{\mathrm{dd}}$ similarly. This would include short calls and puts with the exercise price between Su^2 and Sud.

Theorem 7. *Consider a two-period binomial model incorporating transaction costs with parameters S, u, d, R, μ, and λ. For every contingent claim $\{(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}), (\Delta_{\mathrm{ud}}, B_{\mathrm{ud}}), (\Delta_{\mathrm{dd}}, B_{\mathrm{dd}})\}$ satisfying Equation* (3), *there always exists a least cost superreplicating portfolio which belongs to one of the following four types (note that in all cases transactions are carried out at the terminal nodes so that the final share holdings are Δ_{uu}, Δ_{ud}, Δ_{du}, Δ_{dd} in* *states uu, ud, du, and dd, respectively):*

(I) *the initial holdings are* $(\Delta_{\rm dd}, B_{\rm dd}/R^2)$ *and the only additional share transaction is selling* $(\Delta_{\rm dd} - \Delta_{\rm uu})$ *shares in state* u *(this type arises only if* $R < d(1+\lambda)$ *and* $B_{\rm uu} - B_{\rm dd} - Sud(1-\mu)(\Delta_{\rm dd} - \Delta_{\rm uu}) < 0$*);*

(II) *the initial holdings are* (δ, B)*, where* $\delta \leq \Delta_{\rm uu}$ *and* (δ, B) *is such that* $BR - B_{\rm uu}/R$ *is just enough to carry out the only additional share transaction of buying back* $(\Delta_{\rm uu} - \delta)$ *shares of stocks in state* u*; there are two possibilities*:

  (a) $\delta = \Delta_{\rm uu}$ *and the terminal holdings in the* du *state are* $(\Delta_{\rm uu}, B_{\rm uu})$ *(this case only arises if* $B_{\rm uu} - B_{\rm dd} - Sd^2(1+\lambda)(\Delta_{\rm dd} - \Delta_{\rm uu}) \geq 0$*);*

  (b) $\delta < \Delta_{\rm uu}$ *and the terminal holdings in the* dd *state are* $(\Delta_{\rm dd}, B_{\rm dd})$ *(this case only arises if* $B_{\rm uu} - B_{\rm dd} - Sd^2(1+\lambda)(\Delta_{\rm dd} - \Delta_{\rm uu}) < 0$*);*

(III) *the initial holdings are* $(\alpha, B/R)$*, where* $\alpha > \Delta_{\rm uu}$ *and* (α, B) *are the initial holdings in a replicating portfolio for the one-period portion* $\{d, du, dd\}$*, and the only additional share transaction is selling* $(\alpha - \Delta_{\rm uu})$ *shares in state* u *(this case only arises if* $R < d(1+\lambda)$*);*

(IV) *a replicating portfolio for the whole two-period model.*

Proof. It follows from the remark after Theorem 5 that we need only determine the $\Delta_{\rm d}$ which yields the least cost for the one-period contingent claim $\{(\Delta_{\rm uu}, B_{\rm uu}/R), (\Delta_{\rm d}, b_{\rm d}(\Delta_{\rm d})/R)\}$ with initial stock price S and then determine a least cost superreplicating portfolio for this one-period claim. For this claim, we have

$$a_{\rm u} = a_{\rm u}(\Delta_{\rm d}) = a_{\rm u}(\Delta_{\rm uu}, \Delta_{\rm d}) = (\Delta_{\rm d} - \Delta_{\rm uu})S\overline{u} + \frac{b_{\rm d}(\Delta_{\rm d}) - B_{\rm uu}}{R},$$

$$a_{\rm d} = a_{\rm d}(\Delta_{\rm d}) = a_{\rm d}(\Delta_{\rm uu}, \Delta_{\rm d}) = (\Delta_{\rm d} - \Delta_{\rm uu})S\overline{d} + \frac{b_{\rm d}(\Delta_{\rm d}) - B_{\rm uu}}{R},$$

where

$$\overline{u} = \begin{cases} u(1+\lambda) & \text{if } \Delta_{\rm uu} \geq \Delta_{\rm d}, \\ u(1-\mu) & \text{if } \Delta_{\rm uu} < \Delta_{\rm d}, \end{cases} \qquad \overline{d} = \begin{cases} d(1-\mu) & \text{if } \Delta_{\rm uu} \geq \Delta_{\rm d}, \\ d(1+\lambda) & \text{if } \Delta_{\rm uu} < \Delta_{\rm d}, \end{cases}$$

and

$$b_{\rm d}(\Delta_{\rm d}) = \max\{B_{\rm uu} + Sude(\Delta_{\rm d} - \Delta_{\rm uu}), B_{\rm dd} + Sd^2e(\Delta_{\rm d} - \Delta_{\rm dd})\}.$$

Also

$$f_{\rm d}(\Delta_{\rm d}) = B_{\rm uu} + Sude(\Delta_{\rm d} - \Delta_{\rm uu}) - B_{\rm dd} - Sd^2e(\Delta_{\rm d} - \Delta_{\rm dd}).$$

Note that $(\Delta_{\rm d}, B_{\rm d})$ is a replicating portfolio for the one-period portion $\{d, du, dd\}$ if and only if $f_{\rm d}(\Delta_{\rm d}) = 0$ and $B_{\rm d} = b_{\rm d}(\Delta_{\rm d})/R$. Further observe that the continuous function $f_{\rm d}(\Delta_{\rm d})$ is decreasing and linear for $\Delta_{\rm d} \leq \Delta_{\rm uu}$, $\Delta_{\rm d} \geq \Delta_{\rm dd}$ and linear and decreasing, constant, or increasing for $\Delta_{\rm uu} < \Delta_{\rm d} < \Delta_{\rm dd}$ depending on the sign of $u(1-\lambda) - d(1+\mu)$. Note also that

$$f_{\rm d}(\Delta_{\rm uu}) = B_{\rm uu} - B_{\rm dd} + Sd^2(\Delta_{\rm uu} - \Delta_{\rm dd})(1+\lambda),$$

and that

$$a_{\rm u}(\Delta_{\rm uu}) = a_{\rm d}(\Delta_{\rm uu}) = \frac{[f_{\rm d}(\Delta_{\rm uu})]^-}{R}.$$

