

# Chapter 1

## Introduction

The classical theta function, defined for  $\text{Im } \tau > 0$  by

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau},$$

satisfies the modular transformation law

$$\theta(-1/\tau) = (\tau/i)^{\frac{1}{2}} \theta(\tau). \quad (1.1)$$

Perhaps the best-known way to derive the functional equation of the Riemann zeta function  $\zeta(s)$ ,

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{(s-1)/2} \Gamma(\{1-s\}/2) \zeta(1-s), \quad (1.2)$$

is by way of (1.1) [107, p. 22]. Conversely, it is not difficult to show that (1.2) implies (1.1), but this derivation requires the use of the Phragmén-Lindelöf Theorem [105, §5.65]. In 1921 H. Hamburger [38] showed that, under certain auxiliary analytic conditions,  $\zeta(s)$  is essentially the only solution to the functional equation (1.2). For a more transparent proof, see C. L. Siegel's paper [99], [100, pp. 154–156]. More specifically, they proved that if  $f(s)$  is a Dirichlet series satisfying the aforementioned auxiliary restrictions, and if

$$R(s) = \pi^{-s} \Gamma(s) f(2s), \quad R(s) = R\left(\frac{1}{2} - s\right), \quad (1.3)$$

then  $f(s)$  is a constant multiple of  $\zeta(s)$ . See also [107, pp. 31–32]. That  $f(2s)$ , as opposed to  $f(s)$ , appears in (1.3) guarantees *a priori* that the inverse Mellin transform of  $R(s)$ , an exponential series, has its coefficient sequence supported on integral squares, and thus it has the general shape of

$\theta(\tau) - 1$ . The proof of Hamburger's Theorem is then completed by showing that this inverse Mellin transform is in fact a constant multiple of  $\theta(\tau) - 1$ .

Of even greater interest within the context of the present work is a second, distinct version, due to Hecke, of the Hamburger theorem. This version is, in fact, a direct consequence of a general correspondence theorem proved by Hecke in 1936 [47] (the "main correspondence theorem" of Chapter 2, below) and the fact that, under certain conditions of regularity,  $\theta(\tau)$  is the only solution to (1.1) that is periodic (with period 2). For further details concerning the two formulations of Hamburger's theorem, see the introduction to Hecke's final published paper [49], [62, esp. pp. 201–207], and the **Application** following Remark 7.4.

Throughout the sequel we let  $\tau = x + iy$  and  $s = \sigma + it$  with  $x, y, \sigma$ , and  $t$  real. We denote the upper half-plane,  $\{\tau : y > 0\}$ , by  $\mathcal{H}$ . The set of all complex numbers will be denoted by  $\mathbb{C}$ , the set of all real numbers by  $\mathbb{R}$ , the set of all rational numbers by  $\mathbb{Q}$ , and the set of all rational integers by  $\mathbb{Z}$ . We adopt the following *argument convention*: for  $w \in \mathbb{C}$ ,  $w \neq 0$ , and  $k \in \mathbb{R}$ ,  $w^k$  is defined by

$$w^k = |w|^k e^{ik \arg w}, \quad -\pi \leq \arg w < \pi. \quad (1.4)$$

The summation sign  $\sum$  with no indices always means  $\sum_{n=1}^{\infty}$ . We write  $\int_{(c)}$  for  $\int_{c-i\infty}^{c+i\infty}$ , where  $c$  is real and the path of integration is the straight line from  $c - i\infty$  to  $c + i\infty$ . We often use the symbol  $A$  to denote a positive constant, not necessarily the same with each occurrence.