

Chapter 1

Lattice Structures and Discretizations

Quantum systems on discrete spaces have received much interest in many areas like electrons on lattices under the influence of external fields, quantum optics, electronics, neuronal networks, signal processing, mesoscopic systems and superlattices. Starting with the discrete space or time one gets faced with the formulation of appropriate discretization schemes. For this purpose discrete variables $x \in \mathbf{Z}$ are almost useful. Other related equations, such as q -difference ones are able to be implied by modern mathematical developments such as quantum groups, the covariant calculus on noncommutative spaces, and R -matrix descriptions (see Chari and Pressley (1994), Faddeev et al (1990)). Here $0 < q < 1$, stands for the generic deformation parameter, but complex q -values and especially roots of unity can also be considered. In addition, there are q -difference equations which are incorporated naturally into the description of actual physical systems such as Bloch-electrons on two- or three-dimensional lattices threaded by a magnetic field, as already referred to above. Moreover, many systems are able to be characterized by inherent length scales which can be used, at least in principle, for subsequent discretizations, too.

1.1 Discrete derivatives

Assuming a function on a lattice with the spacing a , say $f(x)$, we may introduce left and right hand difference quotients as

$$\delta_- f(x) = \frac{f(x) - f(x-a)}{a} = \frac{1}{a} \left(1 - \exp \left(-a \frac{d}{dx} \right) \right) f(x) \quad (1.1)$$

and

$$\delta_+ f(x) = \frac{f(x+a) - f(x)}{a} = \frac{1}{a} \left(\exp\left(a \frac{d}{dx}\right) - 1 \right) f(x) \quad (1.2)$$

respectively. Such quotients, which can also be viewed as discrete derivatives, reproduce the usual one to $O(a^2)$ -order. The discretization proceeds in terms of the identification $x = n_i a$, where x_i is an integer, i.e. $n_{i+1} = n_i + 1$. Then the above quotients are replaced by the discrete derivatives

$$\nabla f(x) = f(x) - f(x-1) \quad (1.3)$$

and

$$\Delta f(x) = f(x+1) - f(x) \quad (1.4)$$

respectively, where x stands hereafter for n_i . Accordingly

$$\nabla \Delta f(x) = \Delta \nabla f(x) = f(x+1) - 2f(x) + f(x-1) \quad (1.5)$$

represents the discretized version of the second derivative. On the other hand one starts from the common assumption that the usual dimensionless momentum operator $i\partial/\partial x$ is Hermitian, so that the discrete derivative Δ behaves as $\Delta^+ = -\nabla$ under Hermitian conjugation. Accordingly, the second order discrete derivative operator $\nabla \Delta$ displayed above is itself Hermitian.

We have to remark that the Leibniz-rule for the differentiation of a product is modified as follows

$$\nabla (f(x)g(x)) = f(x)\nabla g(x) + g(x-1)\nabla f(x) \quad (1.6)$$

and

$$\Delta (f(x)g(x)) = f(x)\Delta g(x) + g(x+1)\Delta f(x) \quad (1.7)$$

respectively, whereas

$$\Delta^n f(x) = \sum_{k=0}^n \frac{n! (-1)^k}{(n-k)! k!} f(x+n-k) \quad (1.8)$$

and

$$\nabla^n f(x) = \sum_{k=0}^n \frac{n! (-1)^k}{k! (n-k)!} f(x-k) \quad . \quad (1.9)$$

Such formulae are quite useful for concrete computations. In this context hypergeometric type second-order difference equations like

$$\sigma(x)\nabla\Delta P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0 \quad (1.10)$$

where $\sigma(x)$ ($\tau(x)$) is an at most second-degree (first degree) polynomial, have been studied in some more detail by Nikiforov, Suslov and Uvarov (1991).

It is understood that (1.10) can be rewritten equivalently as a second-order discrete equation:

$$(\sigma(x) + \tau(x)) P_n(x+1) + \sigma(x) P_n(x-1) + (\lambda - \tau(x) - 2\sigma(x)) P_n(x) = 0 \quad (1.11)$$

which produces hypergeometric type solutions like Hahn, Chebyshev, Meixner, Krawtchouk or Charlier polynomials for selected forms of $\sigma(x)$ - and $\tau(x)$ -functions (Nikiforov, Suslov and Uvarov (1991)). Main formulae concerning such polynomials are presented in section 1.7 as well as in Appendix A.