The signs of $a_{\rm u}$ and $a_{\rm d}$: We start by examining the signs of $a_{\rm u}$ and $a_{\rm d}$. First we show that when $\Delta_{\rm d} < \Delta_{\rm uu}$, then $a_{\rm d} > 0$. This follows because

$$b_{\rm d}(\Delta_{\rm d}) \geq B_{\rm uu} - Sud(\Delta_{\rm d} - \Delta_{\rm uu})(1+\lambda),$$

and so

$$\begin{aligned} a_{\rm d} &\geq (\Delta_{\rm d} - \Delta_{\rm uu})Sd(1-\mu) - \frac{Sud(\Delta_{\rm d} - \Delta_{\rm uu})(1+\lambda)}{R} \\ &= \frac{Sd(\Delta_{\rm d} - \Delta_{\rm uu})}{R}[R(1-\mu) - u(1+\lambda)] > 0. \end{aligned}$$

Suppose now that $\Delta_{\rm d} > \Delta_{\rm uu}$ and $R(1+\lambda) > u(1-\mu)$. Then as

$$b_{\rm d}(\Delta_{\rm d}) \geq B_{\rm uu} - Sud(\Delta_{\rm d} - \Delta_{\rm uu})(1-\mu),$$

we have

$$\begin{aligned} a_{\rm d} &\geq (\Delta_{\rm d} - \Delta_{\rm uu})Sd(1+\lambda) - \frac{Sud(\Delta_{\rm d} - \Delta_{\rm uu})(1-\mu)}{R} \\ &= \frac{Sd(\Delta_{\rm d} - \Delta_{\rm uu})}{R}[R(1+\lambda) - u(1-\mu)] > 0. \end{aligned}$$

Assume next that $\Delta_{\rm d} > \Delta_{\rm uu}$, $f_{\rm d}(\Delta_{\rm uu}) \leq 0$, and $R(1+\lambda) \leq u(1-\mu)$. The latter implies that $d(1+\lambda) < u(1-\mu)$ and so $f_{\rm d}(\Delta_{\rm d})$ is strictly decreasing.

Then as $f_{\rm d}(\Delta_{\rm uu}) \leq 0$, we have $f_{\rm d}(\Delta_{\rm d}) < 0$ for $\Delta_{\rm d} > \Delta_{\rm uu}$ and so

$$b_{\rm d}(\Delta_{\rm d}) = B_{\rm dd} + Sd^2 e(\Delta_{\rm d} - \Delta_{\rm dd}),$$

and

$$a_{\rm d} = (\Delta_{\rm d} - \Delta_{\rm uu})Sd(1+\lambda) + \frac{B_{\rm dd} - B_{\rm uu} + Sd^2 e(\Delta_{\rm d} - \Delta_{\rm dd})}{R}.$$

It follows that

$$a_{\rm d}(\Delta_{\rm uu}) = \frac{B_{\rm dd} - B_{\rm uu} - Sd^2(1+\lambda)(\Delta_{\rm uu} - \Delta_{\rm dd})}{R} \geq 0$$

and

$$a'_{\rm d}(\Delta_{\rm d}) = \frac{Sd}{R}\begin{cases} (R-d)(1+\lambda) & \text{if } \Delta_{\rm d} < \Delta_{\rm dd} \\ R(1+\lambda) - d(1-\mu) & \text{if } \Delta_{\rm d} > \Delta_{\rm dd} \end{cases} > 0.$$

Hence, in this case, we still have

$$a_{\rm d} > 0$$

for $\Delta_{\rm d} > \Delta_{\rm uu}$.

Hence we are left with the case $\Delta_{\rm d} > \Delta_{\rm uu}$, $f_{\rm d}(\Delta_{\rm uu}) > 0$, and $R(1+\lambda) \leq u(1-\mu)$. In this case, there exists a unique $\gamma > \Delta_{\rm uu}$ such that

$$f_{\rm d}(\Delta_{\rm d}) \begin{cases} > 0 & \text{if } \Delta_{\rm d} < \gamma, \\ = 0 & \text{if } \Delta_{\rm d} = \gamma, \\ < 0 & \text{if } \Delta_{\rm d} > \gamma. \end{cases}$$

Also note that $a_{\rm d}(\Delta_{\rm uu}) = 0$. It follows as in the previous case that $a'_{\rm d}(\Delta_{\rm d}) > 0$ if $\Delta_{\rm d} > \gamma$. However, if $\Delta_{\rm uu} < \Delta_{\rm d} < \gamma$, then

$$b_{\rm d}(\Delta_{\rm d}) = B_{\rm uu} - Sud(1-\mu)(\Delta_{\rm d} - \Delta_{\rm uu})$$

and

$$a_{\rm d} = (\Delta_{\rm d} - \Delta_{\rm uu})Sd(1+\lambda) - \frac{Sud(1-\mu)(\Delta_{\rm d} - \Delta_{\rm uu})}{R},$$

so that

$$a'_{\rm d}(\Delta_{\rm d}) = \frac{Sd}{R}[R(1+\lambda) - u(1-\mu)] \leq 0.$$

Then there exists $\tilde{\delta} \geq \gamma$ such that

$$a_{\mathrm{d}} \begin{cases} \leq 0 & \text{if } \Delta_{\mathrm{uu}} < \Delta_{\mathrm{d}} \leq \tilde{\delta}, \\ = 0 & \text{if } \Delta_{\mathrm{d}} = \tilde{\delta}, \\ > 0 & \text{if } \Delta_{\mathrm{d}} > \tilde{\delta}. \end{cases}$$

Now we examine the sign of a_{u}. First note that if $\Delta_{\mathrm{d}} > \Delta_{\mathrm{uu}}$, then

$$\begin{aligned} a_{\mathrm{u}}(\Delta_{\mathrm{d}}) &= (\Delta_{\mathrm{d}} - \Delta_{\mathrm{uu}})Su(1-\mu) + \frac{b_{\mathrm{d}}(\Delta_{\mathrm{d}}) - B_{\mathrm{uu}}}{R} \\ &\geq (\Delta_{\mathrm{d}} - \Delta_{\mathrm{uu}})Su(1-\mu) - \frac{Sud(\Delta_{\mathrm{d}} - \Delta_{\mathrm{uu}})(1-\mu)}{R} \\ &= \frac{Su}{R}(\Delta_{\mathrm{d}} - \Delta_{\mathrm{uu}})(R-d)(1-\mu) > 0. \end{aligned}$$

If $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) \geq 0$, then $f_{\mathrm{d}}(\Delta_{\mathrm{d}}) > 0$ for $\Delta_{\mathrm{d}} < \Delta_{\mathrm{uu}}$ and so for $\Delta_{\mathrm{d}} < \Delta_{\mathrm{uu}}$,

$$a_{\mathrm{u}} = \frac{Su}{R}(\Delta_{\mathrm{d}} - \Delta_{\mathrm{uu}})(R-d)(1+\lambda) < 0.$$

Also as $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) \geq 0$, $a_{\mathrm{u}}(\Delta_{\mathrm{uu}}) = 0$.

If $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) < 0$, there exists a unique $\gamma < \Delta_{\mathrm{uu}}$ such that $f_{\mathrm{d}}(\gamma) = 0$. We show as in the case $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) \geq 0$ that $a_{\mathrm{u}} < 0$ if $\Delta_{\mathrm{d}} \leq \gamma$. Now as $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) < 0$, $a_{\mathrm{u}}(\Delta_{\mathrm{uu}}) > 0$. Then as a_{u} is a linear function of Δ_{d} in the interval $[\gamma, \Delta_{\mathrm{uu}}]$, it follows that there exists a unique δ in $(\gamma, \Delta_{\mathrm{uu}})$ such that $a_{\mathrm{u}}(\delta) = 0$. Note also that $f_{\mathrm{d}}(\delta) < 0$. Thus, if $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) < 0$,

$$a_u \begin{cases} < 0 & \text{if } \Delta_{\mathrm{d}} < \delta, \\ = 0 & \text{if } \Delta_{\mathrm{d}} = \delta, \\ > 0 & \text{if } \delta < \Delta_{\mathrm{d}} \leq \Delta_{\mathrm{uu}}. \end{cases}$$

We now consider four different cases.

1. Suppose first that $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) < 0$. Then $a_{\mathrm{d}} > 0$ for all Δ_{d}, $a_{\mathrm{u}} > 0$ for $\Delta_{\mathrm{d}} > \delta$, and $a_{\mathrm{u}}(\delta) = 0$ and $a_{\mathrm{u}} < 0$ for $\Delta_{\mathrm{d}} < \delta$. Also $f_{\mathrm{d}}(\gamma) = 0$ has at most three solutions. As the function $b_{\mathrm{d}}(\Delta_{\mathrm{d}})$ is linear in any interval not containing Δ_{uu}, Δ_{dd}, or any of the γ's, we see from Remark 1 that the cost function $C(\Delta_{\mathrm{uu}}, \Delta_{\mathrm{d}}) = C(\Delta_{\mathrm{d}})$ is linear in any interval not containing δ, Δ_{uu}, Δ_{dd}, or any of the γ's. So the minimum must be achieved at one of these points.

Suppose the minimum occurs at δ. At δ, $a_{\mathrm{d}} > 0$ and $a_{\mathrm{u}} = 0$ and so the one-period claim $\{(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}/R), (\delta, b_{\mathrm{d}}(\delta)/R)\}$ is in Case 2 of Remark 1 so that the replicating portfolio is unique and by Theorem 4 is the least cost superreplicating portfolio. However, the condition $a_{\mathrm{u}}(\delta) = 0$ implies that

$(\delta, b_{\rm d}(\delta)/R^2)$ is a replicating portfolio for this one-period claim. So the initial holdings are $(\delta, B) = (\delta, b_{\rm d}(\delta)/R^2)$, where $(BR - B_{\rm uu}/R)$ is just enough to buy back $(\Delta_{\rm uu} - \delta)$ shares of stocks in state u. Moreover, as $f_{\rm d}(\delta) < 0$, $b_{\rm d}(\delta) = B_{\rm dd} + (\Delta_{\rm dd} - \delta)Sd^2(1+\lambda)$ and so

$$b_{\rm d}(\delta) - (\Delta_{\rm dd} - \delta)Sd^2(1+\lambda) = B_{\rm dd},$$

that is, the final holdings in the dd state are $(\Delta_{\rm dd}, B_{\rm dd})$. This is type (II)(b) of Theorem 7

We now show that in this case the minimum is either not attained at $\Delta_{\rm uu}$ or if it is, then it is also attained at $\Delta_{\rm dd}$ or at one of the solutions of $f_{\rm d}(\Delta_{\rm d}) = 0$. Let γ be the least number greater than $\Delta_{\rm uu}$ such that $f_{\rm d}(\gamma) = 0$ (take $\gamma = \infty$ if no such γ exists). Set $\tilde{\gamma} = \min\{\gamma, \Delta_{\rm dd}\}$. Then in the interval $(\delta, \tilde{\gamma}]$, $a_{\rm u}$ and $a_{\rm d}$ are positive and $f_{\rm d}(\Delta_{\rm d}) \le 0$. So the one-period claim $\{(\Delta_{\rm uu}, B_{\rm uu}/R), (\Delta_{\rm d}, b_{\rm d}(\Delta_{\rm d})/R)\}$ is in one of Cases 4, 7, or 12 of Remark 1 for $\Delta_{\rm d}$ in $(\Delta_{\rm uu}, \tilde{\gamma}]$ and in Case 1 for $\Delta_{\rm d}$ in $(\delta, \Delta_{\rm uu}]$.