1.2 The Jackson derivative

Proceeding in a different manner, Jackson (1909) proposed long ago the generalized q -derivative

$$\partial_x^{(q)} f(x) \equiv \frac{d_q(x)}{d_q x} = \frac{f(qx) - f(x)}{x(q-1)} \quad (1.12)$$

working irrespective of x , which reproduces the usual one if $q \rightarrow 1$. Now the Leibniz-rule is modified as

$$\partial_x^{(q)} (f(x)g(x)) = f(x)\partial_x^{(q)} g(x) + g(qx)\partial_x^{(q)} f(x) \quad (1.13)$$

which looks like (1.6). In particular the q -derivative of a power-like function x^n is given by

$$\partial_x^{(q)} x^n = [[n]]_q x^{n-1} \quad (1.14)$$

where

$$[[n]]_q = \frac{q^n - 1}{q - 1} \quad (1.15)$$

denotes the related quantum number. It is clear that $[[n]]_q \rightarrow n$ as $q \rightarrow 1$. Relatedly, one says that under q -deformation the c -number n is replaced by the quantum-number $[[n]]_q$. We have to be aware that (1.12) opens the way to the formulation of q -analysis. Accordingly special functions are substituted by q ones, and the same concerns, of course the hypergeometric series (Gasper and Rahman (1990)). One realizes that the q -derivative written down above is not unique. Indeed, we can also introduce the symmetrized derivative (see e.g. Kulish and Damaskinsky (1990))

$$\mathcal{D}_x^{(q)} f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})} \quad (1.16)$$

which yields

$$\mathcal{D}_x^{(q)} x^n = [n]_q x^{n-1} \quad (1.17)$$

instead of (1.14), where now

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (1.18)$$

However, one has

$$[n]_q = q^{1-n} [[n]]_{q^2} \quad (1.19)$$

which shows that such quantum numbers are actually inter-related. Moreover, we have to remark that

$$\Delta f(x) = z(q - 1) \partial_z^{(q)} \varphi(z) \quad (1.20)$$

where $z = q^x$ and $f(x) = \varphi(z)$. This relationship is able to serve to the conversion of discrete equations into a q -difference ones and conversely.

Needless to say that q -analysis is far from being trivial. So, the q -binomial theorem reads (see Bailey (1935))

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}} \quad (1.21)$$

in which “ a ” denotes a constant parameter. Accordingly, a q -analog of the gamma function is given by

$$\Gamma_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1 - q)^{1-x} \quad (1.22)$$

which has also been expressed before in terms of the q -integral proposed by Jackson (1910). For the convergence sake one should consider that $0 < q < 1$ both in (1.21) and (1.22). In the above formulae $(a; q)_n$ denotes the q -shifted factorial

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \quad (1.23)$$

where $(a; q)_0 = 1$ so that

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n) \quad . \quad (1.24)$$

We have to remark that the present q -Gamma function satisfies the typical functional equation

$$\Gamma_q(x + 1) = [[x]]_q \Gamma_q(x) \quad (1.25)$$

which confirms in turn the relevance of its definition via (1.22). The infinite products quoted above yield a shifted factorial as follows

$$(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}} \quad (1.26)$$

which is now valid for arbitrary complex n -values.

Before proceeding further we would like to say that the Jackson derivative has also been rediscovered recently in terms of the radial reduction of the covariant derivative characterizing the quantum-group $SO_q(N)$, as illustrated by q -deformed radial Schrödinger equations established before (Carow-Watamura and S. Watamura (1997), Papp (1997)).

1.3 The q -integral

In order to perform scalar products and suitable normalizations we have to apply the Jackson q -integral (Gasper and Rahman (1990), Jackson (1910)). So one has

$$\int_0^a f(x) d_q x = a(1-q) \sum_{k=0}^{\infty} f(aq^k) q^k \quad (1.27)$$

if $0 < q < 1$, but

$$\int_0^a f(x) d_q x = a(1-q) \sum_{k=0}^{\infty} \frac{1}{q^{k+1}} f\left(\frac{a}{q^{k+1}}\right) \quad (1.28)$$

if $q > 1$. Inserting e.g. $f(x) = x^n$ yields

$$\int_0^a x^n d_q x = \frac{a^{n+1}}{[[n+1]]_q} \quad (1.29)$$

in both cases. One sees that (1.14) and (1.29) express inverse operations, as one might expect. Furthermore, there is

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{k=-\infty}^{+\infty} f(q^k) q^k \quad (1.30)$$

if $0 < q < 1$, which can be generalized on the whole real axis as

$$\int_{-\infty}^{+\infty} f(x) d_q x = (1-q) \sum_{k=-\infty}^{+\infty} [f(q^k) + f(-q^k)] q^k \quad (1.31)$$

Of course, one realizes that convergence conditions $0 < q < 1$ and $q > 1$ mentioned above can be generalized as $|q| < 1$ and $|q| > 1$, respectively.