If $R \ge d(1+\lambda)$, it follows from Theorem 4 and Remark 1 that the cost function $C(\Delta_{\rm uu}, \Delta_{\rm d}) = C(\Delta_{\rm d})$ in these two intervals is given by

$$C(\Delta_{\rm d}) = \frac{p}{R}[\Delta_{\rm uu} Su(1+\lambda) + B_{\rm uu}] + \frac{1-p}{R}\left[\Delta_{\rm d} Sd(1+\lambda) + \frac{b_{\rm d}(\Delta_{\rm d})}{R}\right],$$

where

$$b_{\rm d}(\Delta_{\rm d}) = B_{\rm dd} - Sd^2(1+\lambda)(\Delta_{\rm d} - \Delta_{\rm dd}), \qquad 0 < p = \frac{R - d(1+\lambda)}{(u-d)(1+\lambda)} < 1.$$

We see that for $\delta < \Delta_{\rm d} < \tilde{\gamma}$,

$$C'(\Delta_{\rm d}) = \frac{(1-p)Sd(R-d)(1+\lambda)}{R^2} > 0.$$

Hence there is no minimum at $\Delta_{\rm uu}$ if $R \ge d(1+\lambda)$.

On the other hand, if $R < d(1+\lambda)$, then it follows from Theorem 4 that

$$C(\Delta_{\rm d}) = \Delta_{\rm d} + \frac{b_{\rm d}(\Delta_{\rm d})}{R^2},$$

which is linear in $(\delta, \tilde{\gamma}]$. Hence if there is a minimum at $\Delta_{\rm uu}$, there is also one at $\tilde{\gamma}$ and hence at $\Delta_{\rm dd}$ or at a solution of $f_{\rm d}(\gamma) = 0$.

So the conclusion in this case is that the minimum of $C(\Delta_{\rm d})$ occurs at one of the points δ, giving type (II)(b) of Theorem 7, or at $\Delta_{\rm dd}$ or at one of the solutions of $f_{\rm d}(\Delta_{\rm d}) = 0$.

2. We consider next the case $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) = 0$. Then $a_{\mathrm{d}} > 0$ for all $\Delta_{\mathrm{d}} \neq \Delta_{\mathrm{uu}}$, and $a_{\mathrm{u}} > 0$ for $\Delta_{\mathrm{d}} > \Delta_{\mathrm{uu}}$, and $a_{\mathrm{u}} < 0$ for $\Delta_{\mathrm{d}} < \Delta_{\mathrm{uu}}$ and $a_{\mathrm{u}}(\Delta_{\mathrm{uu}}) = a_{\mathrm{d}}(\Delta_{\mathrm{uu}}) = 0$. If $f_{\mathrm{d}}(\Delta_{\mathrm{dd}}) = 0$, then $f_{\mathrm{d}}(\Delta_{\mathrm{d}}) = 0$ if and only if $\Delta_{\mathrm{uu}} \leq \Delta_{\mathrm{d}} \leq \Delta_{\mathrm{dd}}$. As the function $b_{\mathrm{d}}(\Delta_d)$ is linear in any interval not containing Δ_{uu} or Δ_{dd}, we see from Remark 1 that the cost function $C(\Delta_{\mathrm{uu}}, \Delta_{\mathrm{d}}) = C(\Delta_{\mathrm{d}})$ is linear in any such interval. So the minimum must be achieved at one of these two points. If $f_{\mathrm{d}}(\Delta_{\mathrm{dd}}) \neq 0$, then $f_{\mathrm{d}}(\gamma) = 0$ has at most one more solution in addition to Δ_{uu}. Again the cost function $C(\Delta_{\mathrm{uu}}, \Delta_{\mathrm{d}}) = C(\Delta_{\mathrm{d}})$ is linear in any interval not containing Δ_{uu}, Δ_{dd}, or any of the γ's. So the minimum must be achieved at one of these points.

Suppose it is achieved at $\Delta_{\mathrm{d}} = \Delta_{\mathrm{uu}}$. Then the one-period claim

$$\{(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}/R), (\Delta_{\mathrm{d}}, b_{\mathrm{d}}(\Delta_{\mathrm{d}})/R)\} = \{(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}/R), (\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}/R)\}$$

is in Case 2 of Remark 1 so that by Theorem 4 the unique replicating portfolio $(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}/R^2)$ is the least cost superreplicating portfolio. This is type (II)(a) of Theorem 7.

3. We consider next the case $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) > 0$ and $R(1+\lambda) > u(1-\mu)$ so that f_{d} is strictly decreasing. Then $a_{\mathrm{d}} > 0$ for all $\Delta_{\mathrm{d}} \neq \Delta_{\mathrm{uu}}$, and $a_{\mathrm{u}} > 0$ for $\Delta_{\mathrm{d}} > \Delta_{\mathrm{uu}}$, and $a_{\mathrm{u}} < 0$ for $\Delta_{\mathrm{d}} < \Delta_{\mathrm{uu}}$ and $a_{\mathrm{u}}(\Delta_{\mathrm{uu}}) = a_{\mathrm{d}}(\Delta_{\mathrm{uu}}) = 0$. Then $f_{\mathrm{d}}(\gamma) = 0$ has exactly one solution γ which is greater than Δ_{uu}. Again we see from Remark 1 that the cost function $C(\Delta_{\mathrm{uu}}, \Delta_{\mathrm{d}}) = C(\Delta_{\mathrm{d}})$ is linear in any interval not containing Δ_{uu}, Δ_{dd}, or γ. So the minimum must be achieved at one of these points.

If the minimum is achieved at Δ_{uu}, we show as in the previous case that it is of type (II)(a) of Theorem 7.

4. We consider next the case $f_{\mathrm{d}}(\Delta_{\mathrm{uu}}) > 0$ and $R(1+\lambda) \leq u(1-\mu)$. Then $f_{\mathrm{d}}(\gamma) = 0$ has exactly one solution γ which is greater than Δ_{uu}, and there exists $\tilde{\delta} \geq \gamma$ ($\tilde{\delta} = \gamma$ if and only if $R(1+\lambda) = u(1-\mu)$) such that $a_{\mathrm{d}} > 0$ for $\Delta_{\mathrm{d}} < \Delta_{\mathrm{uu}}$, $a_{\mathrm{d}} \leq 0$ for $\Delta_{\mathrm{uu}} < \Delta_{\mathrm{d}} < \tilde{\delta}$, $a_{\mathrm{d}}(\tilde{\delta}) = 0$, $a_{\mathrm{d}} > 0$ for $\Delta_{\mathrm{d}} > \tilde{\delta}$. Also $a_{\mathrm{u}} > 0$ for $\Delta_{\mathrm{u}} > \Delta_{\mathrm{uu}}$, $a_{\mathrm{u}} < 0$ for $\Delta_{\mathrm{u}} < \Delta_{\mathrm{uu}}$, and $a_{\mathrm{u}}(\Delta_{\mathrm{uu}}) = a_{\mathrm{d}}(\Delta_{\mathrm{uu}}) = 0$. Again we see from Remark 1 that the cost function $C(\Delta_{\mathrm{uu}}, \Delta_{\mathrm{d}}) = C(\Delta_{\mathrm{d}})$ is linear in any interval not containing Δ_{uu}, Δ_{dd}, $\tilde{\delta}$, or γ. So the minimum must be achieved at one of these points.

If the minimum is achieved at Δ_{uu}, we show as in the previous case that it is of type (II)(a).

Suppose a minimum occurs at $\tilde{\delta}$. As $d(1+\lambda) < u(1-\mu)$ and taking into account the signs of a_{u} and a_{d}, the one-period claim

$\{(\Delta_{uu}, B_{uu}/R), (\Delta_d, b_d(\Delta_d)/R)\}$ is in Case 4 for Δ_d in $(\tilde{\delta}, \infty)$ and in Case 5 for Δ_d in $(c, \tilde{\delta}]$, where we take $c = \gamma$ if $R(1+\lambda) < u(1-\mu)$ and $c = \Delta_{uu}$ if $R(1+\lambda) = u(1-\mu)$. We also note that $f_d(\Delta_d) < 0$ in (γ, ∞) and $f_d(\Delta_d) > 0$ in (Δ_{uu}, γ).

If $R \geq d(1+\lambda)$, then the cost function $C(\Delta_{uu}, \Delta_d) = C(\Delta_d)$ in the two intervals $(c, \tilde{\delta}]$ and $(\tilde{\delta}, \infty)$ is given by

$$C(\Delta_d) = \frac{p}{R}[\Delta_{uu} S\bar{u} + B_{uu}] + \frac{1-p}{R}\left[\Delta_d Sd(1+\lambda) + \frac{b_d(\Delta_d)}{R}\right],$$

where

$$b_d(\Delta_d) = \begin{cases} B_{dd} + e(\Delta_d - \Delta_{dd})Sd^2 & \text{if } f(\Delta_d) < 0, \\ B_{uu} - Sud(1-\mu)(\Delta_d - \Delta_{uu}) & \text{if } f(\Delta_d) > 0, \end{cases}$$

$$0 < p = \frac{R - d(1+\lambda)}{\overline{u} - d(1+\lambda)} < 1, \qquad \overline{u} = \begin{cases} u(1-\mu) & \text{if } \Delta_d \in (c, \tilde{\delta}], \\ u(1+\lambda) & \text{if } \Delta_d \in (\tilde{\delta}, \infty). \end{cases}$$

So if $R(1+\lambda) < u(1-\mu)$, there is no minimum at $\tilde{\delta}$, because in (γ, ∞)

$$C'(\Delta_d) = \frac{(1-p)Sd}{R^2}\begin{cases} (R-d)(1+\lambda) & \text{if } \Delta_d < \Delta_{dd} \\ R(1+\lambda) - d(1-\mu) & \text{if } \Delta_d > \Delta_{dd} \end{cases} > 0.$$

If $R(1+\lambda) = u(1-\mu)$,

$$C'(\Delta_d) = \frac{(1-p)Sd}{R^2}[R(1+\lambda) - u(1-\mu)] = 0$$

in the interval $(\Delta_{uu}, \gamma]$ and so if there is a minimum at $\tilde{\delta}$, there is also one at γ or at Δ_{uu} which is type (II)(a) of Theorem 7.

That leaves us with the case $R < d(1+\lambda)$. Then for $\Delta_d > c$ we have the cost function

$$C(\Delta_d) = \Delta_d S + \frac{b_d(\Delta_d)}{R^2},$$

which is linear in any interval in (c, ∞) which does not contain Δ_{dd} or γ. Hence the minimum is also attained at γ or Δ_{dd} or Δ_{uu}. Thus, we conclude that if the minimum is attained at $\tilde{\delta}$, then it is also attained at γ or Δ_{dd} or Δ_{uu}, the latter being of type (II)(a) of Theorem 7.