The symmetrized versions of (1.27) and (1.28) are given by

$$\int_0^a f(z) D_q z = \left(\frac{1}{q} - q\right) \sum_{k=0}^{\infty} aq^{2k+1} f(aq^{2k+1}) \quad (1.32)$$

and

$$\int_0^a f(z) D_q z = \left(q - \frac{1}{q}\right) \sum_{k=0}^{\infty} \frac{a}{q^{2k+1}} f\left(\frac{a}{q^{2k+1}}\right) \quad (1.33)$$

if $|q| < 1$ and $|q| > 1$, respectively. So, one obtains

$$\int_0^a z^n D_q z = \frac{a^{n+1}}{[n+1]_q} \quad (1.34)$$

which is valid again in both cases. This reproduces, of course, (1.29) up to the selection of the appropriate quantum number. Integrations by parts can also be easily done. Using (1.13) one finds e.g.

$$\int_0^a f d_q g = f(x) g(x)|_0^a - \int_0^a g(qx) d_q f \quad (1.35)$$

which works both for $|q| < 1$ and $|q| > 1$. One remarks that the discrete counterpart of (1.35) reads

$$\sum_{x=a}^{b-1} f(x) \Delta g(x) = f(x) g(x)|_a^b - \sum_{x=a}^{b-1} g(x+1) \Delta f(x) \quad (1.36)$$

where x stands again for the discrete variable and which proceeds in accord with (1.7). A similar equation can be derived for the left difference operator ∇ .

1.4 Generalized q -hypergeometric functions

Many solutions of discrete equations are expressed in terms of the generalized hypergeometric function (Gasper and Rahman (1990), Gradshteyn and Ryzhik (1965))

$${}_{p_1}F_{p_2} \left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{p_1} \\ \beta_1, \beta_2, \dots, \beta_{p_2} \end{matrix} \middle| z \right) = \quad (1.37)$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_{p_1})_k}{(\beta_1)_k \cdots (\beta_{p_2})_k} \frac{z^k}{k!}$$

where p_1 and p_2 are indices and where α_j and β_j are parameters. The classical shifted factorial is given by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1) \quad (1.38)$$

as usual, where $(a)_0 = 1$. Recall that the usual Gaussian hypergeometric function is

$$F(\alpha, \beta, \gamma; z) = {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} \middle| z\right) \quad (1.39)$$

whereas the confluent one is given by

$$F(\alpha, \gamma; z) = {}_1F_1\left(\begin{matrix} \alpha \\ \gamma \end{matrix} \middle| z\right) \quad (1.40)$$

Resorting for convenience, to the quantum number (1.15), we shall perform the q -generalization of (1.37) in terms of the substitutions

$$(a)_k \rightarrow \frac{(q^a, q)_k}{(1-q)^k} = \prod_{l=0}^{k-1} [[a+l]]_q \quad (1.41)$$

and

$$k! \rightarrow [[k]]_q! = \frac{(q, q)^k}{(1-q)^k} = \prod_{l=1}^k [[l]]_q \quad (1.42)$$

This leads to the q -generalization

$$\begin{aligned} & {}_{p_1}F_{p_2}^{(q)}\left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{p_1} \\ \beta_1, \beta_2, \dots, \beta_{p_2} \end{matrix} \middle| \tilde{z}\right) = \\ & = \sum_{k=0}^{\infty} \frac{(q^{\alpha_1}, q)_k \cdots (q^{\alpha_{p_1}}, q)_k}{(q^{\beta_1}, q)_k \cdots (q^{\beta_{p_2}}, q)_k} \frac{\tilde{z}^k}{(q, q)_k} \end{aligned} \quad (1.43)$$

which reproduces (1.37) as $q \rightarrow 1$, where

$$\tilde{z} = (1-q)^{p_2-p_1+1} z \quad (1.44)$$

In order to identify the generalized q -hypergeometric function it is convenient to rewrite (1.43) as follows

$$\begin{aligned}
 {}_{p_1}F_{p_2}^{(q)} \left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{p_1} \\ \beta_1, \beta_2, \dots, \beta_{p_2} \end{matrix} \middle| \gamma \tilde{z} \right) &= \quad (1.45) \\
 &= \sum_{k=0}^{\infty} C_k \tilde{z}^k
 \end{aligned}$$

where γ is an additional factor. One would then obtain the recurrence relation

$$C_{k+1} = \gamma \frac{(1-q)^{p_2-p_1+1} (1-q^{\alpha_1+k}) \dots (1-q^{\alpha_{p_1}+k})}{(1-q^{k+1}) (1-q^{\beta_1+k}) \dots (1-q^{\beta_{p_2}+k})} C_k \quad (1.46)$$

which can also be used in order to check the appearance of polynomial solutions for which $C_{n+1} = 0$. The q -exponential function is also of a special interest, but it will be discussed in section 3.6 in a close connection with the q -deformation of the time evolution.

1.5 The discrete space-time: a short retrospect

The basic idea about a discrete space and/or time is as old as the very beginning of the scientific thinking. The old Greeks speculated about the generalization of the atomistic structure of matter to space-time, as illustrated in an excellent volume by Vialtzew (1965). Such steps deserve an actual appreciation even from the perspective of contemporary physics. With the advent of quantum mechanics, the question of whether subdivisions of space and time intervals can be indefinitely performed or not, has focussed increasing interest. The latter alternative, which leads to the introduction of space-time quanta proceeding in a close connection with ultimate accuracies of space-time measurements, is looks almost promising. Pioneering work along this direction has been done by many authors, but here we would like just to remember specifically contributions done by Poincaré (1913), Planck (1913), Ambarzumian and Iwanenko (1930), March (1937), Heisenberg (1938a,1938b), Wheeler (1957), Brill and Gowdy (1970) and Finkelstein (1997). In ‘‘Physics and Reality’’ Einstein said:

‘‘To be sure, it has been pointed out that the introduction of a spacetime continuum may be considered as contrary to nature in view of the molecular

structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraical method of description of nature, that is to the elimination of continuum functions from physics. Then, however, we must also give up, by principle, the spacetime continuum" (see also Atakishiyev and Suslov (1990)).

Further developments rely on a celebrated paper by Snyder (1947) dealing with the introduction of a Lorentz invariant space-time. We then have to consider a discrete space-time in which there exists a smallest unit of length, say $a \neq 0$. The basic expectation is that accounting for a such length leads to the elimination of divergence difficulties plaguing quantum field theory. Introducing a such length in space-time yields, however, non-commutative relationships between x, y, z and t , so that space-time coordinates have to be considered by now as observables. More exactly, they are identified with the generators of the group of transformations leaving invariant the five-dimensional de Sitter quadratic form (see also Yang (1947))

$$R_5^2 = -\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 \quad (1.47)$$

in which the η 's are assumed to be real variables. One would then obtain Hermitian space-time generators as follows

$$X_1 = ia \left(\eta_4 \frac{\partial}{\partial \eta_1} - \eta_1 \frac{\partial}{\partial \eta_4} \right) \quad (1.48)$$

$$X_2 = ia \left(\eta_4 \frac{\partial}{\partial \eta_2} - \eta_2 \frac{\partial}{\partial \eta_4} \right) \quad (1.49)$$

$$X_3 = ia \left(\eta_4 \frac{\partial}{\partial \eta_3} - \eta_3 \frac{\partial}{\partial \eta_4} \right) \quad (1.50)$$

and

$$\tilde{T}_0 = \frac{ia}{c} \left(\eta_4 \frac{\partial}{\partial \eta_0} - \eta_0 \frac{\partial}{\partial \eta_4} \right) . \quad (1.51)$$

In addition, there are three operators like

$$L_k = -i\hbar \varepsilon_{kls} \eta_l \frac{\partial}{\partial \eta_s} \quad (1.52)$$

where $k, l, s = 1, 2, 3$ and where the usual summation convention is assumed. These latter operators correspond to the . Further three operators like

$$M_1 = i\hbar \left(\eta_0 \frac{\partial}{\partial \eta_1} + \eta_1 \frac{\partial}{\partial \eta_0} \right) \quad (1.53)$$

$$M_2 = i\hbar \left(\eta_0 \frac{\partial}{\partial \eta_2} + \eta_2 \frac{\partial}{\partial \eta_0} \right) \quad (1.54)$$

and

$$M_3 = i\hbar \left(\eta_0 \frac{\partial}{\partial \eta_3} + \eta_3 \frac{\partial}{\partial \eta_0} \right) \quad (1.55)$$