By considering the above four cases, we have shown that there is always a minimum of type (II) or the minimum occurs at Δ_{dd} or at a solution of $f_d(\gamma) = 0$. We now consider the latter two possibilities in detail.

1. Suppose the minimum is assumed at $\Delta_d = \Delta_{dd}$ but $f_d(\Delta_{dd}) \neq 0$. If $f_d(\Delta_{dd}) > 0$, then, as $f'_d(\Delta_d) < 0$ for $\Delta_d > \Delta_{dd}$, there exists a unique $\gamma > \Delta_{dd}$ such that $f_d(\gamma) = 0$. This implies that there is a positive number ε such that $\Delta_{uu} < \Delta_{dd} - \varepsilon$ and such that $f_d(\Delta_d) \geq 0$ in $(\Delta_{dd} - \varepsilon, \gamma]$. Also in this interval $a_u > 0$ and throughout the interval either $a_d > 0$ or $a_d \leq 0$. So in the interval we are in one of the Cases 4, 5, 7, or 12 of Remark 1 and as also $b_d(\Delta_d)$ is linear in the interval, it follows also that $C(\Delta_d)$ must be linear and hence constant if the minimum is at Δ_{dd}. Therefore, if $f_d(\Delta_{dd}) > 0$ there is also a minimum at γ for which $f_d(\gamma) = 0$, which is the other possibility to be considered presently.

Suppose now that $f_d(\Delta_{dd}) < 0$. Then there exists $\tilde{\delta} \geq \Delta_{uu}$ such that $a_d \leq 0$ for $\Delta_{uu} < \Delta_d \leq \tilde{\delta}$ and $a_d > 0$ for $\Delta_d > \tilde{\delta}$. If $\tilde{\delta} < \Delta_{dd}$, we choose ε so that $\tilde{\delta} < \Delta_{dd} - \varepsilon$. Also we choose ε so that $\Delta_{uu} < \Delta_{dd} - \varepsilon$ and $f_d(\Delta_d) < 0$ in $(\Delta_{dd} - \varepsilon, \Delta_{dd})$. So throughout the latter interval, $a_d > 0$ when $\tilde{\delta} < \Delta_{dd}$ and $a_d \leq 0$ when $\tilde{\delta} \geq \Delta_{dd}$. As we also know that $a_u > 0$ for $\Delta_d > \Delta_{uu}$, it follows that in the interval $(\Delta_{dd} - \varepsilon, \Delta_{dd})$, we are in one of Cases 4, 7, or 12 if $a_d > 0$, and Case 5 if $a_d \leq 0$. Note also that $R < u(1-\mu)$ if $a_d \leq 0$, because we know that $R(1+\lambda) > u(1-\mu)$ implies that $a_d > 0$ for $\Delta_d > \Delta_{uu}$. Hence, reasoning as we did for the interval $(c, \tilde{\delta}]$ in Case 4 above, we find that $C'(\Delta_d) > 0$ in the interval $(\Delta_{dd} - \varepsilon, \Delta_{dd})$ if $R > d(1+\lambda)$. Then we must have $R \leq d(1+\lambda)$, in which case the initial holdings of the least cost superreplicating portfolio are $(\Delta_{dd}, B_{dd}/R^2)$. This is of type (I).

2. The final possibility is that the cost function $C(\Delta_d)$ has its minimum at $\Delta_d = \gamma$ for which $(\gamma, b_d(\gamma)/R)$ is a replicating portfolio for the one-period contingent claim $\{(\Delta_{uu}, B_{uu}), (\Delta_{dd}, B_{dd})\}$ with initial stock price Sd. If $\gamma < \Delta_{uu}$, then $f_d(\Delta_{uu}) < 0$ and so $a_u(\gamma) < 0$. Also we know $a_d(\gamma) > 0$. If $\gamma = \Delta_{uu}$, then $f_d(\Delta_{uu}) = 0$ and so $a_u(\gamma) = a_d(\gamma) = 0$. Hence, if $\gamma \leq \Delta_{uu}$, the one-period claim $\{(\Delta_{uu}, B_{uu}/R), (\gamma, b_d(\gamma)/R)\}$ is in Case 2 of Remark 1 and so the initial holdings in the least cost superreplicating portfolio are those for the unique replicating portfolio for this one-period claim. This is of type (IV).

In contrast, if $\gamma > \Delta_{uu}$, then $a_u(\gamma) > 0$ and if $a_d(\gamma) \leq 0$ then $R(1+\lambda) \leq u(1-\mu)$ so that $R \leq u(1-\mu)$. Thus the one-period claim $\{(\Delta_{uu}, B_{uu}/R), (\gamma, b_d(\gamma)/R)\}$ is in one of Cases 4, 7, or 12 of Remark 1 if $a_d > 0$ and in Case 5 if $a_d \leq 0$ with $R < u(1-\mu)$. Therefore if $R \geq d(1+\lambda)$, the initial holdings in the least cost superreplicating portfolio are those for the unique replicating portfolio for this one-period claim.

This is again of type (IV). On the other hand, if $R < d(1+\lambda)$, the initial holdings are $(\gamma, b_{\mathrm{d}}(\gamma)/R^2)$. This is of type (III).

So the proof of the theorem is complete.

5. An Example with Path-Dependent Least Cost Superreplicating Portfolios

In a two-period binomial model with parameters S, u, d, R, λ, and μ, we consider a *short position in a put option* with exercise price K satisfying

$$Sd^2 < K < Sud.$$

This is the contingent claim $\{(0,0), (0,0), (1, -K)\}$. It follows from Theorem 7 that there is a least cost superreplicating portfolio and we need only consider the following possibilities for such a portfolio:

(I) the initial holdings are $(1, -K/R^2)$ (only arises if $R < d(1+\lambda)$ and $K < Sud(1-\mu)$);

(II) the initial holdings are (δ, B), where $\delta \leq 0$ and there are two possibilities: $\delta = 0$ which only occurs if $K \geq Sd^2(1+\lambda)$ and then $B = 0$ also; $\delta < 0$ which only occurs if $K < Sd^2(1+\lambda)$ and then δ and B satisfy $BR = -\delta Su(1+\lambda)$ and $BR^2 - (1-\delta)Sd^2(1+\lambda) = -K$;

(III) the initial holdings are $(\alpha, B/R)$, where $\alpha > 0$ and (α, B) are the initial holdings in a replicating portfolio for the one-period portion $\{d, du, dd\}$ (only arises if $R < d(1+\lambda)$);

(IV) a replicating portfolio for the whole two-period model.

Example.

Consider a two-period model with $u = 1.1$, $d = 0.95$, $R = 1.05$, $\lambda = \mu = 0.06$, and $S = 100$. Consider the put with exercise price 93 which is between 90.25 and 104.50. A short position in this put is the claim $\{(0, 0), (0, 0), (1, -93)\}$.

		121	(0, 0)
	110		
100		104.50	(0, 0)
	95		
		90.25	(1, −93)

Note first that as $R > d(1+\lambda)$, we do not need to consider (I) or (III). We consider (II) first. As $K < Sd^2(1+\lambda)$, the only possibility is that there exist $\delta < 0$ and B such that

$$BR^2 = -\delta SuR(1+\lambda) = (1-\delta)Sd^2(1+\lambda) - K,$$

that is,

$$1.05^2 B = -\delta 110 \times 1.05 \times 1.06 = (1-\delta) \times 90.25 \times 1.06 - 93.$$

We solve the last equation to get $\delta = -0.0996$ and then $B = 12.1940/1.05^2$. Then the initial holdings are $(-0.0996, 12.1940/1.05^2)$, which has cost 1.1003.

The only other possibility is (IV), a replicating portfolio for the whole model. We find that the one-period claim $\{(0,0),(1,-93)\}$ with initial stock price 95 has the unique replicating portfolio $(-0.1764, 18.1627)$. Next we need to determine the replicating portfolio for the one-period claim $\{(0,0),(-0.1764, 18.1627)\}$ with initial stock price 100. It turns out that this has cost 1.16. Hence the least cost is 1.1003. Note that this least cost superreplicating portfolio is path-dependent.

Now we show that the example just given is a special case of a situation in which there is a unique least cost superreplicating portfolio and it is path-dependent.

Theorem 8. *Consider a two-period binomial model with parameters* S, u, d, R, μ*, and* λ *satisfying*

$$d(1+\lambda) < u(1-\mu), \qquad R(1-\mu) < d(1+\lambda) < R.$$

For every contingent claim with terminal holdings $\{(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}), (\Delta_{\mathrm{ud}}, B_{\mathrm{ud}}), (\Delta_{\mathrm{dd}}, B_{\mathrm{dd}})\}$ *satisfying* $\Delta_{\mathrm{uu}} = \Delta_{\mathrm{ud}} < \Delta_{\mathrm{dd}}$, $B_{\mathrm{uu}} = B_{\mathrm{ud}}$*, and*

$$B_{\mathrm{dd}} - B_{\mathrm{uu}} - Sd^2(1+\lambda)(\Delta_{\mathrm{uu}} - \Delta_{\mathrm{dd}}) > 0, \tag{4}$$

there exists a unique least cost superreplicating portfolio. Moreover, this portfolio is path-dependent.

Proof. Let $(\Delta_{\mathrm{u}}, B_{\mathrm{u}})$ and $(\Delta_{\mathrm{d}}, B_{\mathrm{d}})$ be the holdings in a least cost superreplicating portfolio at the end of the first period. Then we know that the initial holdings (Δ, B) form a least cost superreplicating portfolio for the one-period claim $\{(\Delta_u, b_u(\Delta_u)/R), (\Delta_d, b_d(\Delta_d)/R)\}$ with initial stock price S. We have to show that there is just one possibility for $\{(\Delta, B), (\Delta_u, B_u), (\Delta_d, B_d)\}$.

We first show it is necessary that

$$\Delta_u = \Delta_{uu}.$$

As $d(1+\lambda) < u(1-\mu)$, the one-period claim $\{(\Delta_u, b_u(\Delta_u)/R), (\Delta_d, b_d(\Delta_d)/R)\}$ is in one of Cases 1–6 of Remark 1. Denote by $C(\Delta_u, \Delta_d)$ the cost of its least cost superreplicating portfolio. As $R > d(1+\lambda)$, it follows from Theorem 4(a) that the least cost superreplicating portfolio for the one-period claim is either the unique replicating portfolio with corresponding p satisfying $0 < p < 1$ or $(\Delta_u, b_u(\Delta_u)/R^2)$. Hence, referring to the proof of Theorem 5.1 of Chen *et al.* (2004), where here we observe that $\alpha_u = \beta_u = \Delta_{uu}$, we find that for fixed Δ_d,

$$\frac{\partial C}{\partial \Delta_u} \begin{cases} < 0 & \text{if } \Delta_u < \Delta_{uu}, \\ > 0 & \text{if } \Delta_u > \Delta_{uu}, \end{cases}$$

and so we must have $\Delta_u = \Delta_{uu}$ at a minimum. This also means that $b_u(\Delta_u) = b_u(\Delta_{uu}) = B_{uu}$.