bearing on the Lorentz-boosts, have also to be considered. Accordingly, one finds the commutation relations

$$[X_k, X_l] = i \frac{a^2}{\hbar} \varepsilon_{kls} L_s \quad (1.56)$$

and

$$[\tilde{T}_0, X_k] = i \frac{a^2}{\hbar c} M_k \quad (1.57)$$

which are responsible for the discrete space-time and which reproduce the space-time continuum as soon as $a \rightarrow 0$. It is understood that L_k and M_k commutes with the Minkowski quadratic form

$$S_4^2 = c^2 \tilde{T}_0^2 - X_1^2 - X_2^2 - X_3^2 \quad (1.58)$$

provided (1.48)-(1.51) are valid. This means that the discrete space-time defined in this manner is itself Lorentz-invariant.

The eigenvalues, say x'_k , characterizing the discrete space are given by

$$x'_k = n_k a \quad (1.59)$$

where $n_k \in \mathbb{Z}$, which reveals the space discreteness in an explicit manner. The \tilde{T}_0 operator has, however, a continuous spectrum, which shows, so far, that the discreteness of space does not imply necessarily a discrete time. In this context a continuous time can be preserved, however, for reasons of mathematical simplicity. Either case, (1.59) shows that there are actual interplays between relativity and space-discreteness. Several ideas and approaches to the discrete space have also been sketched in an unpublished paper by Heisenberg in 1930 (see also Carazza and Kragh (1995)). In this

context the fundamental length has also been conceived as the absolute minimum of the position uncertainty of a particle.

Other manifestations of the fundamental length have been done by defining covariant space-time and four-momentum operators as (Hellund and Tanaka (1954))

$$X_\mu = \tilde{x}_\mu - a^2 \frac{\partial}{\partial \tilde{x}_\mu} \tilde{x}_\nu \frac{\partial}{\partial \tilde{x}_\nu} \quad (1.60)$$

and

$$P_\mu = \tilde{p}_\mu - i\hbar \frac{\partial}{\partial \tilde{x}_\mu} \quad (1.61)$$

respectively, where \tilde{x}_μ and \tilde{p}_μ are c -numbers like $\tilde{x}_\mu = (x, y, z, ict)$ and $\tilde{p}_\mu = (p_x, p_y, p_z, iE/c)$. For convenience, the same quotations for space-time operators have been preserved. This yields the commutation relations

$$[X_\mu, P_\nu] = i\hbar \left(\delta_{\mu\nu} + \frac{a^2}{\hbar^2} P_\mu P_\nu \right) \quad (1.62)$$

in which case Heisenberg's uncertainty principle is generalized as follows

$$\Delta X_\mu \Delta P_\nu \geq \hbar \left(1 + \frac{a^2}{\hbar^2} P_\mu^2 \right) \quad (1.63)$$

for $\mu = \nu$ (no summation). Moreover, there is

$$\Delta X_1 \Delta \tilde{T}_0 \geq \frac{a^2}{c} \quad (1.64)$$

which shows that elementary space-time volumes have also to be considered.

Other evolutions could also be mentioned (see Kadyshevsky (1961)), but the main point is that accounting for (1.59) leads to the introduction of discrete-equations serving as starting points for an improved theoretical descriptions in several respects. Interestingly enough, similar results are provided by q -deformed oscillators for which q is a root of unity, as shown by Bonatsos et al (1994). Then position and momentum operators have discrete eigenvalues, which are the roots of certain q -deformed Hermite polynomials. This leads, in general, to a non-equidistant phase-space lattice. An exception is the parafermionic oscillator, for which an equidistant space-phase lattice can be readily derived.

In other words, there are theoretical supports to the existence of a discrete space in which the fundamental length plays a role which is similar, in a way or another, to the lattice spacing in solid state physics. Accordingly, discrete derivatives written down in section 1.1 are well motivated from a quantum mechanical point of view. From the mathematical perspective, similar developments can be traced back to the introduction of q -hypergeometric functions and Jackson-derivatives (see e.g. chapter 1 in Gasper and Rahman (1990)).