Next we determine the zeros of the function f_d. As $d(1+\lambda) < u(1-\mu)$, $f_d(\Delta_d)$ is strictly decreasing and Equation (4) says that $f_d(\Delta_{uu}) < 0$. Hence there is a unique γ such that $f_d(\gamma) = 0$ and $\gamma < \Delta_{uu}$. So

$$b_d(\Delta_d) = \begin{cases} B_{uu} - Sud(1+\lambda)(\Delta_d - \Delta_{uu}) & \text{if } \quad \Delta_d \le \gamma, \\ B_{dd} - Sd^2(1+\lambda)(\Delta_d - \Delta_{dd}) & \text{if } \gamma \le \Delta_d \le \Delta_{dd}, \\ B_{dd} - Sd^2(1-\mu)(\Delta_d - \Delta_{dd}) & \text{if } \quad \Delta_{dd} \le \Delta_d. \end{cases}$$

Next it follows from the proof of Theorem 7 that there is a δ with $\gamma < \delta < \Delta_{dd}$ such that

$$a_u(\Delta_d) = \begin{cases} < 0 & \text{if } \Delta_d < \delta, \\ = 0 & \text{if } \Delta_d = \delta, \\ > 0 & \text{if } \delta < \Delta_d, \end{cases}$$

and $a_d(\Delta_d) > 0$ for all Δ_d.

Hence if $\Delta_d \le \delta$, we have $a_d > 0$ and $a_u < 0$ and the one-period claim $\{(\Delta_{uu}, B_{uu}/R), (\Delta_d, b_d(\Delta_d)/R)\}$ is in Case 2 of Remark 1. If $\delta < \Delta_d$, we have $a_d > 0$ and $a_u > 0$ and the claim is in Case 1 or 4 of Remark 1. In all cases, it follows from Theorem 4 and Remark 1 that the least cost superreplicating portfolio for the claim is the unique replicating portfolio so that the least cost C is given by

$$C = \frac{p}{R}\left[\Delta_{\mathrm{uu}} Su(1+\lambda) + \frac{B_{\mathrm{uu}}}{R}\right] + \frac{1-p}{R}\left[\Delta_{\mathrm{d}} S\bar{d} + \frac{b_{\mathrm{d}}(\Delta_{\mathrm{d}})}{R}\right],$$

where

$$p = \frac{R-\bar{d}}{u(1+\lambda)-\bar{d}}, \quad \bar{d} = \begin{cases} d(1-\mu) & \text{if } \Delta_{\mathrm{d}} \leq \delta, \\ d(1+\lambda) & \text{if } \Delta_{\mathrm{d}} \geq \delta. \end{cases}$$

Hence

$$\frac{\partial C}{\partial \Delta_{\mathrm{d}}} = \frac{1-p}{R^2}\begin{cases} SRd(1-\mu) - Sud(1+\lambda) < 0 & \text{if } \Delta_{\mathrm{d}} < \gamma, \\ SRd(1-\mu) - Sd^2(1+\lambda) < 0 & \text{if } \gamma < \Delta_{\mathrm{d}} < \delta, \\ SRd(1+\lambda) - Sd^2(1+\lambda) > 0 & \text{if } \delta < \Delta_{\mathrm{d}} < \Delta_{\mathrm{dd}}, \\ SRd(1+\lambda) - Sd^2(1-\mu) > 0 & \text{if } \Delta_{\mathrm{d}} > \Delta_{\mathrm{dd}}. \end{cases}$$

It follows that the unique minimum is achieved at δ.

Thus, we have shown that $C(\Delta_{\mathrm{u}}, \Delta_{\mathrm{d}})$ achieves its unique minimum at $(\Delta_{\mathrm{uu}}, \delta)$. This means that a least cost superreplicating portfolio for our two-period claim has share holdings Δ_{uu} and δ at the end of the first period and initial holdings which constitute a least cost superreplicating portfolio for the one-period claim $\{(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}/R), (\delta, b_{\mathrm{d}}(\delta)/R)\}$. Morever, as seen above, the least cost superreplicating portfolio for this one-period claim is the unique replicating portfolio $(\delta, b_{\mathrm{d}}(\delta)/R^2)$.

Now we show that this portfolio, consisting of initial holdings $(\delta, b_{\mathrm{d}}(\delta)/R^2)$ and end of first-period holdings $(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}/R)$ and $(\delta, b_{\mathrm{d}}(\delta)/R)$, is path-dependent. If it were path-independent, there would be terminal holdings (Δ, B) in the ud state with $\Delta \geq \Delta_{\mathrm{uu}}$, $B \geq B_{\mathrm{uu}}$ such that when the stock price moves from Su to Sud we could rebalance the holdings $(\Delta_{\mathrm{uu}}, B_{\mathrm{uu}}/R)$ in state u to get (Δ, B), and when the stock price moves from Sd to Sud we could rebalance the holdings $(\delta, b_{\mathrm{d}}(\delta)/R)$ in state d to get (Δ, B) also. Thus, we would need

$$B = B_{\mathrm{uu}} - (\Delta - \Delta_{\mathrm{uu}})Sud(1+\lambda) = b_{\mathrm{d}}(\delta) - (\Delta - \delta)Sud(1+\lambda).$$

As $\Delta \geq \Delta_{\mathrm{uu}}$ and $B \geq B_{\mathrm{uu}}$, the first equation implies that $\Delta = \Delta_{\mathrm{uu}}$, $B = B_{\mathrm{uu}}$ and so the second equation can be written as

$$B_{\mathrm{uu}} = b_{\mathrm{d}}(\delta) - (\Delta_{\mathrm{uu}} - \delta)Sud(1+\lambda).$$

However, $a_{\mathrm{u}}(\delta) = 0$. Thus,

$$(\delta - \Delta_{\mathrm{uu}})Su(1+\lambda) + \frac{b_{\mathrm{d}}(\delta) - B_{\mathrm{uu}}}{R} = 0.$$

From the last two equations we find that $R = d$. This contradiction shows that the least cost superreplicating portfolio is path-dependent.

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