1.6 Quick inspection of q -deformed Schrödinger equations

The q -deformed Schrödinger-equation on the non-commutative quantum Euclidian space has been analyzed for the harmonic oscillator (see e.g. Carow-Watamura and Watamura (1994)), for the Coulomb potential (Song and Liao (1992)), Feigenbaum and Freund (1996), Micu (1999)), as well as for the free particle (Hebecker and Weich (1992), Ocampo (1996), Papp (1997)). For this purpose one resorts to the q -deformed Laplacian, which has been derived in terms of the $SO_q(N)$ -covariant calculus (Carow-Watamura and S. Watamura (1994)). Just mention that the non-commutative N -dimensional quantum Euclidian space is characterized by typical coordinate dependent realizations of $[x_j, x_k]$ commutators in terms of bilinear or quadratic monomials such as given e.g. by the $SO_q(3)$ relationships $[x_1, x_2] = (q - 1)x_2x_1$, $[x_2, x_3] = (q - 1)x_3x_2$ and $[x_1, x_3] = (1 - q)x_2^2/\sqrt{q}$. In addition, there are well defined q -difference realizations of annihilation and creation operators, such as done in the study of a different version of the q -deformed harmonic oscillator (Atakishiyev and Suslov (1990), Li and Sheng (1992)). An interesting q -deformation of the Schrödinger-equation has been established by resorting to a q -deformation of the Witt algebra (Twarock (1999)), in which case the non-linear Schrödinger-equation derived by Doebner and Goldin (1992) gets reproduced as $q \rightarrow 1$. Other issues, such as q -deformations of the phase-space (Zumino (1991), Ubriaco (1993), Kehagias and Zoupanos (1994), Celeghini et al (1995), Wess (1997)), can also be mentioned. We have to recognize, however, that a sound confirmation of the physical relevance of q -deformations referred to above is still lacking. Nevertheless, there are parameter dependent difference equations in which the q -parameter is implemented from the very beginning in terms of an appropriate physical description, such as the q -symmetrized Harper-equation (see chapter 9).

1.7 Orthogonal polynomials of hypergeometric type on the discrete space

Classical orthogonal polynomials of a discrete variable are of much interest for several applications. A systematic study of such polynomials has already been done by Nikiforov, Suslov and Uvarov (1991). These polynomials are specified by the $\sigma(x)$ - and $\tau(x)$ -functions in (1.10). In the case of hypergeometric type polynomials we have to choose $\sigma(x)$ and $\tau(x)$ as polynomials of at most second and of first degrees in x , respectively. So one has e.g.

$$\sigma(x) = x, \quad \tau(x) = \mu(\gamma - x) - x, \quad x \in [0, b] \quad (1.65)$$

for Krawtchouk polynomials, whereas

$$\sigma(x) = x, \quad \tau(x) = \mu(\gamma + x) - x, \quad x \in [0, \infty) \quad (1.66)$$

for Meixner-ones. In the first (second) case one has $\gamma = N_0 = b - 1$, $\mu = p/q$ with $p+q = 1$ ($0 < \mu < 1$ and $\gamma > 0$). Equation (1.10) can also be converted into the self-adjoint form

$$\Delta(\sigma(x)\rho(x)\nabla y(x)) + \lambda\rho(x)y(x) = 0 \quad (1.67)$$

where λ denotes the eigenvalue and where the weight function $\rho(x)$ is implemented via

$$\Delta(\sigma(x)\rho(x)) = \tau(x)\rho(x) \quad . \quad (1.68)$$

In addition, (1.67) can be generalized as

$$\Delta(\sigma(x)\rho_m(x)\nabla v_m(x)) + \mu_m\rho_m(x)v_m(x) = 0 \quad (1.69)$$

where

$$v_m(x) = \Delta^m y(x) \quad (1.70)$$

and where m is a positive integer. Accordingly, (1.68) becomes

$$\Delta(\sigma(x)\rho_m(x)) = \tau_m(x)\rho_m(x) \quad (1.71)$$

where

$$\tau_m(x) = \tau(x + m) + \sigma(x + m) - \sigma(x) \quad . \quad (1.72)$$

The generalized weight function reads

$$\rho_m(x) = \sigma(x + 1)\rho_{m-1}(x) = \rho(x + m) \prod_{l=1}^m \sigma(x + l) \quad (1.73)$$

for which the eigenvalue becomes

$$\mu_m = \mu_{m-1} + \Delta\tau_{m-1}(x) \quad . \quad (1.74)$$

Furthermore, one finds the concrete realizations

$$\tau_m(x) = \tau(x) + m(\sigma'(x) + \tau'(x)) + \frac{m^2}{2}\sigma''(x) \quad (1.75)$$

and

$$\mu_m = \lambda + m\tau'(x) + \frac{m}{2}(m - 1)\sigma''(x) \quad . \quad (1.76)$$

Other details are presented for interested readers in Appendix A.